Sorting by right-to-left minima

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joint work with Rebecca Smith

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Washington, D.C.
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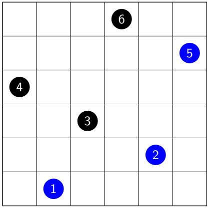
1. Identify right-to-left minima of π.



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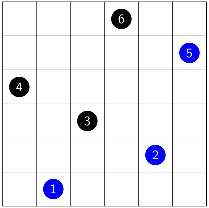

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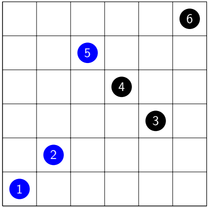

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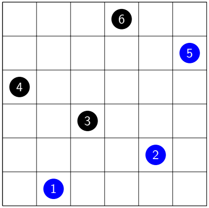

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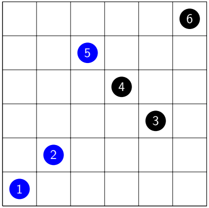

3. Call the result $S(\pi)$. e.g. $S(413625) = 125436$.

Steps

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2. Move them to the front.
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Observation:

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- Any permutation of length n is sorted to increasing order in at most $n-1$ iterations.

Data

$n\backslash k$	0	1	2	3	4	5	6	7
1	1							
2	1	1						
3	1	3	2					
4	1	7	11	5				
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Notation

$t^*_{n,k}$ is number of permutations of length n requiring *exactly* k iterations.

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$t_{n,k}$ is number of permutations of length n requiring *at most* k iterations.

At most one iteration?

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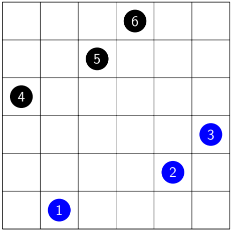

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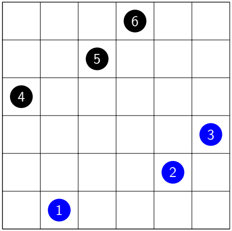

$Av(321, 314\bar{2})$

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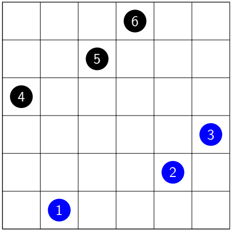

$Av(321, 314\bar{2})$

Corollary

$t^*_{n,1} = t_{n,1} - t_{n,0} = 2^{n-1} - 1$.

At most two iterations?

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Theorem

$$t_{n,2} = \frac{3^n - 2n + 3}{4}. \text{ (A111277)}$$

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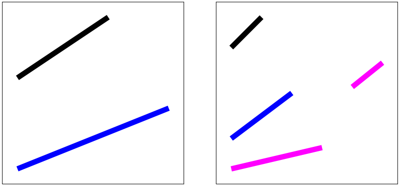

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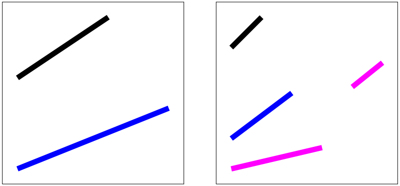

$$t_{n,2} = 1 + \sum_{k=1}^{n-1} 3^{k-1}(n-k) = \frac{3^n - 2n + 3}{4}.$$

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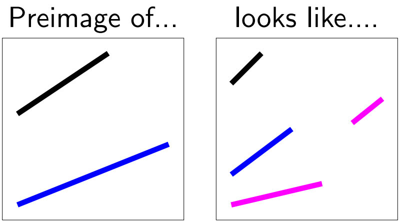

Corollary

$$t^*_{n,2} = t_{n,2} - t_{n,1} = \frac{3^n - 2n + 3}{4} - 2^{n-1} = \frac{3^n - 2^{n+1} - 2n + 3}{4}.$$

At most three iterations?

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$$t_{n,3} = 2^{2n-3} - (n-2)2^{n-2}$$

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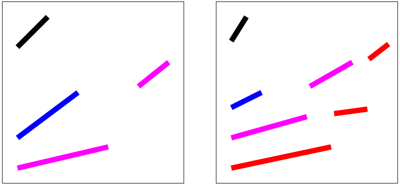

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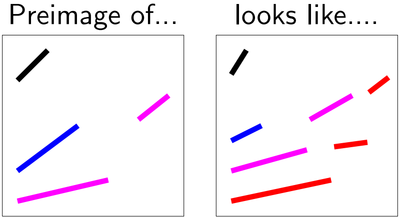

Corollary

$$t^*_{n,3} = t_{n,3} - t_{n,2} = \frac{4^n - 2 \cdot 3^n - (2n-4)2^n + (4n-6)}{8}.$$

Recap

- $t_{n,0} = 1$
- $t_{n,1} = 2^{n-1}$
- $t_{n,2} = \dfrac{3^n - 2n + 3}{4}$
- $t_{n,3} = 2^{2n-3} - (n-2)2^{n-2}$

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In the long run, $t_{n,n-1} = n!$ and $\lim_{n\to\infty} \dfrac{\left(\dfrac{n^n}{2^{n-1}}\right)}{n!} = \infty$.

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In these cases, leading term of $t_{n,k}$ is $\dfrac{(k+1)^n}{2^k}$.

In the long run, $t_{n,n-1} = n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ and $\lim_{n\to\infty} \dfrac{\left(\dfrac{n^n}{2^{n-1}}\right)}{n!} = \infty$.

