Patterns in Permutations

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Valparaiso Women in STEM Research Night
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Definition

A permutation of length n is list of the numbers $1, 2, \ldots, n$ where order matters.

Length 1?

$$1$$

Length 2?

$$12, 21$$

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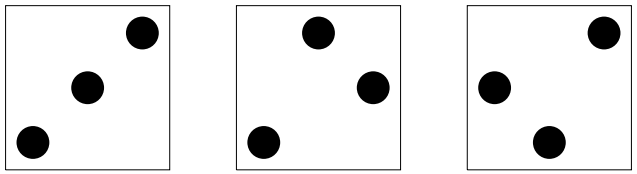
How many permutations of length n?

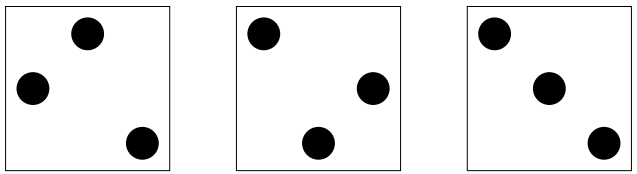
$$n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$$

Note

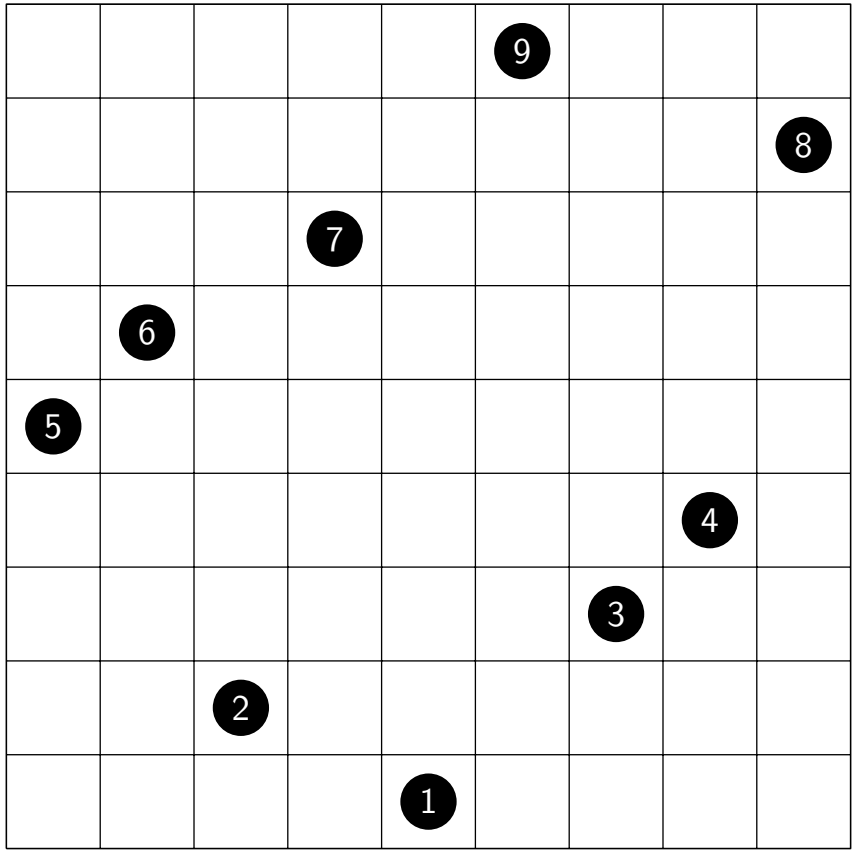
Permutation $p = p_1 p_2 \cdots p_n$ is often visualized by plotting the points (i, p_i) in the xy-plane.



