On the distribution of peaks (and other statistics)

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Motivation

$$\mathrm{des}(\pi) = |\{i|\pi_i > \pi_{i+1}\}|$$

Known:

- $|\{\pi \in \mathcal{S}_n|\mathrm{des}(\pi) = k\}| =$

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$n\backslash k$	0	1	2	3	4	5	6	7
1	1							
2	1	1						
3	1	4	1					
4	1	11	11	1				
5	1	26	66	26	1			
6	1	57	302	302	57	1		
7	1	120	1191	2416	1191	120	1	
8	1	247	4293	15619	15619	4293	247	1

Table: $A(n,k)$ for small values of n and k

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- What is $|\{\pi \in \mathcal{S}_n(\rho)|des(\pi) = k\}|$?
  $\rho \in \mathcal{S}_m$

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$$\mathrm{asc}(\pi) = |\{i|\pi_i < \pi_{i+1}\}| \text{ (ascents)}$$
$$\mathrm{des}(\pi) = |\{i|\pi_i > \pi_{i+1}\}| \text{ (descents)}$$

Ascents and Descents

For $\rho \in \{132, 213, 231, 312\}$,

$$|\{\pi \in \mathcal{S}_n(\rho) | \mathrm{des}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(\rho) | \mathrm{asc}(\pi) = k\}|.$$

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(Follows from bijection between $\mathcal{S}_n(231)$ and Dyck paths)

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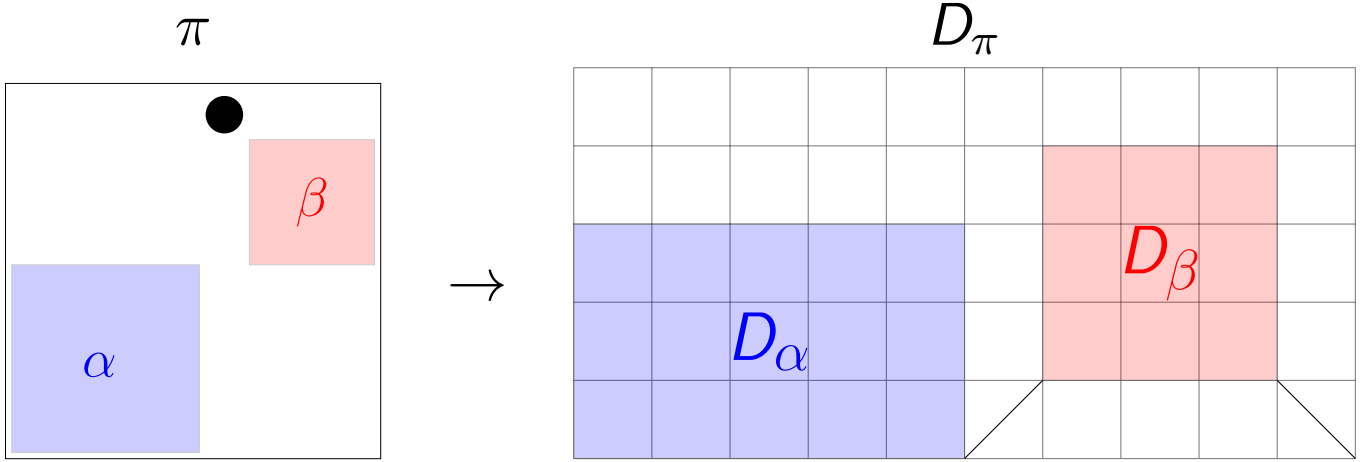

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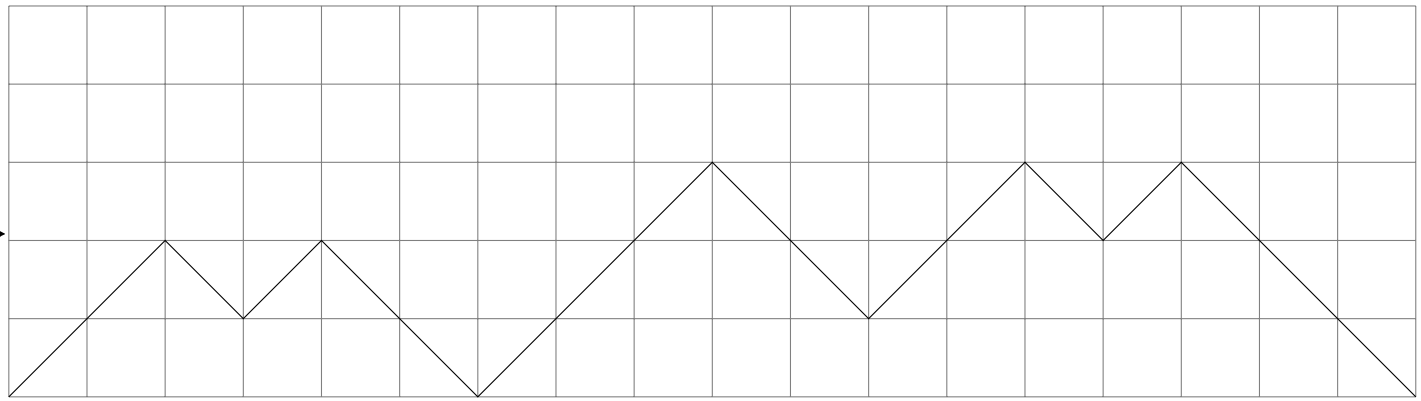

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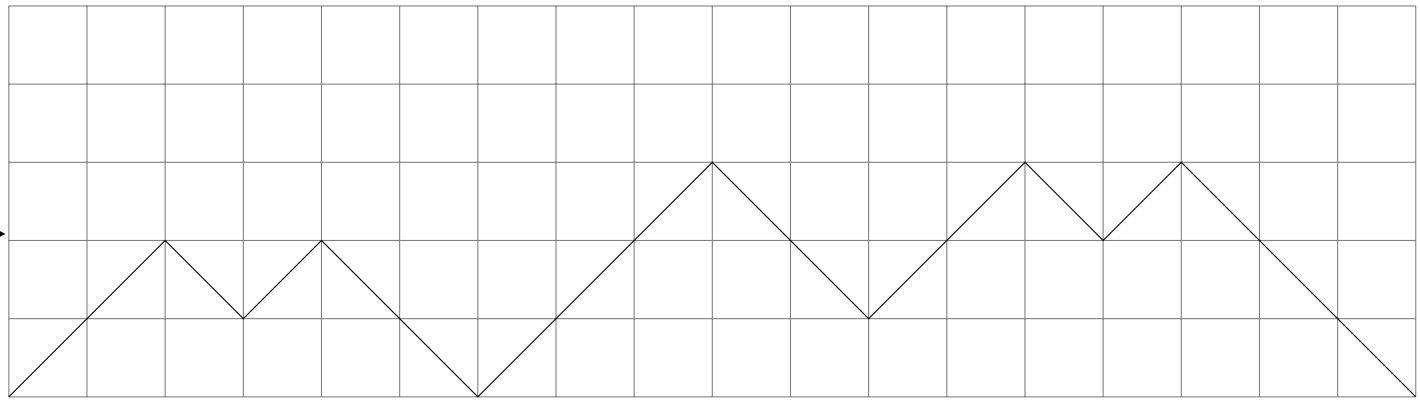

Ascents: 12, 29, 48, 67

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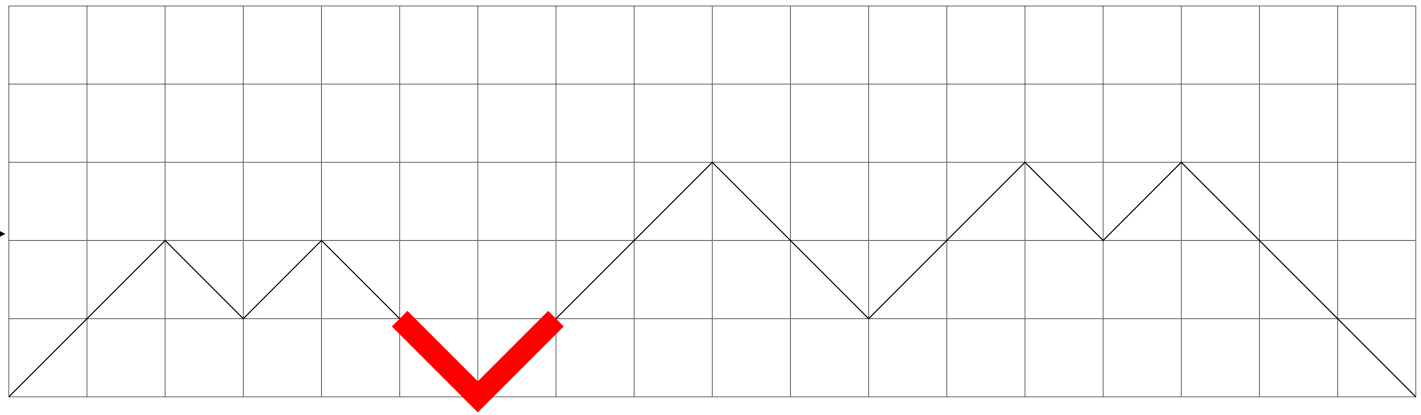

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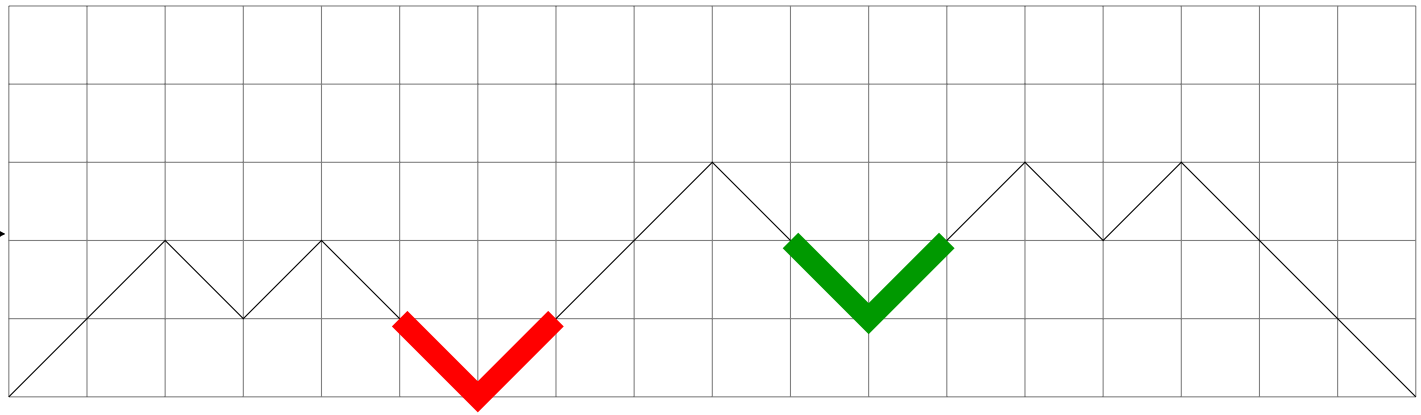

