An Erdős-Szekeres Permutation Game

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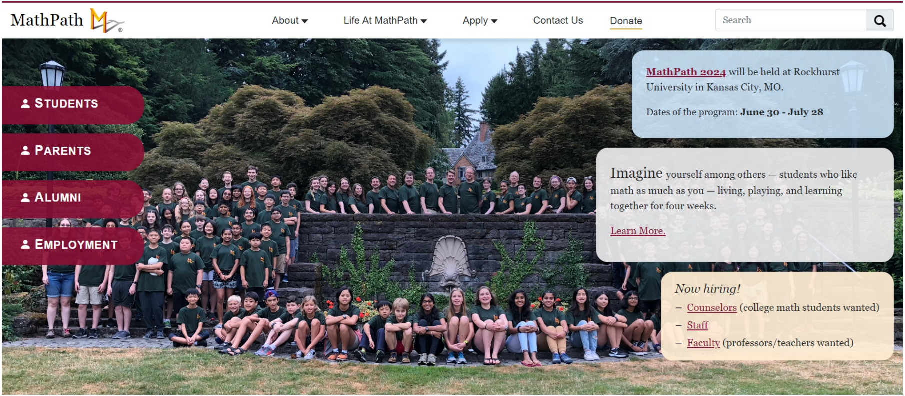
Special Session on Permutation Patterns; AMS Fall Eastern Sectional Meeting
October 20, 2024

Audience: young thinkers



MathPath (mathpath.org)
a national residential summer camp for 11-14 year olds
showing high interest in mathematics.

The Game

The (a, b)-permutation game

- Take turns naming *different* positive numbers.
- The game is over when the permutation you made so far has an increasing subsequence of length a or a decreasing subsequence of length b.

Question: How long can you make the (a, b)-permutation game last?

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Question: How long can you make the (a, b)-permutation game last?

Theorem (Erdős-Szekeres, 1935)

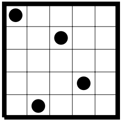
Any permutation of length $(a-1)(b-1)+1$ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

Proof Sketch (Seidenberg, 1959)

- For each turn, keep track of:
  1. the longest increasing sequence ending with the new number.
  2. the longest decreasing sequence ending with the new number.

Example:

Chosen number:	5	1	4	2
increasing:	1	1	2	2
decreasing:	1	2	2	3



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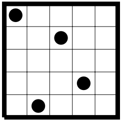

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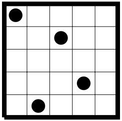

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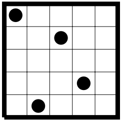

- The corresponding pairs of numbers are different because...
- The game is over when...
- There are $(a-1)(b-1)$ different possible pairs before a game-ending choice must be made.

What happens when you play this competitively?

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$12 \rightarrow 123$
$12 \rightarrow 12\frac{4}{3}$
$12 \rightarrow 12\frac{1}{2}$

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$12 \to 123$

~~$12 \to 12\frac{4}{3}$~~

~~$12 \to 12\frac{1}{2}$~~

$12 \to 132$

$12 \to 231$

What happens when you play this competitively?

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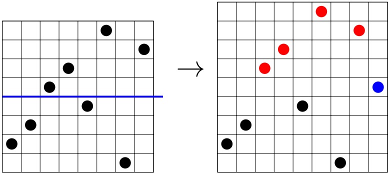
23564817 → 236749185



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  When you complete an increasing subsequence of length a or a decreasing subsequence of length b, *you lose*.

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Question: Who has the winning strategy?

$a \geq 2,\ b = 2$

- Minimum length game? 2 turns
- Maximum length game? a turns

$a \geq 2,\ b = 2$

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- Maximum length game? a turns

Strategy:

- Generate an increasing permutation
- If a is even player 1 wins.
- If a is odd player 2 wins.