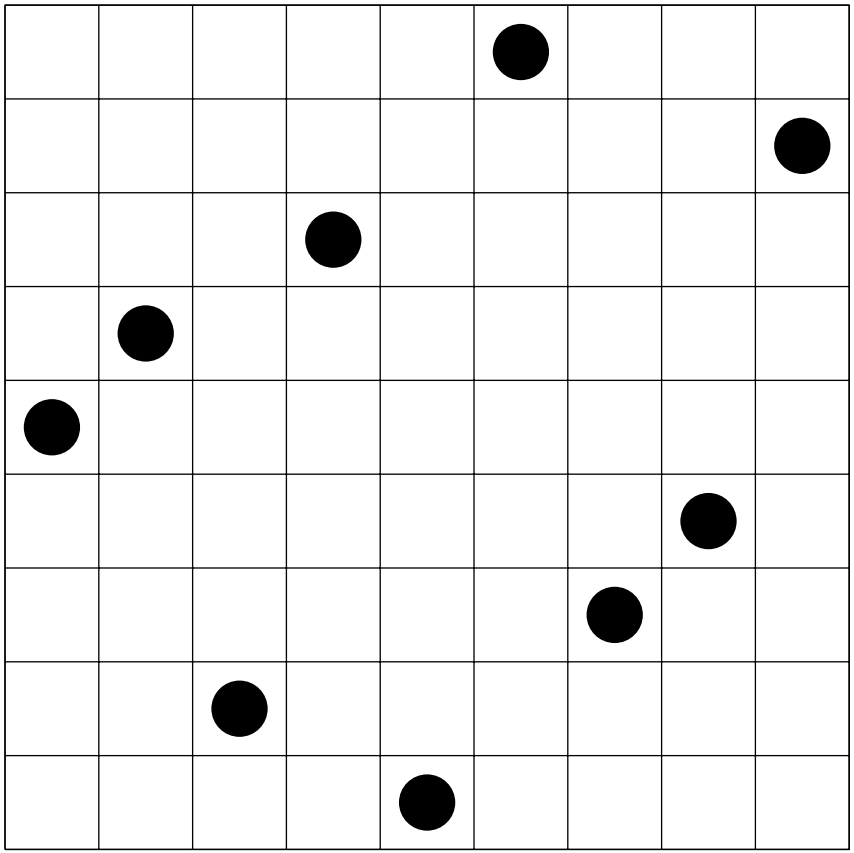
123 132 213

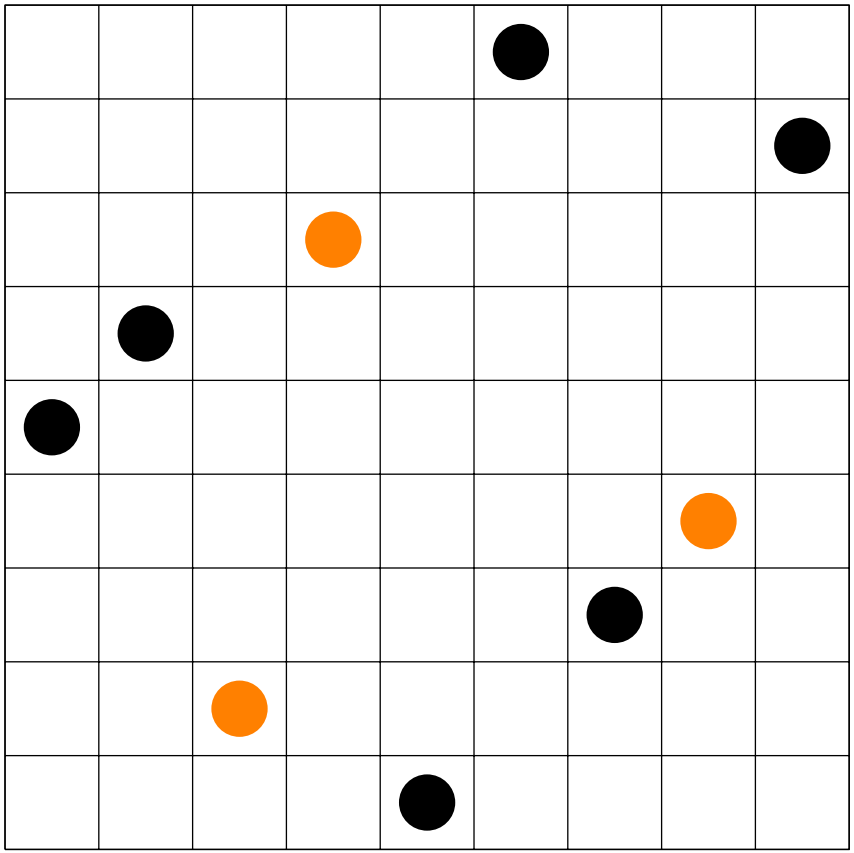


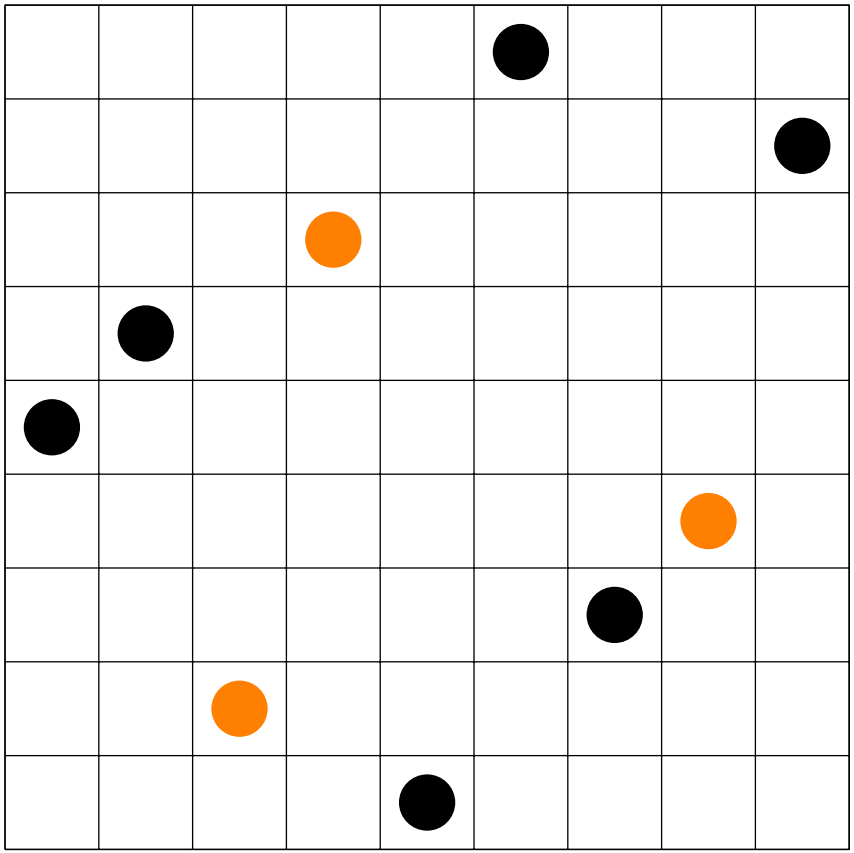
231 312 321

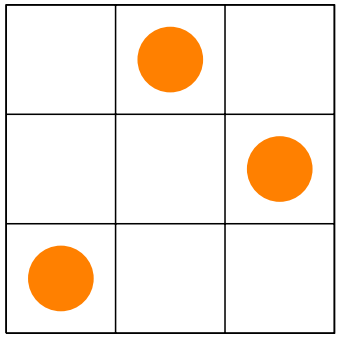