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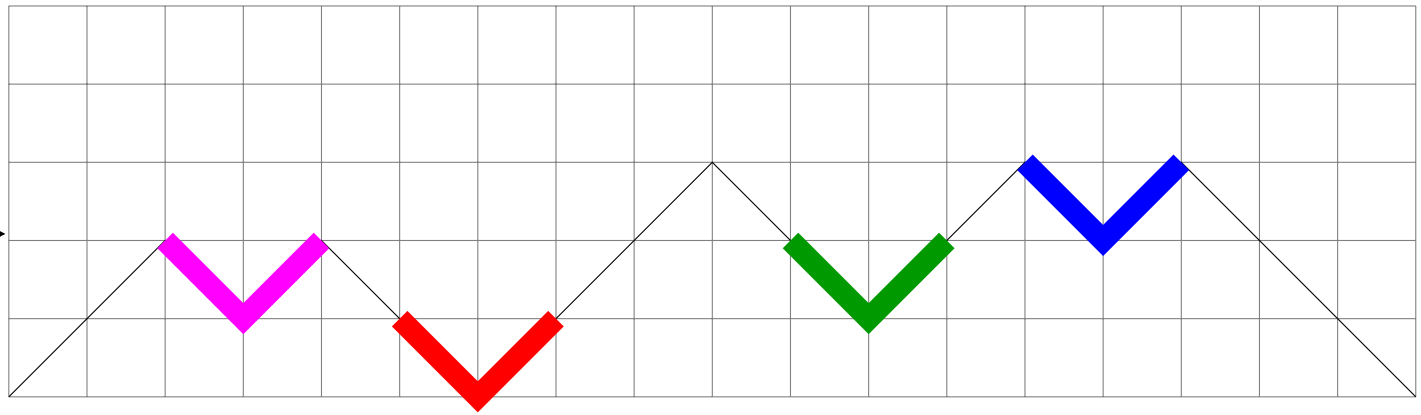

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Ascents and Descents

$$|\{\pi \in \mathcal{S}_n(321) | \mathrm{des}(\pi) = k\}| = |\{\pi \in \mathcal{S}_n(123) | \mathrm{asc}(\pi) = k\}|$$

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A091156: $G(t,z) = \sum_{\pi \in \mathcal{S}_n(321)} t^{\mathrm{des}(\pi)} z^n$ satisfies
$$z(1-z+tz)G^2 - G + 1 = 0.$$

M. Barnabei, F. Bonetti and M. Silimbani, The descent statistic on 123 avoiding permutations, *Sém. Lothar. Combin.* **68** (2010), B63a, 7 pp.

More Statistics

Question: What is $|\{\pi \in \mathcal{S}_n(\rho) | \mathrm{st}(\pi) = k\}|$?

- $\mathrm{asc}(\pi) = |\{i | \pi_i < \pi_{i+1}\}|$ (ascents)
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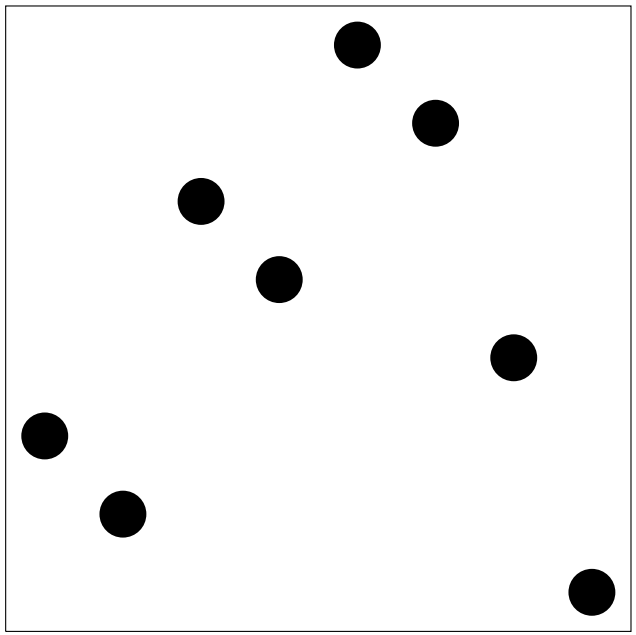
- $\mathrm{asc}(\pi) = |\{i | \pi_i < \pi_{i+1}\}|$ (ascents)
- $\mathrm{des}(\pi) = |\{i | \pi_i > \pi_{i+1}\}|$ (descents)
- $\mathrm{pk}(\pi) = |\{i | \pi_i < \pi_{i+1}$ and $\pi_{i+1} > \pi_{i+2}\}|$ (peaks)
- $\mathrm{vl}(\pi) = |\{i | \pi_i > \pi_{i+1}$ and $\pi_{i+1} < \pi_{i+2}\}|$ (valleys)
- $\mathrm{dasc}(\pi) = |\{i | \pi_i < \pi_{i+1} < \pi_{i+2}\}|$ (double ascents)
- $\mathrm{ddes}(\pi) = |\{i | \pi_i > \pi_{i+1} > \pi_{i+2}\}|$ (double descents)

Data

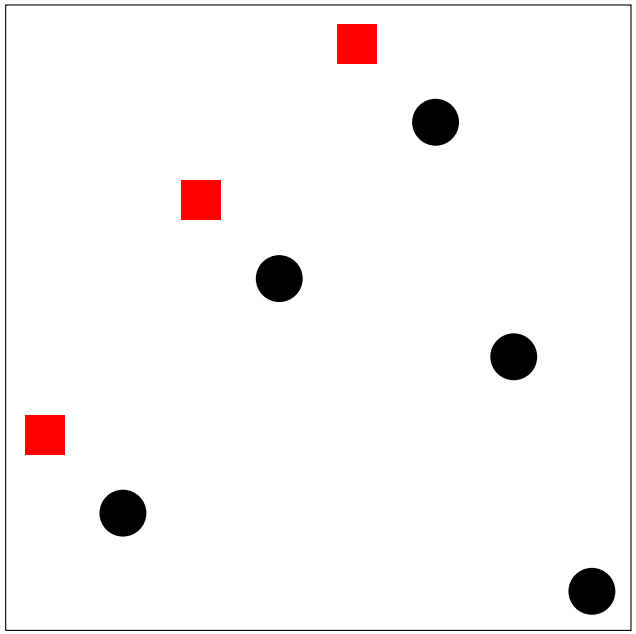
$\rho \setminus \mathrm{st}$	**pk**	**vl**	**dasc**	**ddes**
231	A091894	A236406	A092107	A092107
312	A236406	A091894	A092107	A092107
321	A236406	A236406	new	(none)

$\mathcal{S}_n(312)$ vs. $\mathcal{S}_n(321)$



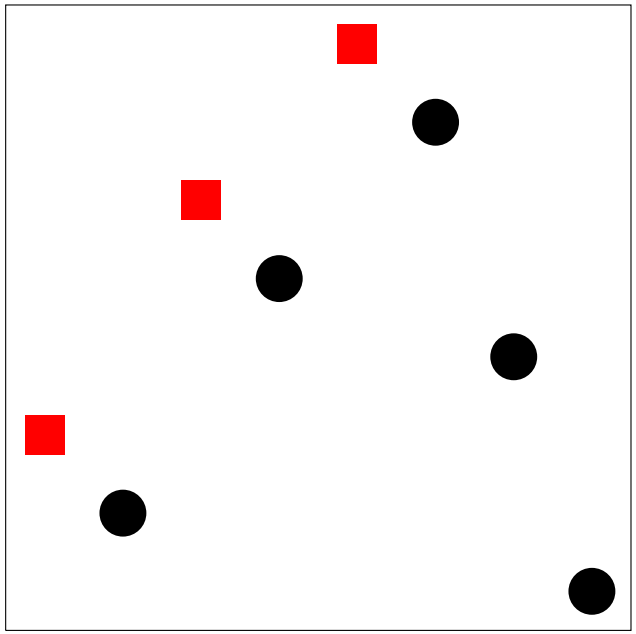
$32658741 \in \mathcal{S}_8(312)$

$\mathcal{S}_n(312)$ vs. $\mathcal{S}_n(321)$



$\underline{3}2\underline{6}5\underline{8}741 \in \mathcal{S}_8(312)$

$\mathcal{S}_n(312)$ vs. $\mathcal{S}_n(321)$



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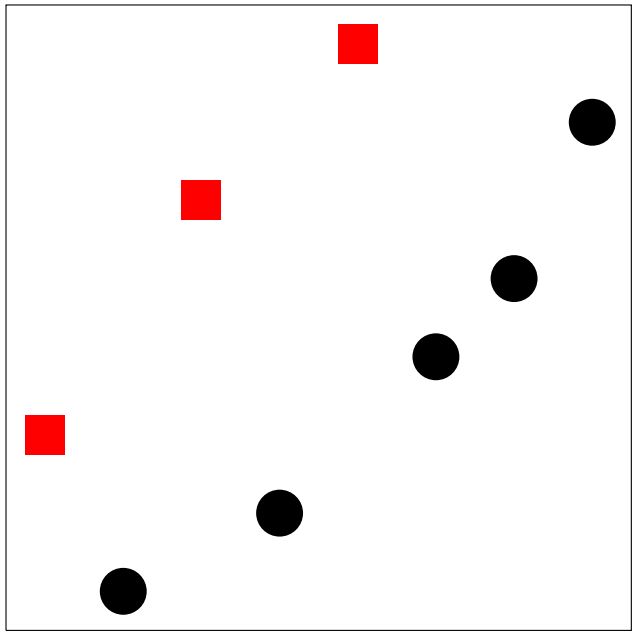
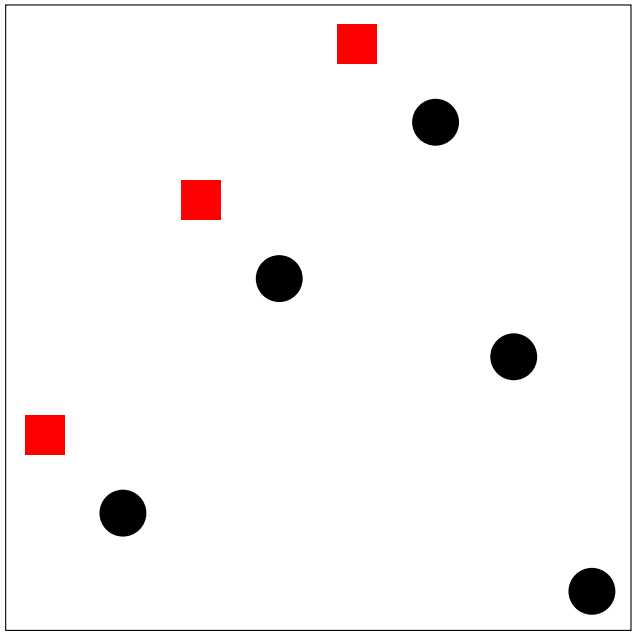
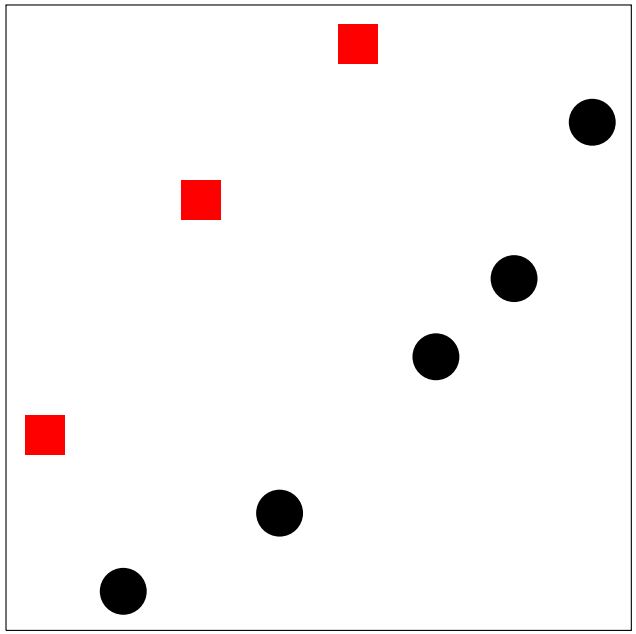