Example:

$$\epsilon \to_1 1 \to_2 12 \to_1 123 \to_2 1234 \to_1 12345 \to_2 \cdots$$

Representing moves visually

Draw a $(b-1) \times (a-1)$ grid, and label cells by ordered pairs as follows:

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	$\cdots$
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	
		$\vdots$			$\ddots$

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Shade a cell when the most recent move has this ordered pair in the Seidenberg proof of Erdős-Szekeres Theorem.

The permutation game, re-framed

Play on a $(b-1) \times (a-1)$ grid.

- First move: Player 1 shades the $(1,1)$ cell.
- Players take turns shading *legal* cells.
- Last move: player who takes the $(a-1, b-1)$ cell wins.

What makes a cell *legal*?

The permutation game, re-framed

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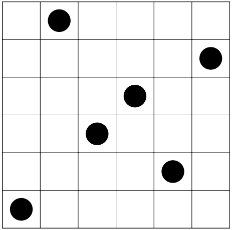
Example:



(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
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(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)

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What makes a cell *legal*?

- Either the row number or column number is strictly larger each previously-shaded cell.

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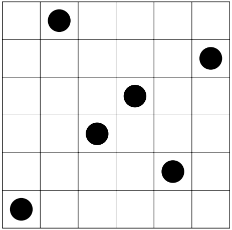
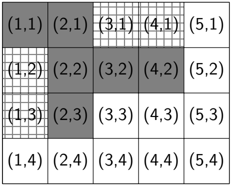


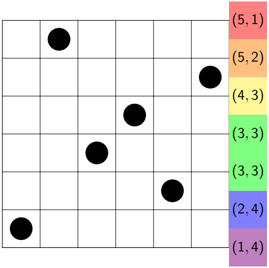


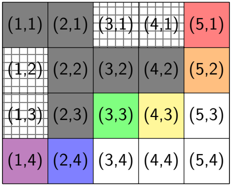
The permutation game, re-framed

What makes a cell *legal*?

- Either the row number or column number is strictly larger each previously-shaded cell.
- Must share an edge with some previously eliminated cell.

Example:





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Play on a $(b-1) \times (a-1)$ grid.

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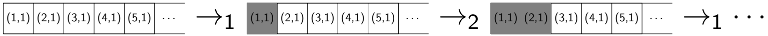
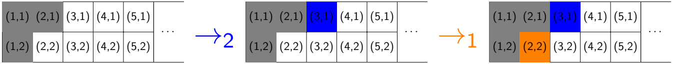
$a \geq 2,\ b = 2$



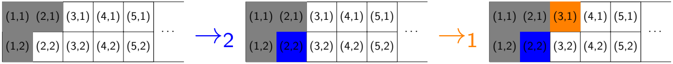
The permutation game, re-framed

$a \geq 3,\ b = 3$

Case 1:



Case 2:



The permutation game, re-framed

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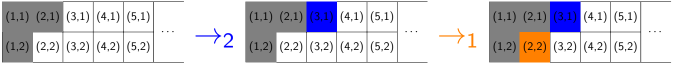
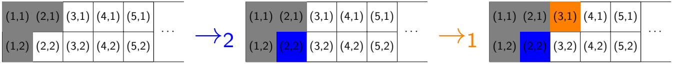


Case 2:

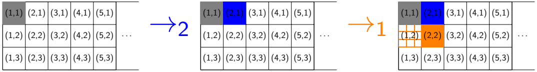


Endgame:

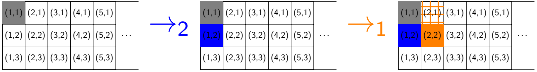
The permutation game, re-framed

$a \geq 4, \ b = 4$

Opening moves:



or



The permutation game, re-framed

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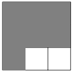
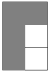
Midgame:

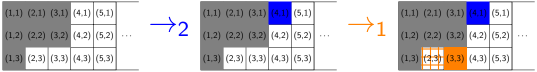
State 1: 

State 2: 

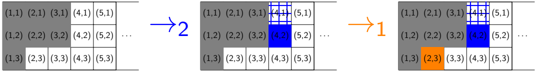
The permutation game, re-framed

$a \geq 4,\ b = 4$

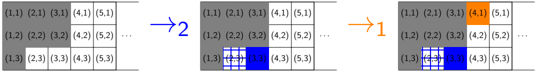
Case 1a:



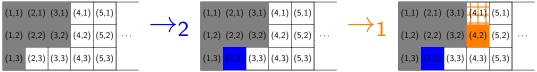
Case 1b:



Case 1c:



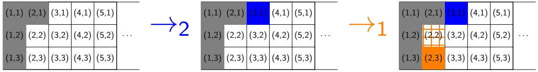
Case 1d:



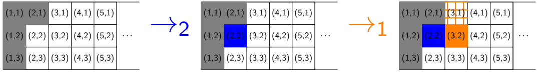
The permutation game, re-framed

$a \geq 4, \; b = 4$

Case 2a:



Case 2b:



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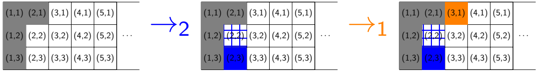

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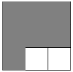
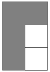
State 1: 

State 2: 

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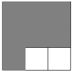
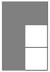
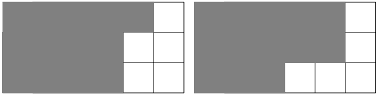
State 1: 

State 2: 

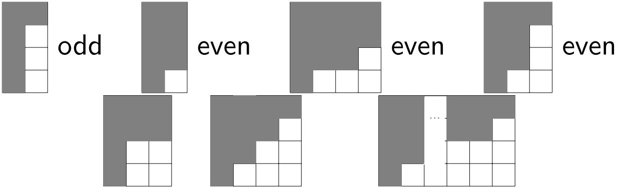
Endgame:



The permutation game, re-framed

$a \geq 5,\ b = 5$

Seven(ish) states:



The permutation game, re-framed

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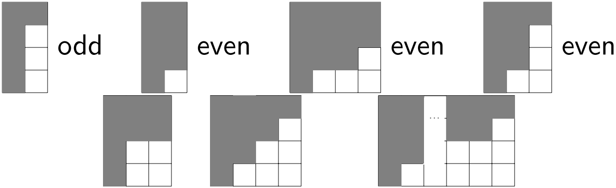

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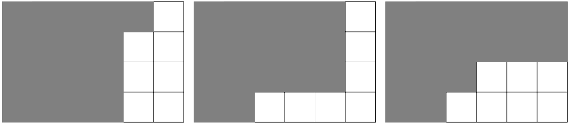

Moves for $a \geq b$ game

b	minimum moves	maximum moves	actual moves using strategy
2	2	a	a
3	3	$2a-1$	$2a-2$
4	4	$3a-2$	$2a-3$ or $2a-1$
5	5	$4a-3$	between $2a-C$ and $4a-6$

What if... ?

- First player to complete $1 \cdots a$ or $b \cdots 1$ *wins*?

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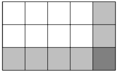
  Same strategy on smaller board



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  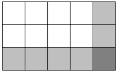
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  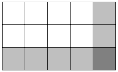
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  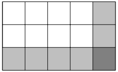

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  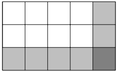
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  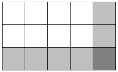
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  Results for $(a-1)(b-1)+1 \leq 15$ (Harary, Sagan, and West); general case open
- More than two players?
  Open
- $b \geq 6$?
  Open, but conjectured first player winning strategy exists.

For more details...

- Frank Harary, Bruce Sagan, and David West, Computer-aided analysis of monotonic sequence games, *Atti Accad. Perolitana Pericolanti Cl. Sci. Fis. Mat. Natur.* **61** (1983), 67–78.
- Lara Pudwell, Catalan Numbers and Permutations, *Mathematics Magazine* **97.3** (2024), 279–283.
- Abraham Seidenberg, A Simple Proof of a Theorem of Erdős and Szekeres, *J. Lond. Math. Soc.* **34.3** (1959), 352.

Thanks for listening!

slides at faculty.valpo.edu/lpudwell