562719348

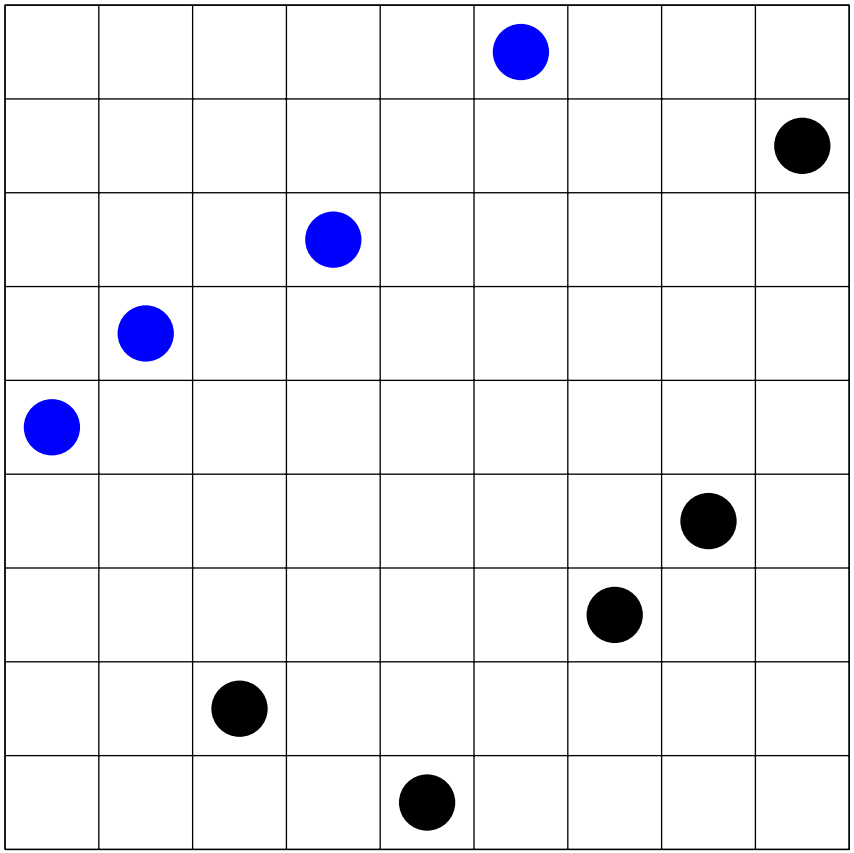


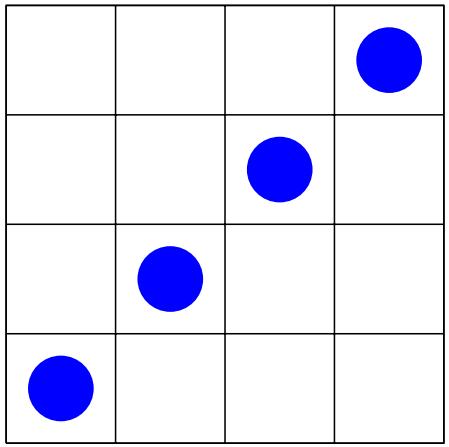




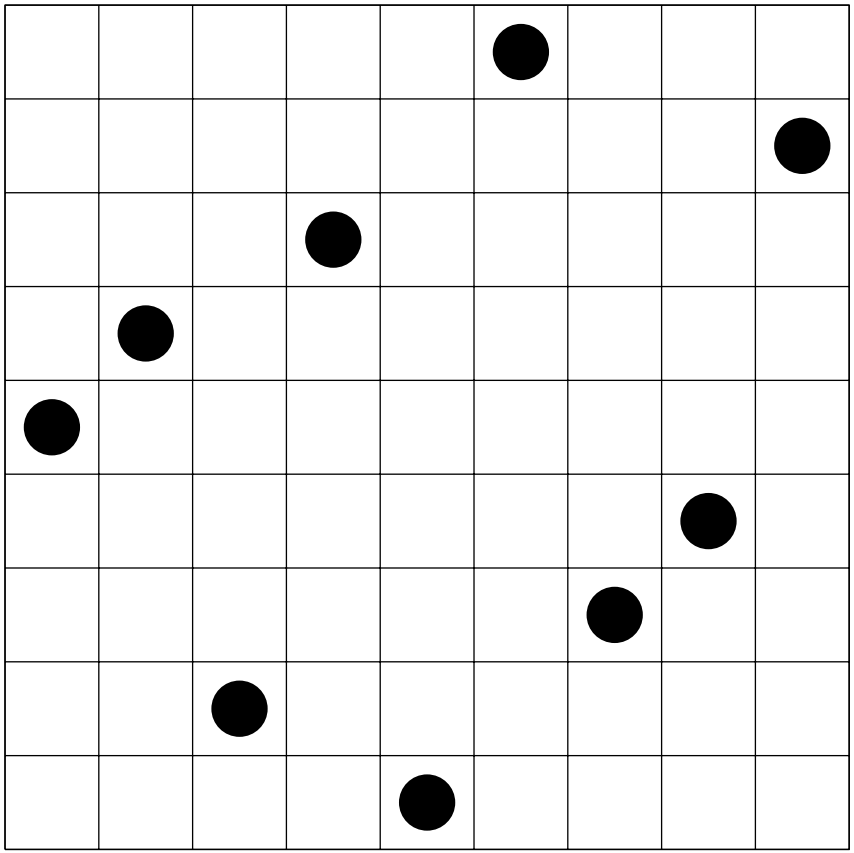


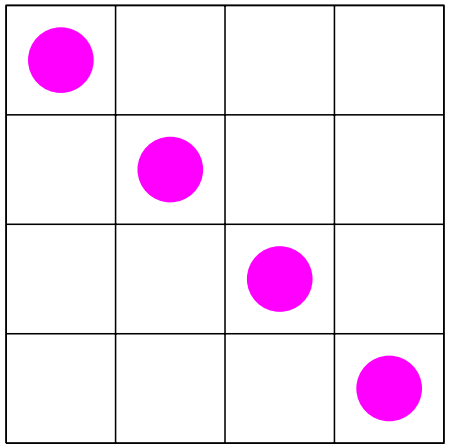
562719348 contains the pattern 132





562719348 contains the pattern 1234





562719348 avoids the pattern 4321

Big question

How many permutations of length n contain the pattern p?