$\underline{3}1\underline{6}2\underline{8}457 \in \mathcal{S}_8(321)$

$\mathcal{S}_n(312)$ vs. $\mathcal{S}_n(321)$



$\underline{3}2\underline{6}5\underline{8}741 \in \mathcal{S}_8(312)$
peaks: $2\underline{6}5$, $5\underline{8}7$



$\underline{3}1\underline{6}2\underline{8}457 \in \mathcal{S}_8(321)$
peaks: $1\underline{6}2$, $2\underline{8}4$

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- A092107 : Dyck paths of semilength n having exactly k UUU's (triple rises), generating function given

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- A091894 : Dyck paths of semilength n, having k DDU's, generating function given

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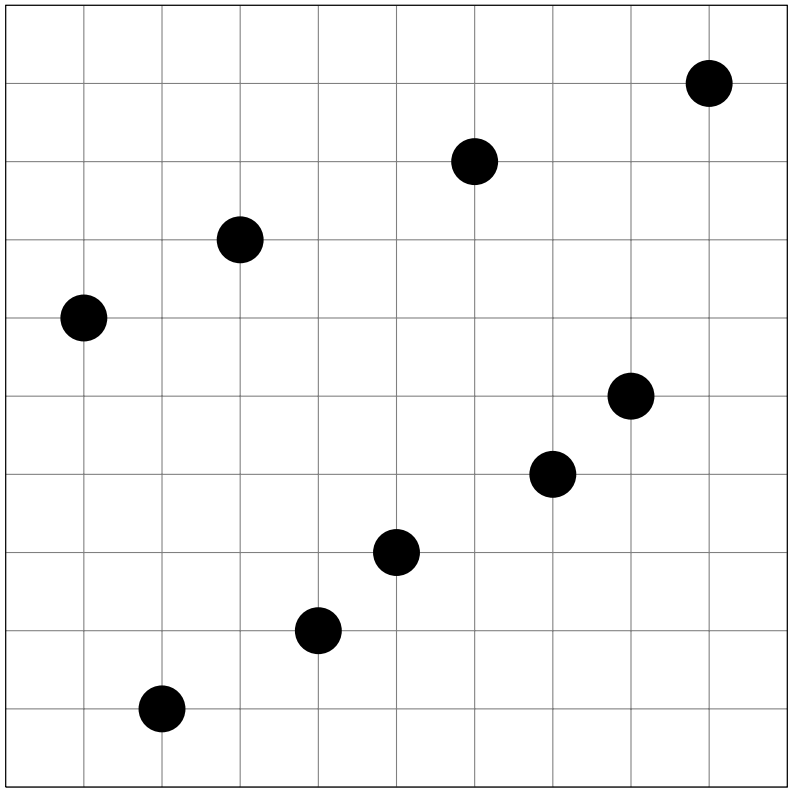
- A092107 : Dyck paths of semilength n having exactly k UUU's (triple rises), generating function given
- A091894 : Dyck paths of semilength n, having k DDU's, generating function given
- A236406 : data from Andrew Baxter's `Statter` algorithm, no formula known

$\mathcal{S}_n(321)$ to (shifted) Dyck paths



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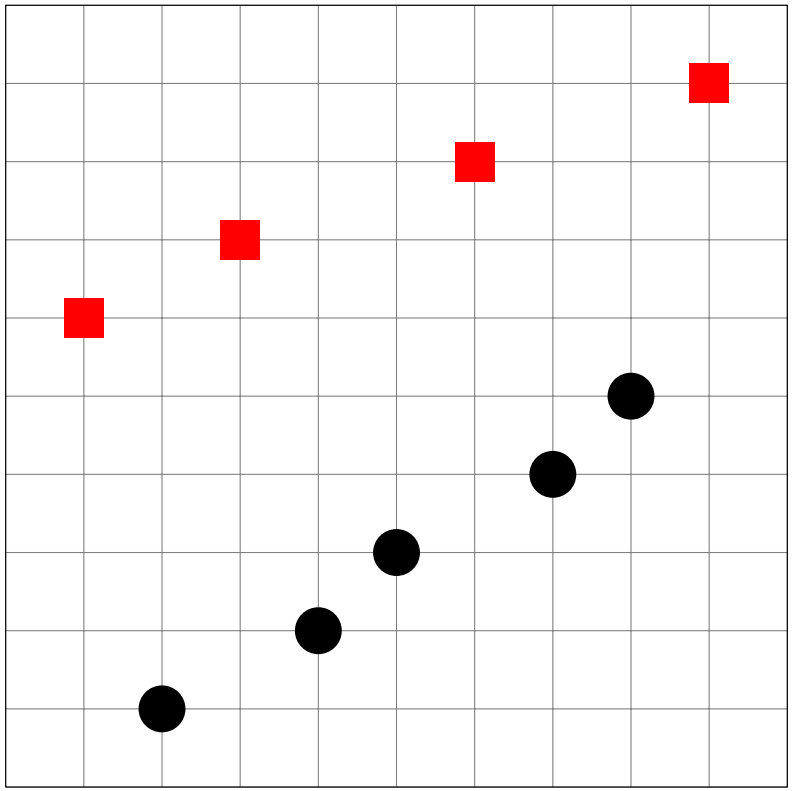

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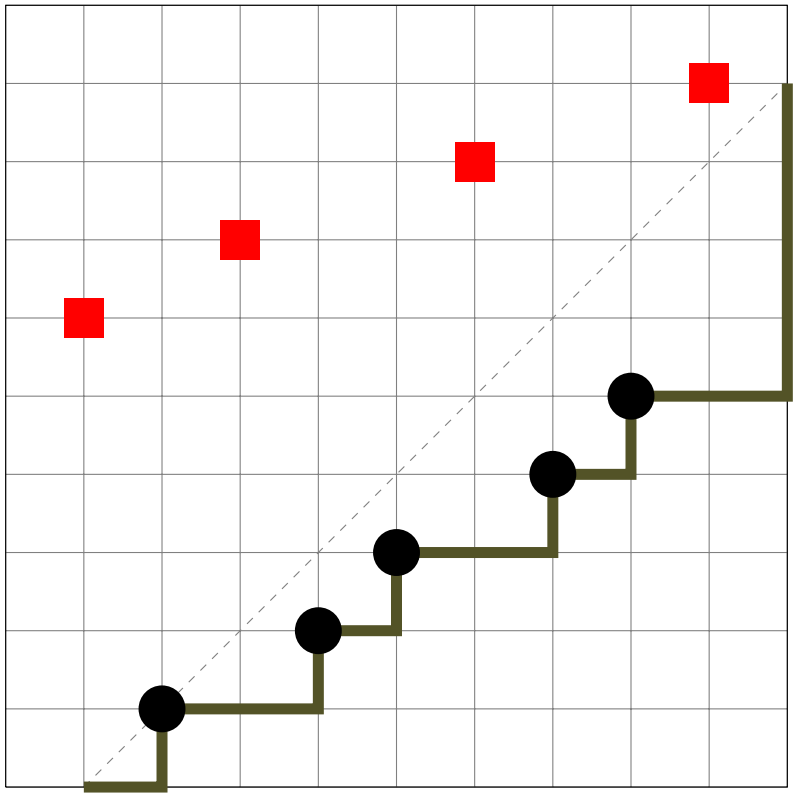

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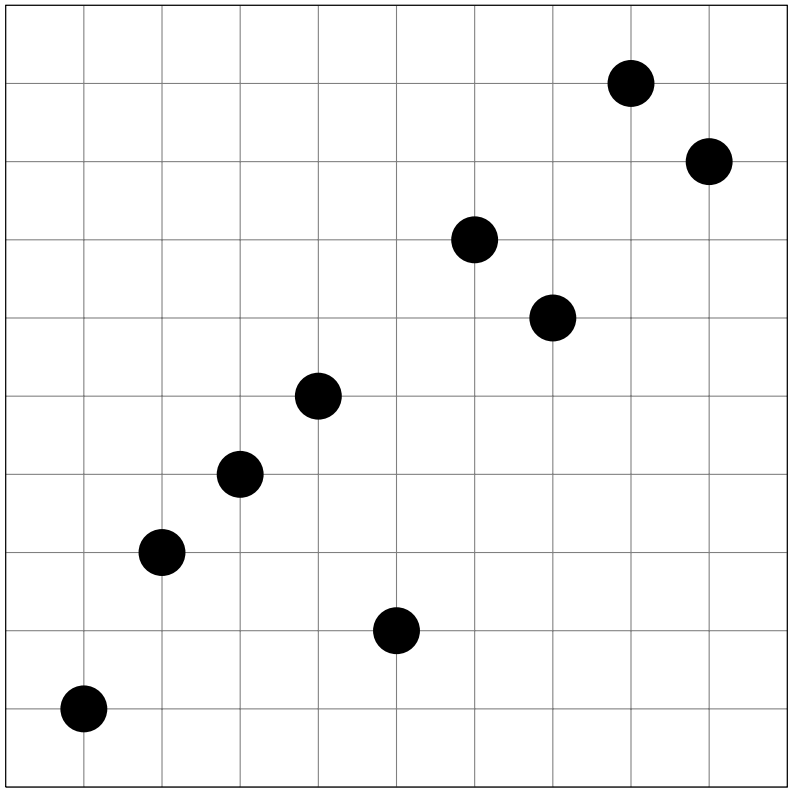