Exactly $n-1$ iterations?

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The permutations counted by $t^*_{n,n-1}$ are exactly those where

- 2 appears before 1, and
- the patterns 123, 231, and 312 are not formed by consecutive *values*.

Intuition:

- If 1 precedes 2...

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Example:

3576241 → 1357624 → 1243576 → 1235647 → ...

Exactly $n-1$ iterations?

Theorem

$$\sum_{n=0}^{\infty} t^*_{n,n-1}\frac{x^n}{n!} = \sec(x) + \tan(x). \text{ (A000111)}$$

Sketch:
Both permutations counted by $t^*_{n,n-1}$ and Euler up-down permutations have have a generating tree with
Root: $(1,1)$
Rule: $(n,k) \to (n+1, n-k+2)(n+1, n-k+3)\cdots(n+1, n+1)$

Up-down permutations

- start with 12
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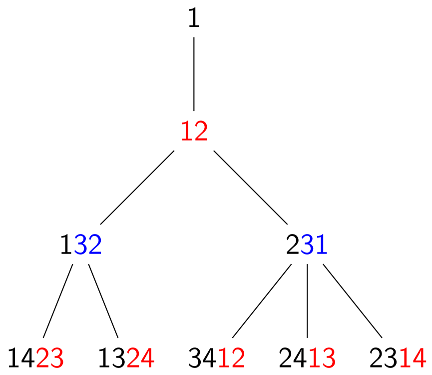

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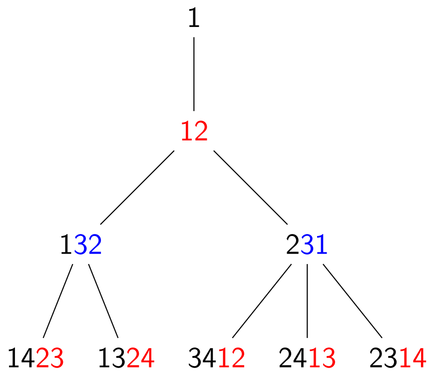

$n-1$ iteration permutations

- lowest values are 21
- no 123, 231, or 312 in consecutive values

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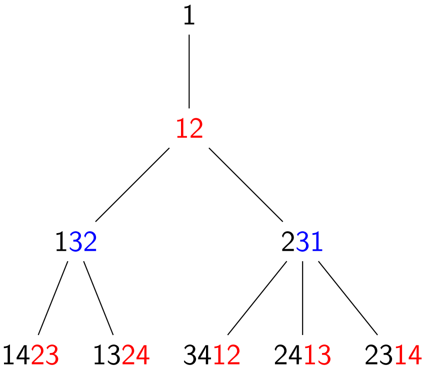

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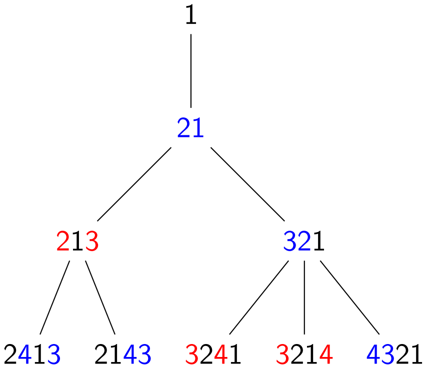
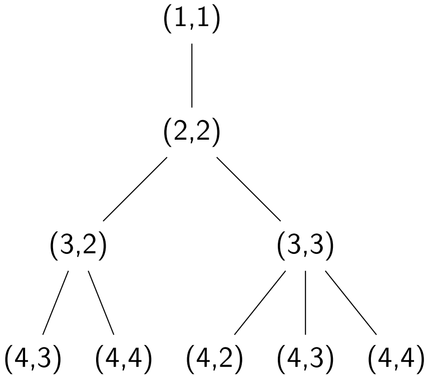


Generating tree



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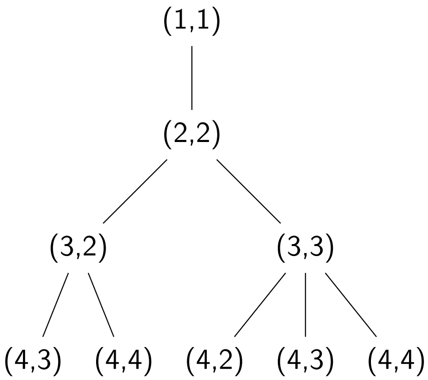

Entringer numbers

$E(n,k) =$
$E(n,k-1) + E(n-1,n-k+1)$

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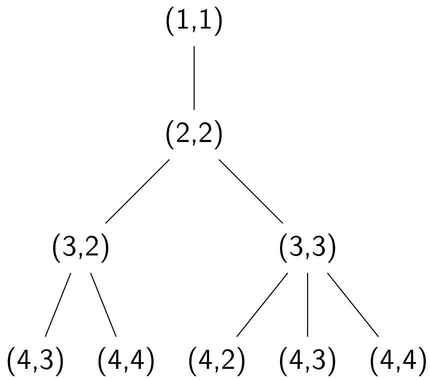

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1	1	0	0	0	0
2	0	1	0	0	0
3	0	1	1	0	0
4	0	1	2	2	0
5	0	2	4	5	5

Exactly $n-2$ iterations?

$n \backslash k$	0	1	2	3	4	5	6	7
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Sketch: make generating tree for these two diagonals together.

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Thanks for listening!
slides at `faculty.valpo.edu/lpudwell`