Or, alternatively...

Big question

How many permutations of length n avoid the pattern p?

(depends on what p is!)

Question

How many permutations of length n avoid the pattern  ?

Length 1?

Question

How many permutations of length n avoid the pattern  ?

Length 1? (1)

Length 2?

Question

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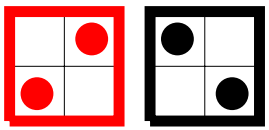
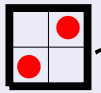


Length 3?

Question

How many permutations of length n avoid the pattern  ?

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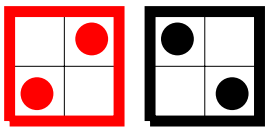
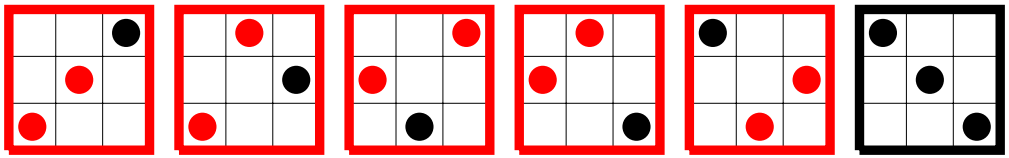


Length 3? (1)



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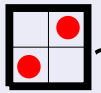
How many permutations of length n avoid the pattern  ?

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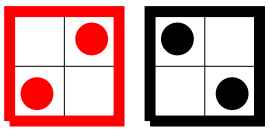

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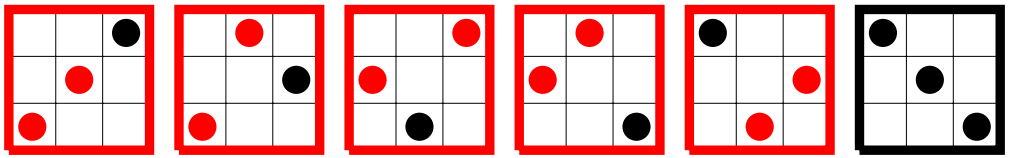

The decreasing permutation is the only permutation of length n that avoids 12.

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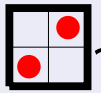
How many permutations of length n avoid the pattern  ?

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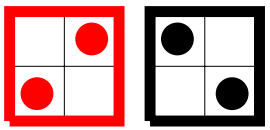
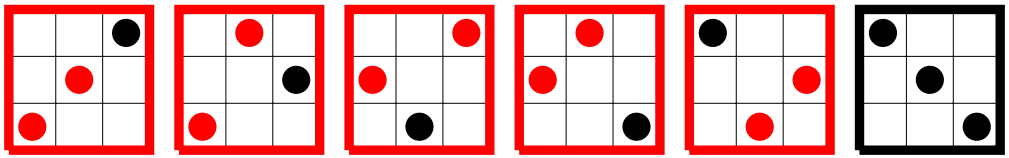


Length 3? (1)



The decreasing permutation is the only permutation of length n that avoids 12.

Similar: the increasing permutation is the only permutation of length n that avoids 21.

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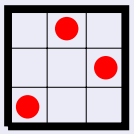
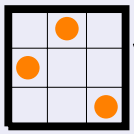
How many permutations of length n avoid the patterns  and  ?

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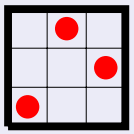
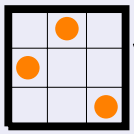
How many permutations of length n avoid the patterns  and  ?

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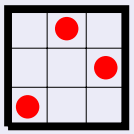
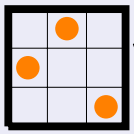
How many permutations of length n avoid the patterns  and  ?

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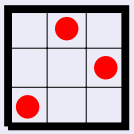
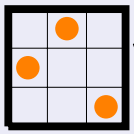
How many permutations of length n avoid the patterns  and  ?

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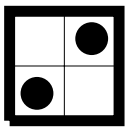
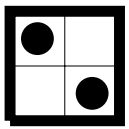
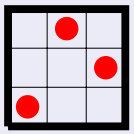
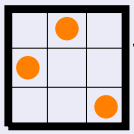
Length 2? (2)

Length 3?