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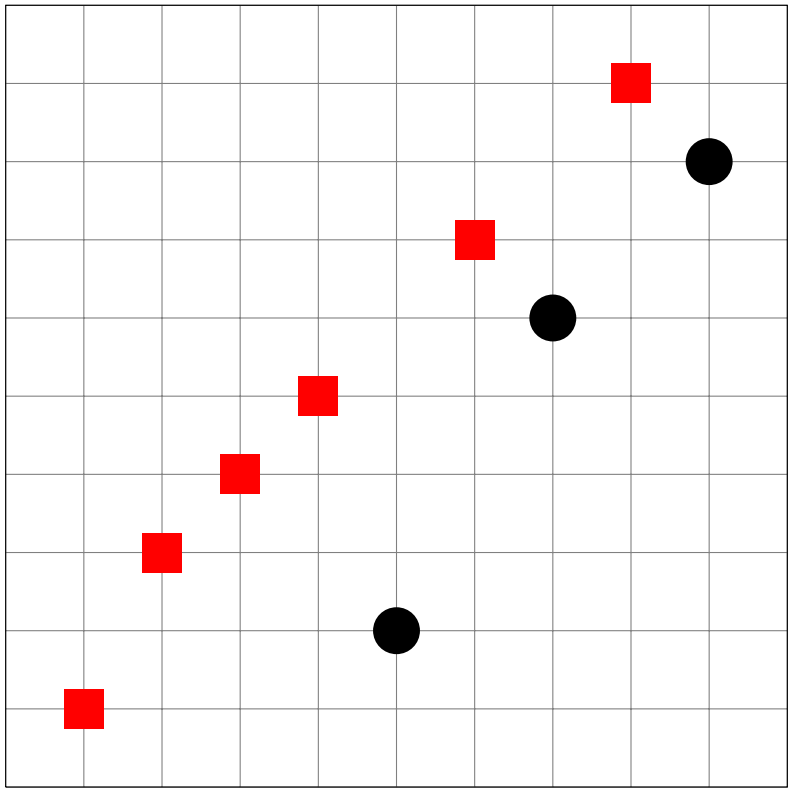

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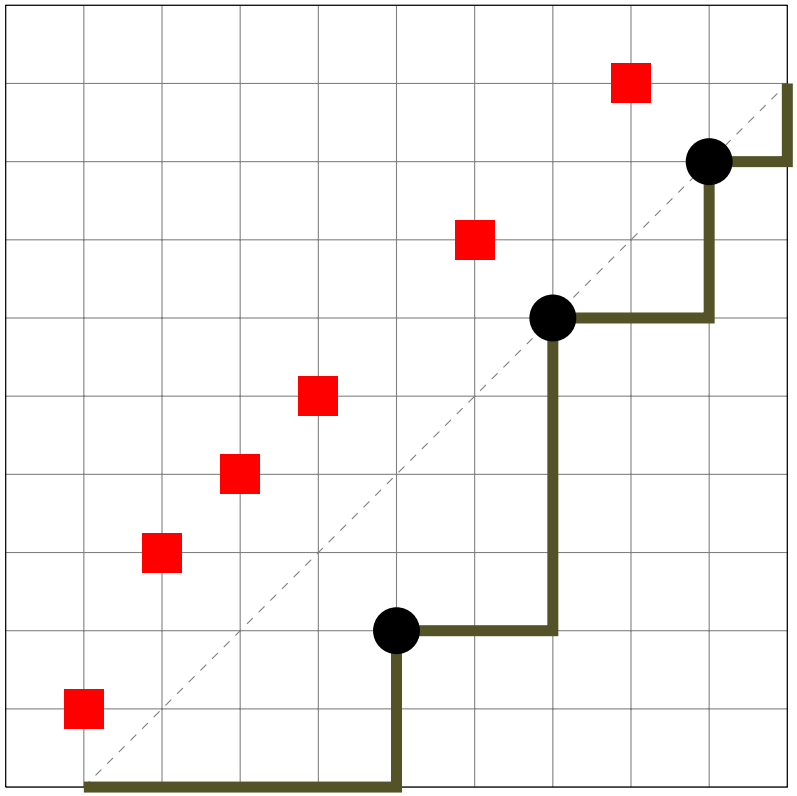

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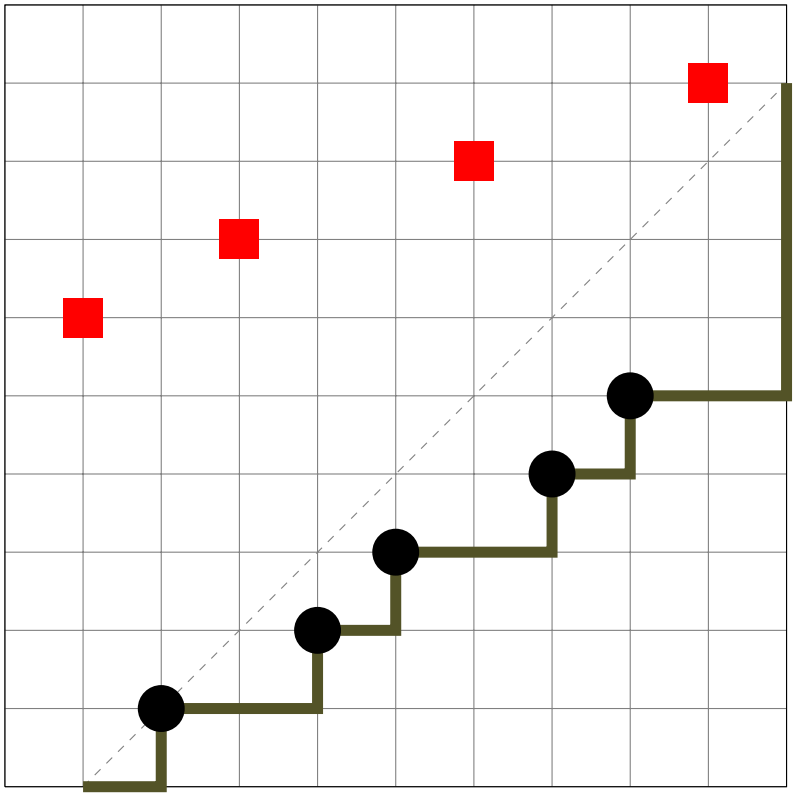
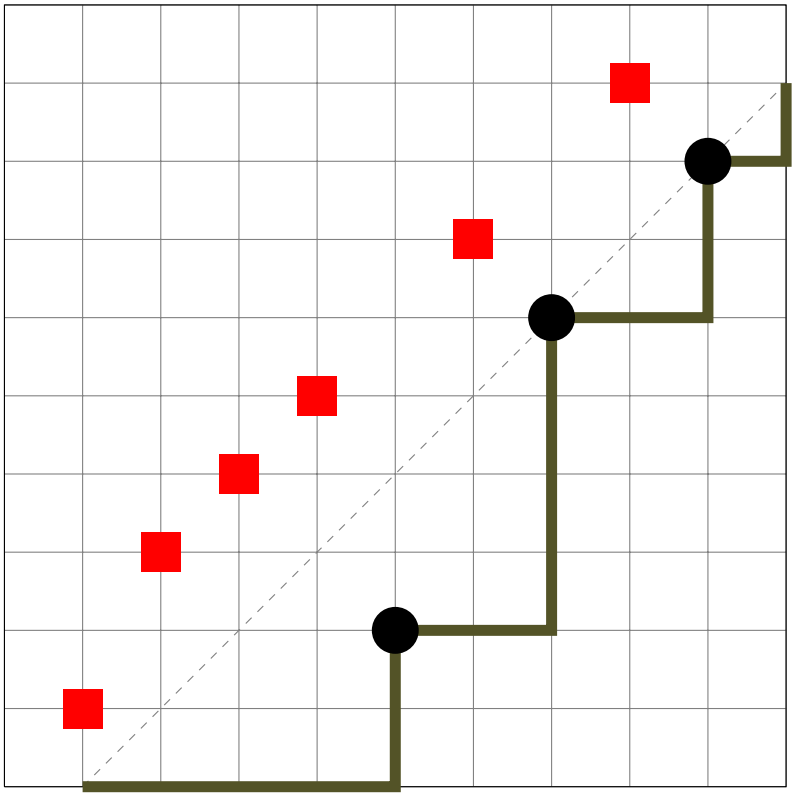


617238459

peaks: 172, 384



134527698

peaks: 452, 276, 698

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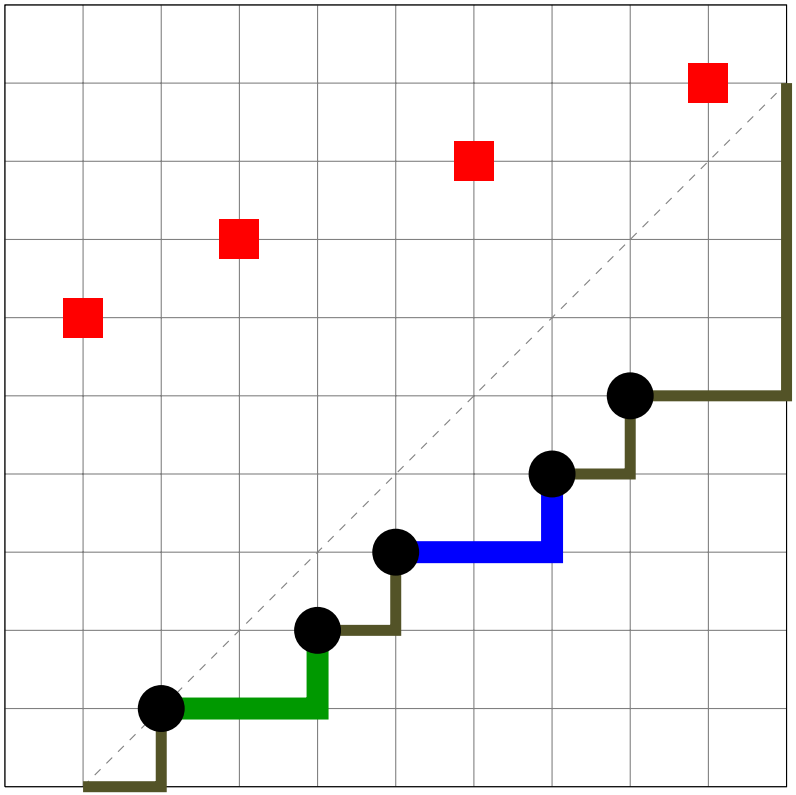

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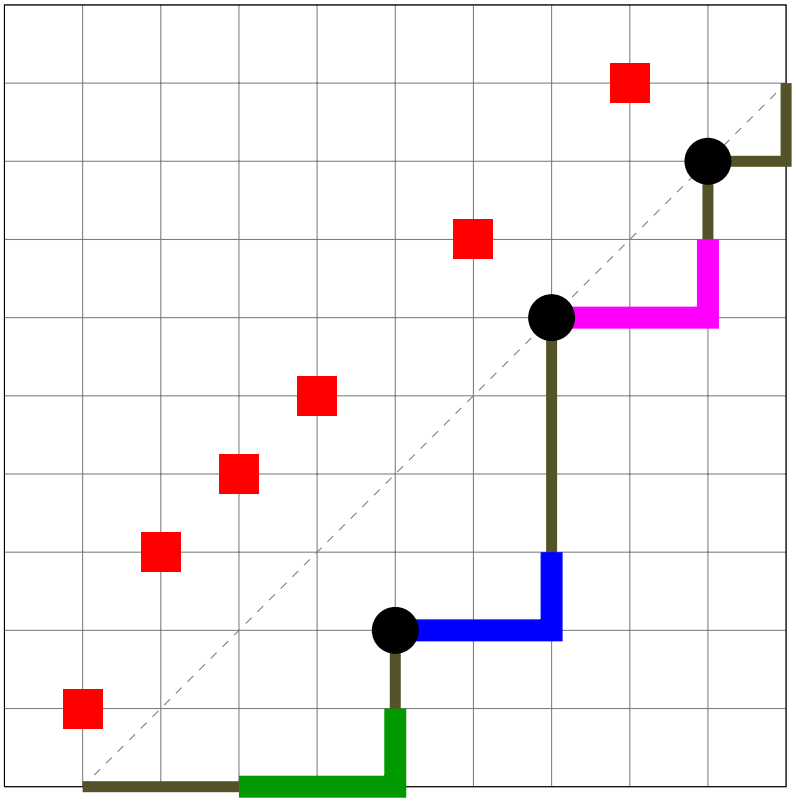

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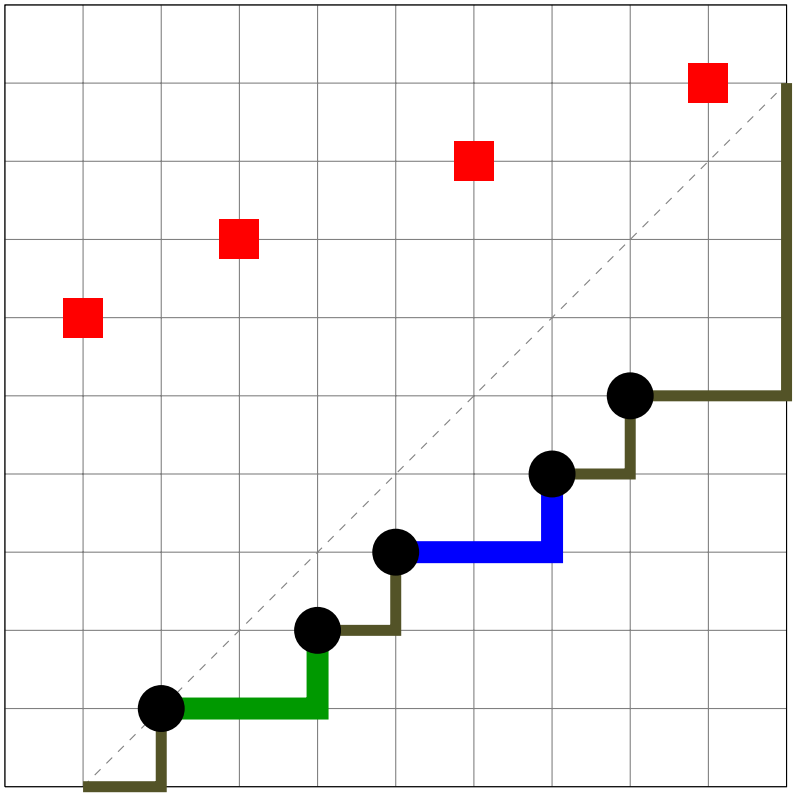

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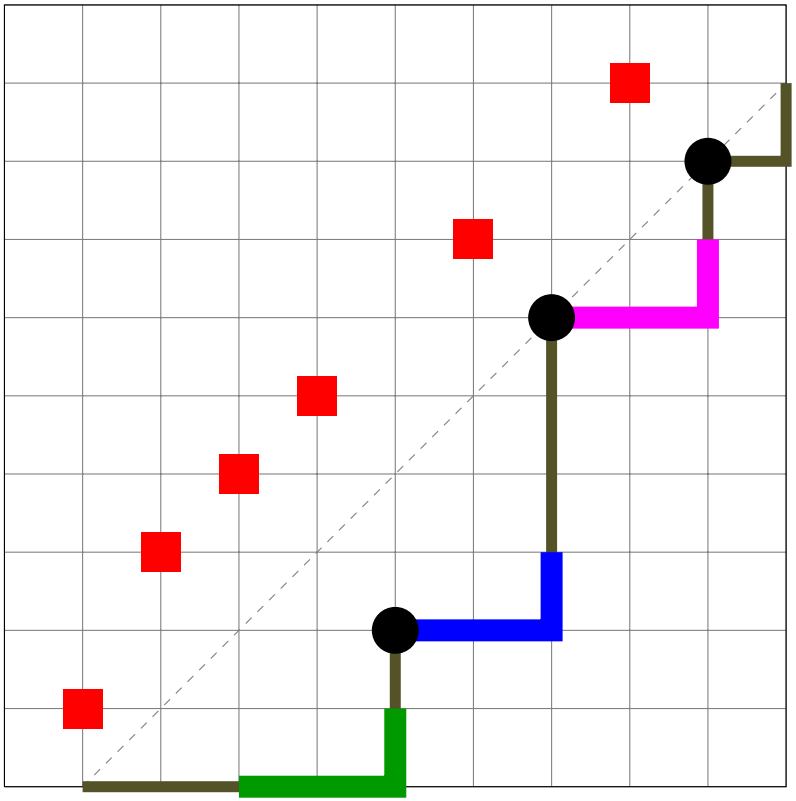

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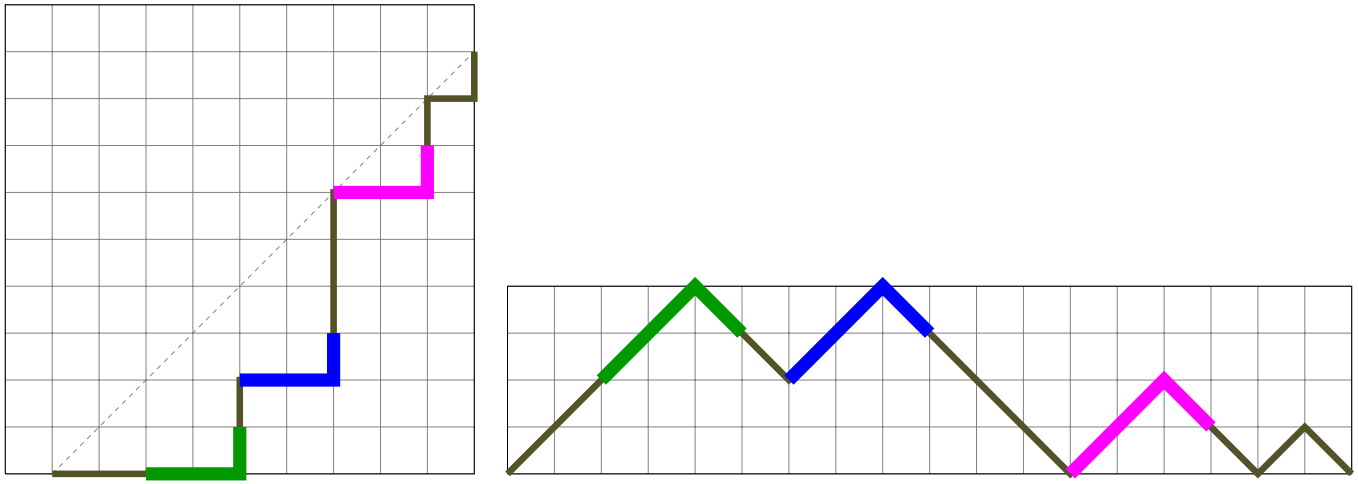
Peaks in π correspond to EEN factors in D_π before the last E.

$\mathcal{S}_n(321)$ to (shifted) Dyck paths, rotated



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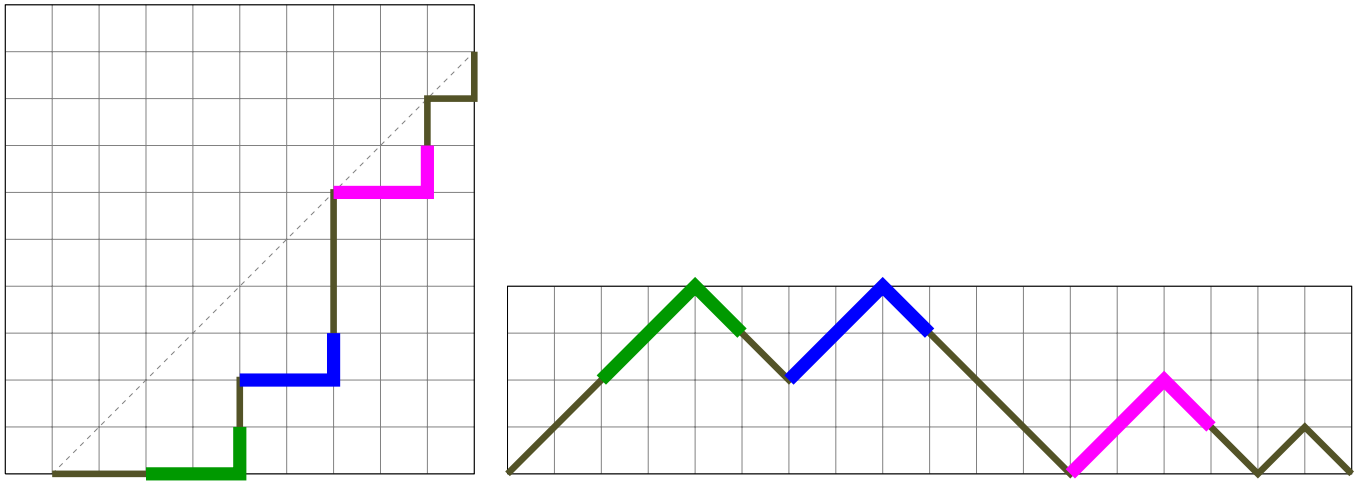

Let

- $\mathcal{D}_n$ be the set of Dyck paths of semi-length n, $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$.
- $\mathrm{st}^*(d)$ be the number of UUD (EEN) factors in Dyck path d that appear before the last U (E).

$$\sum_{d\in\mathcal{D}} q^{\mathrm{st}^*(d)} z^{|d|} = \sum_{\pi\in \mathrm{Av}(321)} q^{\mathrm{pk}(\pi)} z^{|\pi|}$$

st tracks all UUD (EEN) factors. st^* tracks UUD (EEN) factors before the last U (E).