Question

How many permutations of length n avoid the patterns  and  ?

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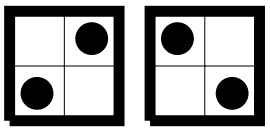
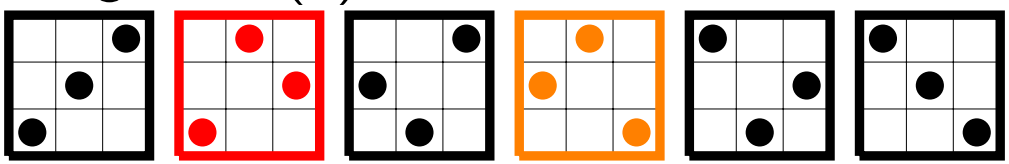


Length 3? (4)



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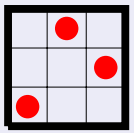
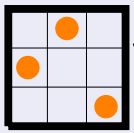
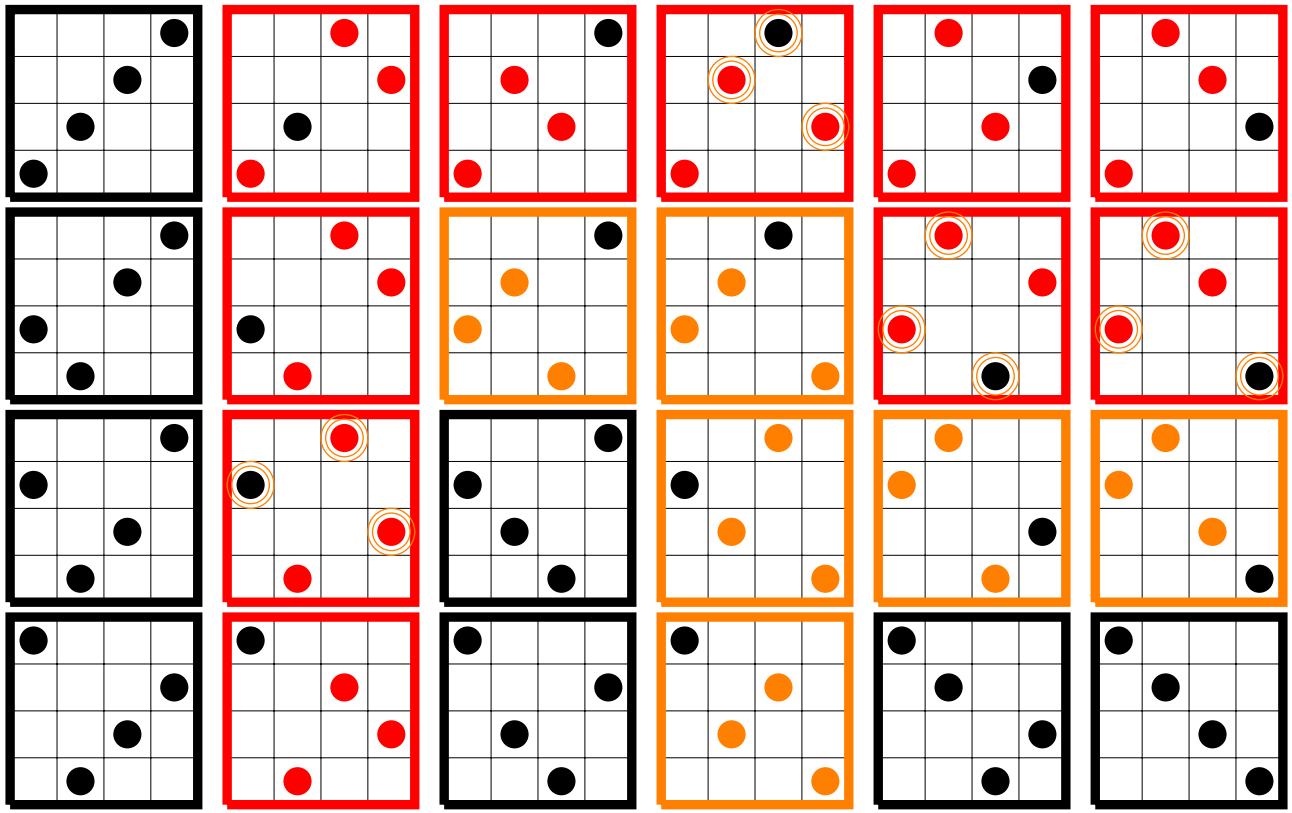
How many permutations of length n avoid the patterns  and  ?