Goal

Determine $\displaystyle\sum_{d\in\mathcal{D}} q^{\mathrm{st}^*(d)} z^{|d|} = \sum_{\pi\in \mathrm{Av}(321)} q^{\mathrm{pk}(\pi)} z^{|\pi|}.$

- $\mathcal{D}_n$ is the set of Dyck paths of semi-length n and $\mathcal{D} = \bigcup_{n\geq 0} \mathcal{D}_n$.
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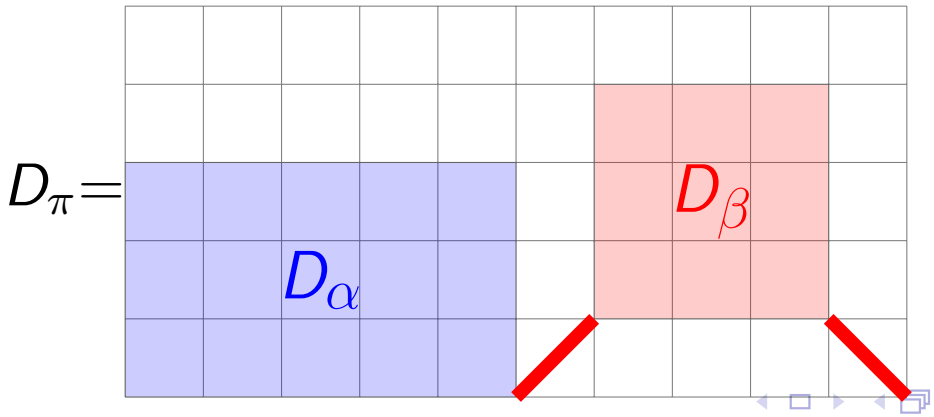
- $\mathcal{ID}_n$ is the set of indecomposable Dyck paths of semi-length n and $\mathcal{ID} = \bigcup_{n\geq 0} \mathcal{ID}_n$.
- $\mathrm{st}(d)$ is the total number of UUD (EEN) factors in Dyck path d.



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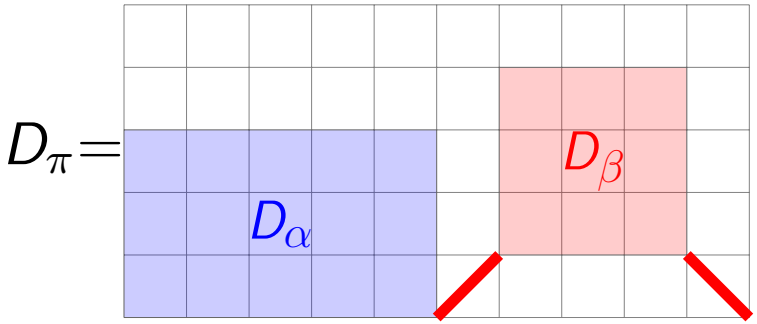
A tale of four generating functions

$$A := \sum_{d \in \mathcal{D}} q^{\mathrm{st}(d)} z^{|d|} \qquad B := \sum_{d \in \mathcal{I}\mathcal{D}} q^{\mathrm{st}(d)} z^{|d|}$$

$$\mathbf{C} := \sum_{\mathbf{d} \in \mathcal{D}} \mathbf{q}^{\mathrm{st}^*(\mathbf{d})} \mathbf{z}^{|\mathbf{d}|} \qquad D := \sum_{d \in \mathcal{I}\mathcal{D}} q^{\mathrm{st}^*(d)} z^{|d|}$$



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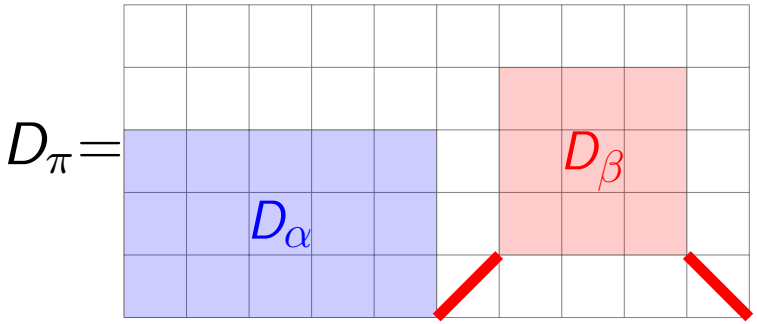

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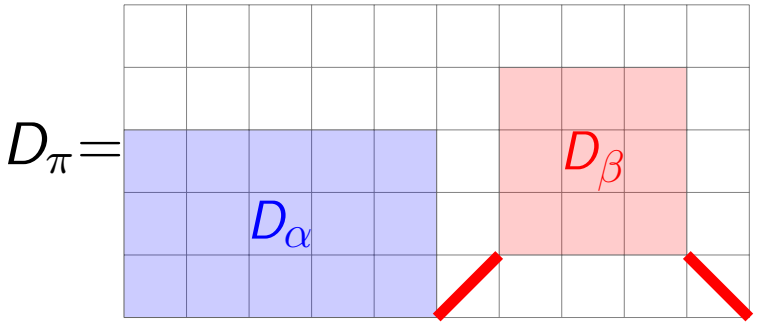

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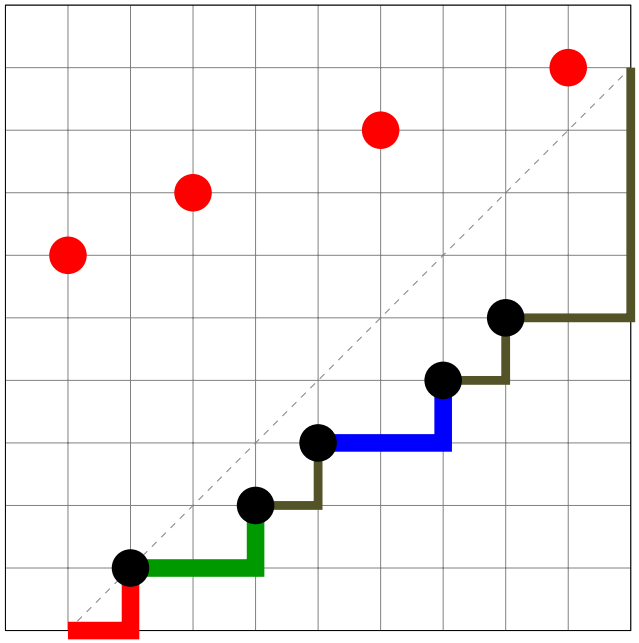
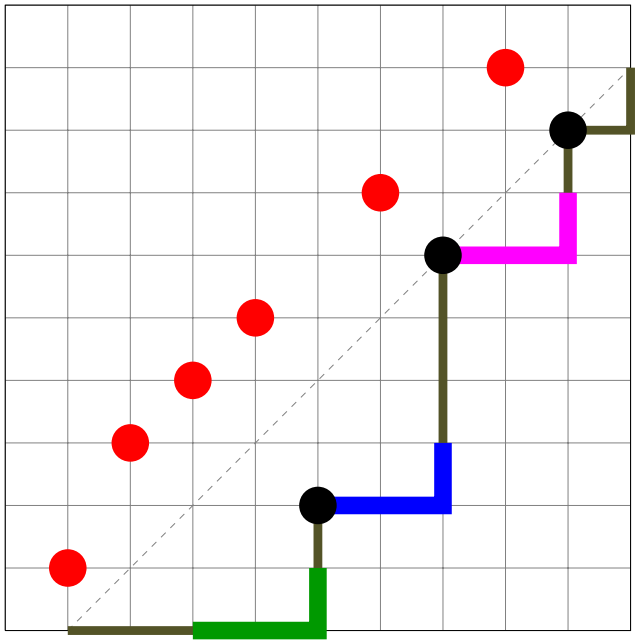
Descents on Dyck paths



617238459

peaks: 172, 384

descents: 61, 72, 84



134527698

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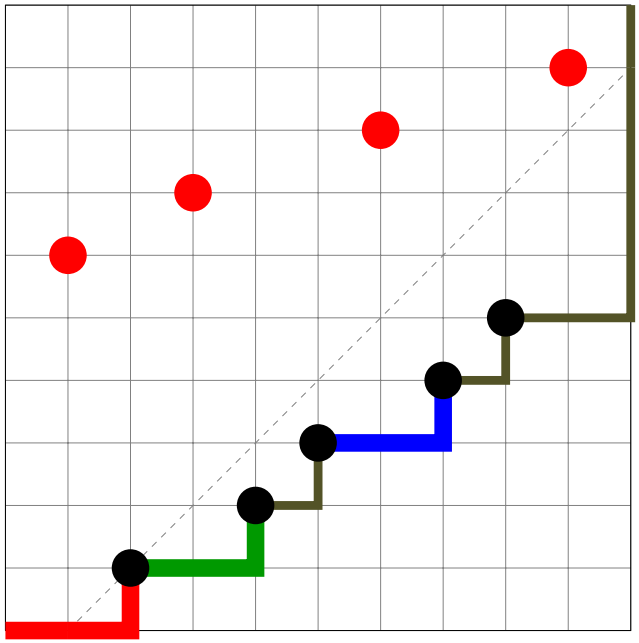
Descents on Dyck paths



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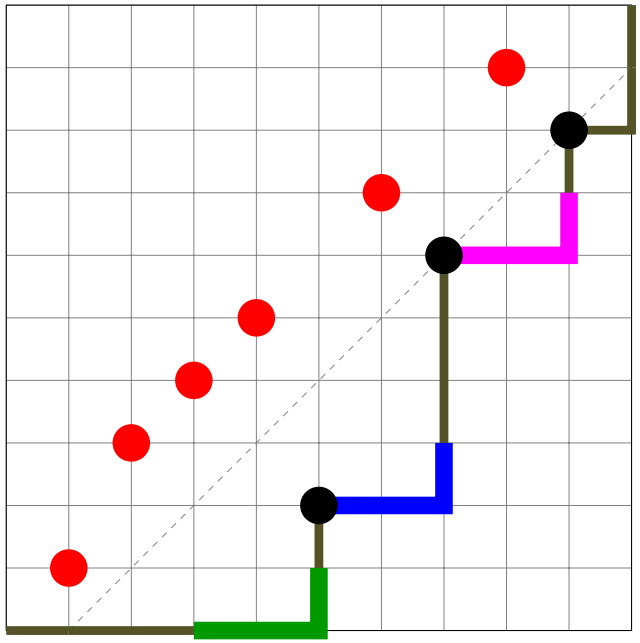

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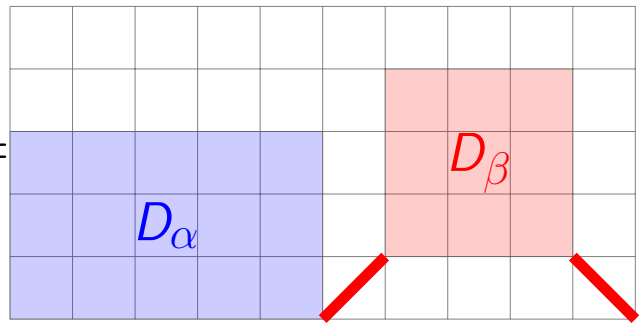
$D_\pi =$ 

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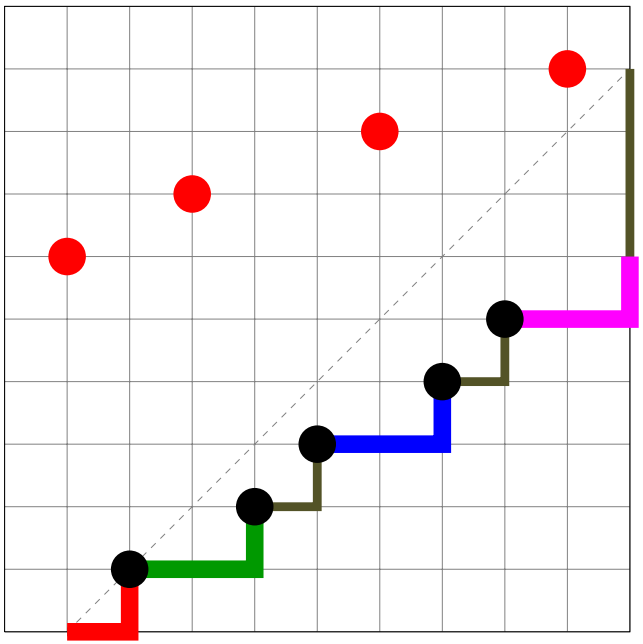
All UUDs (EENs) on all Dyck paths

$$A := \sum_{d \in \mathcal{D}} q^{\mathrm{st}(d)} z^{|d|}$$



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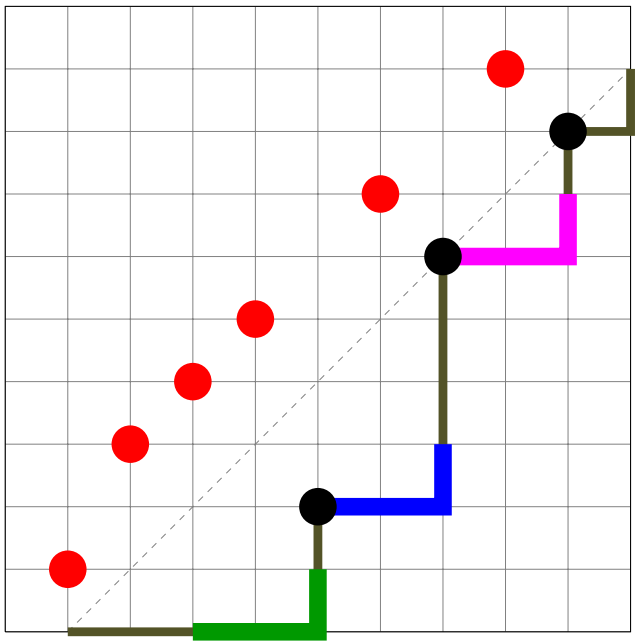

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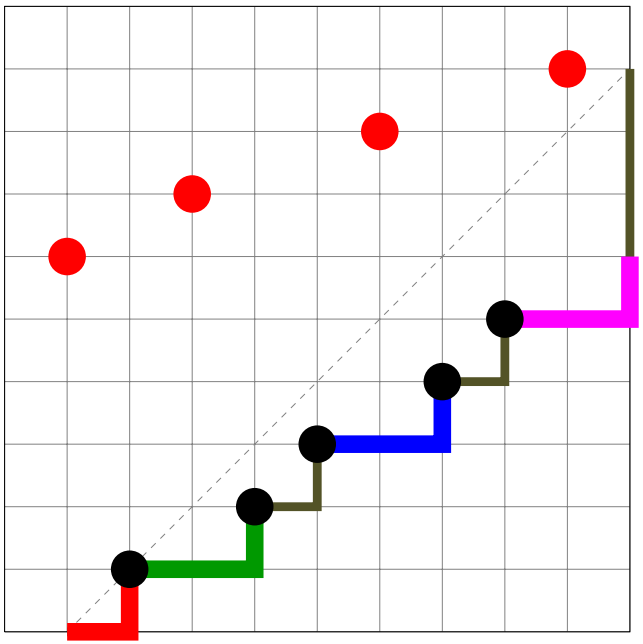

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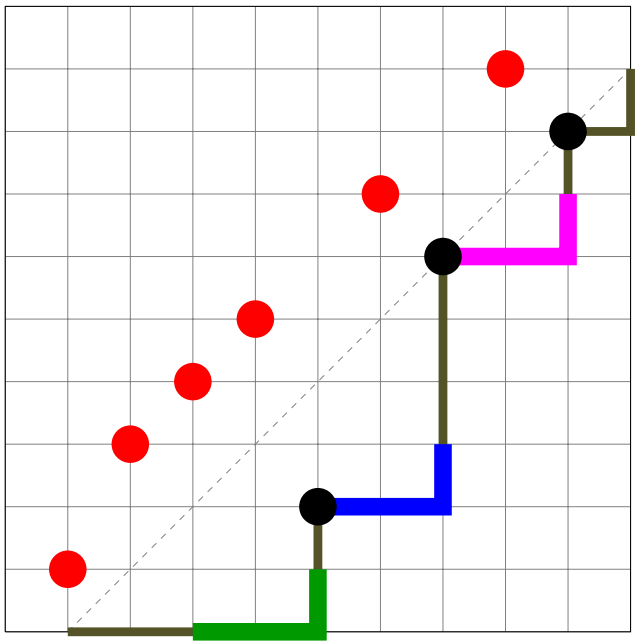
UUDs/EENs: 172, 384, ?



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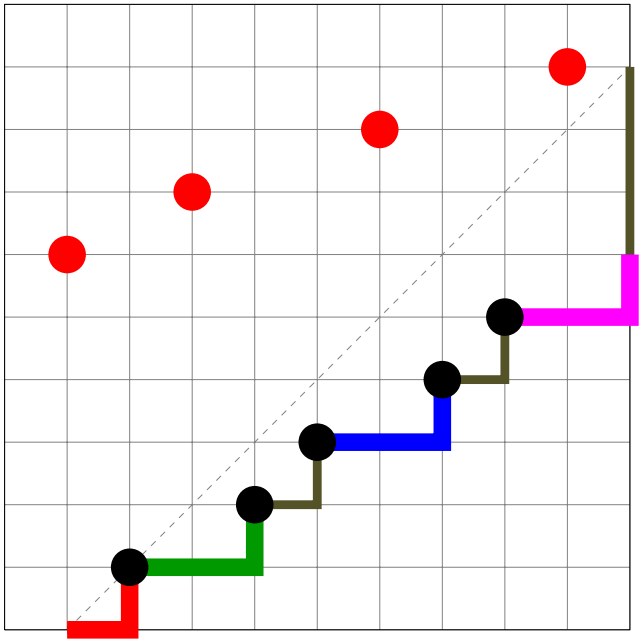

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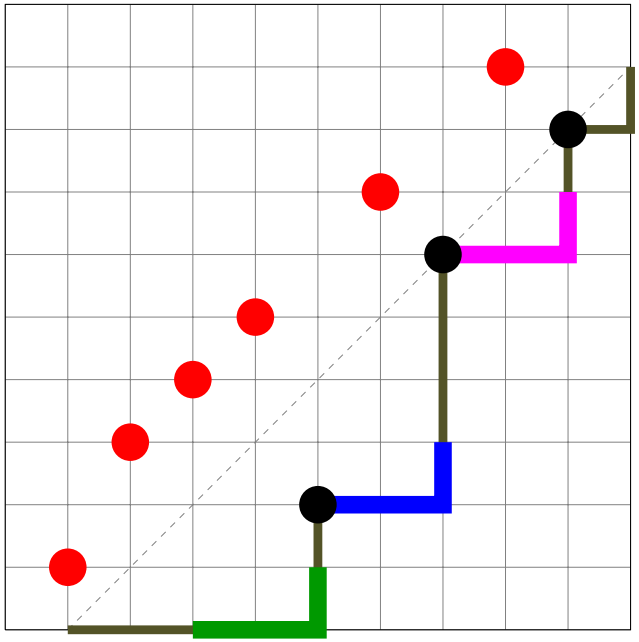

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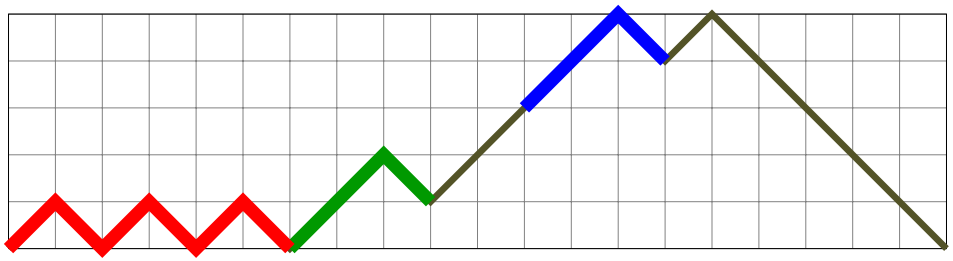

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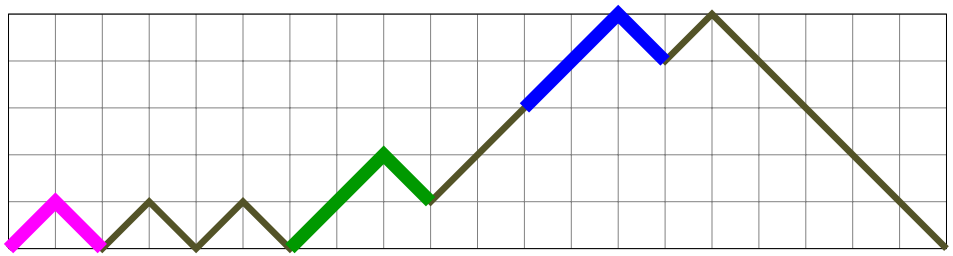