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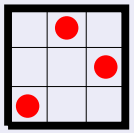
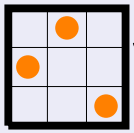
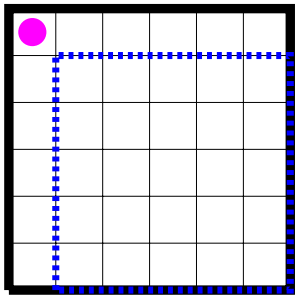
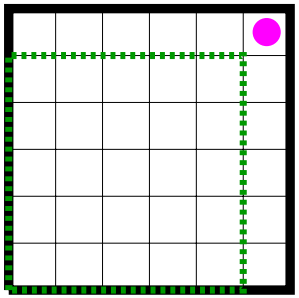
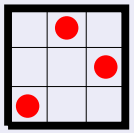
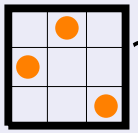
How many permutations of length n avoid the patterns  and  ? (T_n)

or

Question

How many permutations of length n avoid the patterns  and  ? (T_n)

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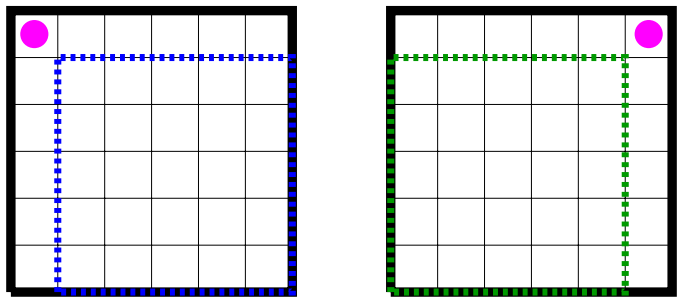

Answer: $T_1 = 1$ and $T_n = T_{n-1} + T_{n-1} = 2T_{n-1}$, so...

$$T_n = 2^{n-1}.$$

How many permutations of length n avoid the pattern(s)...

- 12? $1, 1, 1, 1, 1, 1, 1, \ldots$
- 132 and 231? $1, 2, 4, 8, 16, 32, 64, \ldots$ 2^{n-1}

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- any one pattern of length 3? $1, 2, 5, 14, 42, 132, 429, \ldots$ (Catalan)

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- 1234? $1, 2, 6, 23, 103, 513, 2761, \ldots$ (Gessel, 1990)
- 1342? $1, 2, 6, 23, 103, 512, 2740, \ldots$ (Bóna, 1997)
- 1324? $1, 2, 6, 23, 103, 513, 2762, \ldots$ (open question!)

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Questions:

- What happens if we contain patterns instead of avoid?

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Questions:

- What happens if we contain patterns instead of avoid?
- What happens if we change the definition of pattern?

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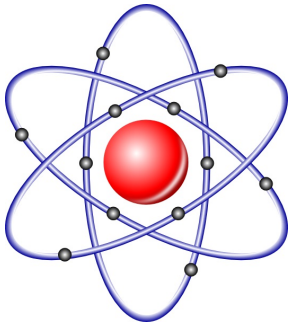
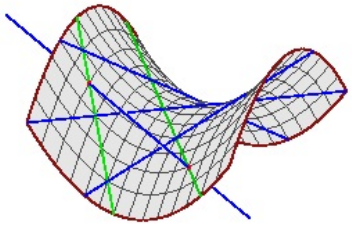
- 1324? $1, 2, 6, 23, 103, 513, 2762, \ldots$ (open question!)

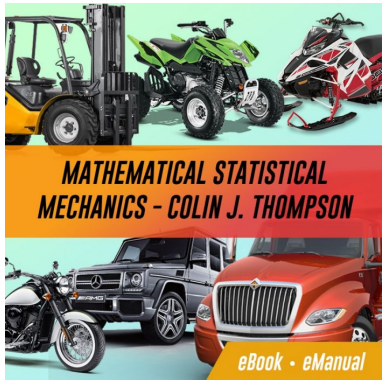
Questions:

- What happens if we contain patterns instead of avoid?
- What happens if we change the definition of pattern?
- What happens if we look for patterns in math objects other than permutations?


A
Z



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