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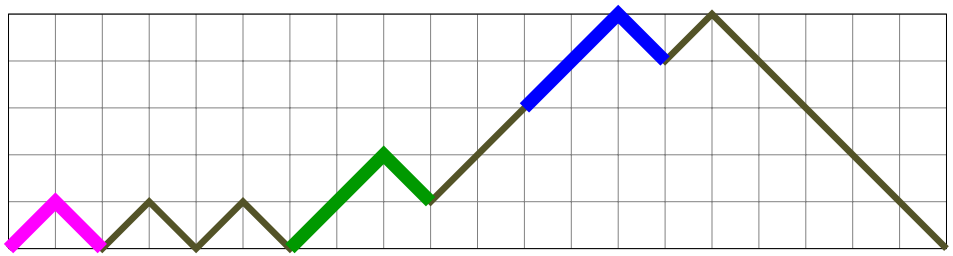

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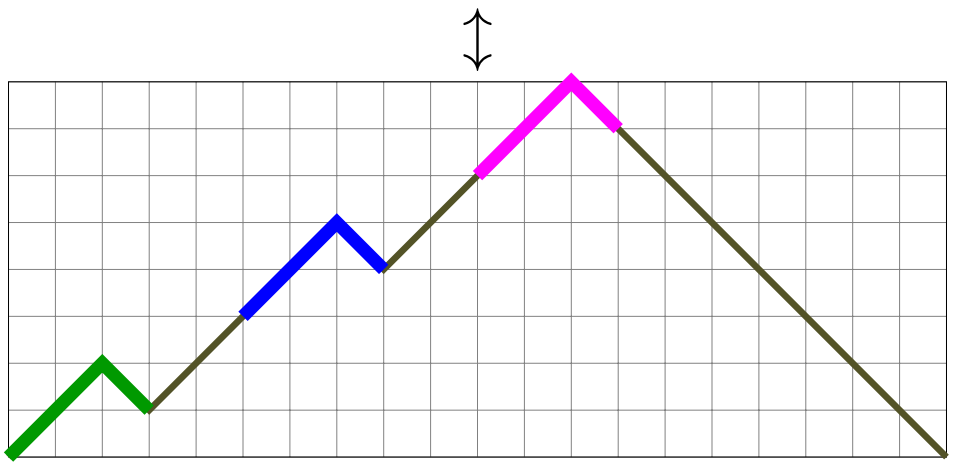

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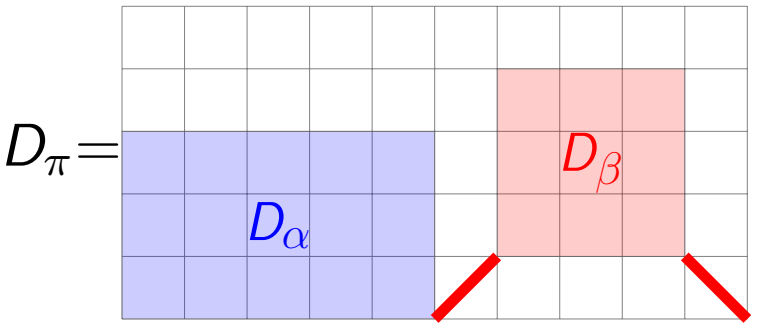

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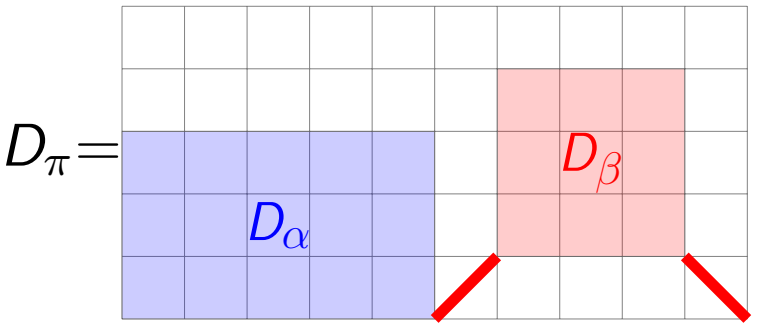

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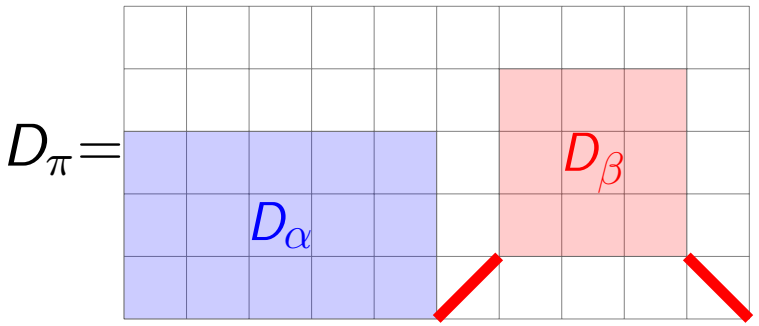

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- $A = \sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{des}(\pi)} z^{|\pi|}$
- $D = zA$
- $C = 1 + zA^2$

The upshot

Known (Barnabei, Bonetti and Silimbani, 2010)

- $$\sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{des}(\pi)} z^{|\pi|} = -\frac{-1+\sqrt{-4z^2q+4z^2-4z+1}}{2z(zq-z+1)}$$
- $$\sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{des}(\pi)} z^{|\pi|} = \sum_{d \in \mathcal{D}} q^{\mathrm{vl}(d)+tf(d)} z^{|d|}$$
  - $vl(d)$ is the number of DU factors and $tf(d)$ is the number of DDD factors of d.

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New:

- $$\sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{pk}(\pi)} z^{|\pi|} = 1 + z\left(-\frac{-1+\sqrt{-4z^2q+4z^2-4z+1}}{2z(zq-z+1)}\right)^2$$
- $$\sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{des}(\pi)} z^{|\pi|} = \sum_{d \in \mathcal{D}} q^{\mathrm{st}(d)} z^{|d|}$$
  - $st(d)$ is the number of UUD factors of d.

But wait! There's more...

- $A := \sum_{d \in \mathcal{D}} q^{\mathrm{st}(d)} z^{|d|} = \sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{des}(\pi)} z^{|\pi|}$
- $B := \sum_{d \in \mathcal{ID}} q^{\mathrm{st}(d)} z^{|d|} =$???
- $C := \sum_{d \in \mathcal{D}} q^{\mathrm{st}^*(d)} z^{|d|} = \sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{pk}(\pi)} z^{|\pi|}$
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$\rho \backslash$ st	**pk**	**vl**	**dasc**	**ddes**
231	A091894	A236406	A092107	A092107
312	A236406	A091894	A092107	A092107
321	A236406	A236406	new	(none)

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- $B := \sum_{d \in \mathcal{ID}} q^{\mathrm{st}(d)} z^{|d|} = z(1-q) + \sum_{\pi \in \mathrm{Av}(231)} q^{\mathrm{pk}(\pi)+1} z^{|\pi|+1}$
- $C := \sum_{d \in \mathcal{D}} q^{\mathrm{st}^*(d)} z^{|d|} = \sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{pk}(\pi)} z^{|\pi|}$
- $D := \sum_{d \in \mathcal{ID}} q^{\mathrm{st}^*(d)} z^{|d|} = \sum_{\pi \in \mathrm{Av}(321)} q^{\mathrm{des}(\pi)} z^{|\pi|+1}$

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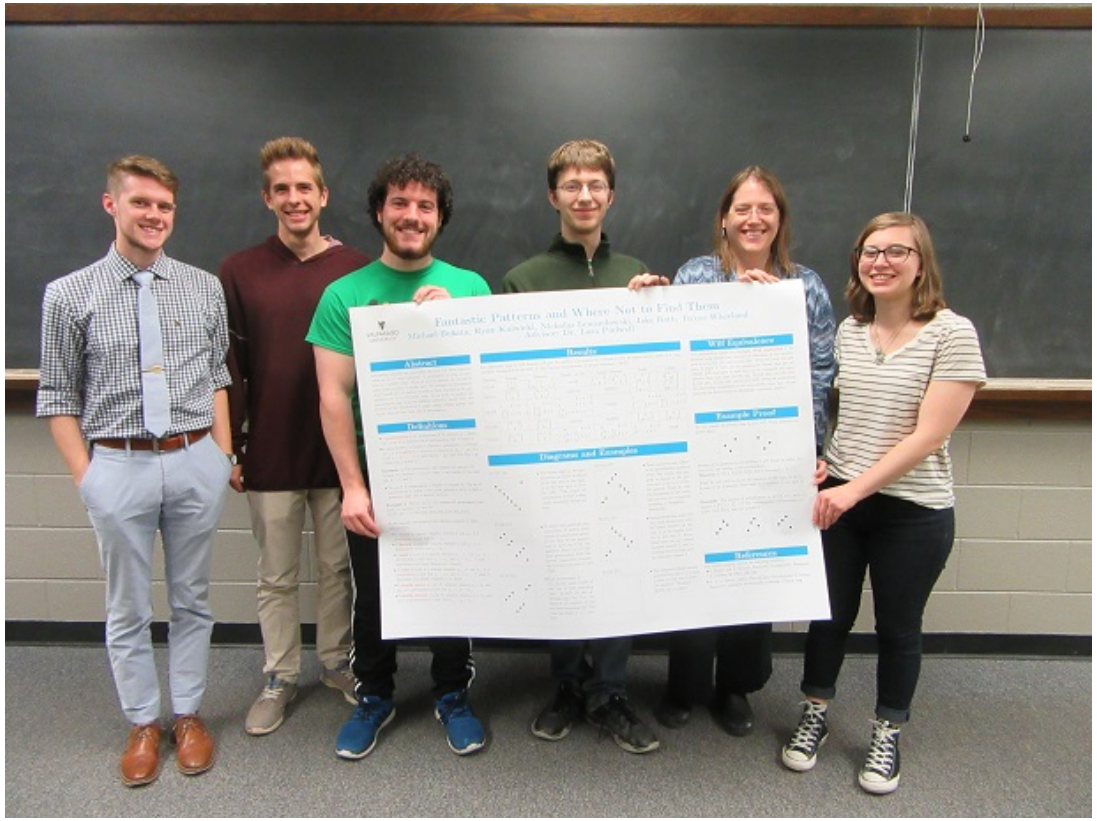
More Patterns?



left to right: Nicholas Lewandowski, Ryan Kulwicki, Jacob Roth*, Michael Bukata, L.P., Teresa Wheeland

Studied statistics over $\mathcal{S}_n(\rho_1, \rho_2)$ with $\rho_1, \rho_2 \in \mathcal{S}_3$.

References

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Thanks for listening!

slides at `faculty.valpo.edu/lpudwell`

email: `Lara.Pudwell@valpo.edu`