Patterns in Permutations

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Chan Stanek Lecture for Students
Mathfest 2023

Definition

A permutation of length n is an ordered list of the numbers $1, 2, \ldots, n$.
$\mathcal{S}_n$ is the set of all permutations of length n.

$$\mathcal{S}_1 = \{1\}$$

$$\mathcal{S}_2 = \{12, 21\}$$

$$\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$$

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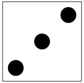
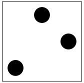
$$|\mathcal{S}_n| = n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$$

Note

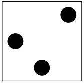
Permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ is often visualized by plotting the points (i, π_i) in the Cartesian plane.



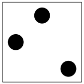
123



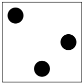
132



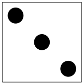
213



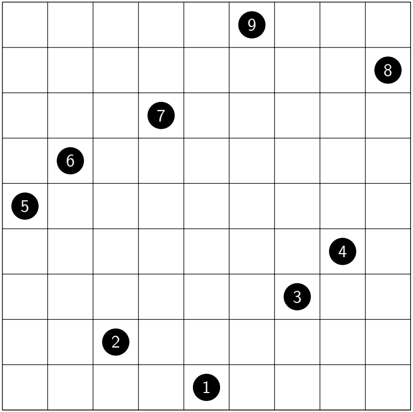
231



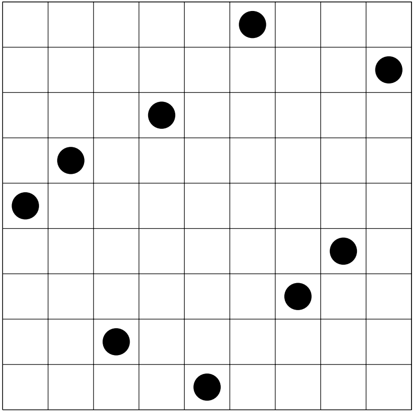
312

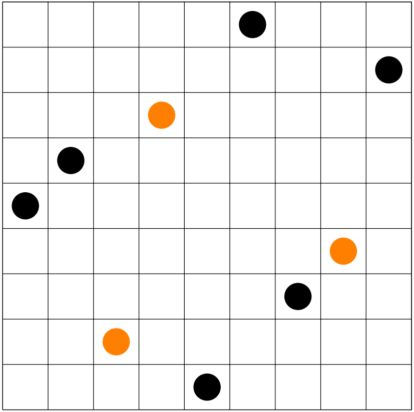


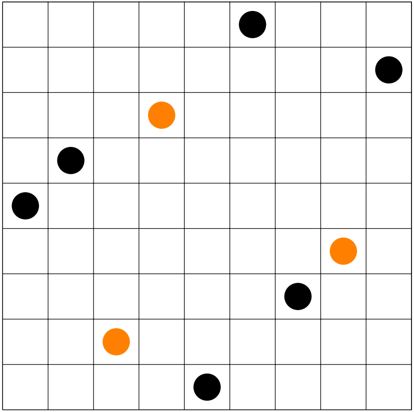
321

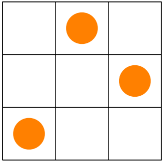


$$\pi = 562719348$$

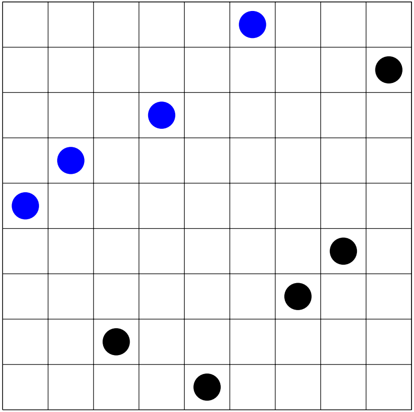


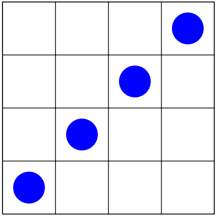




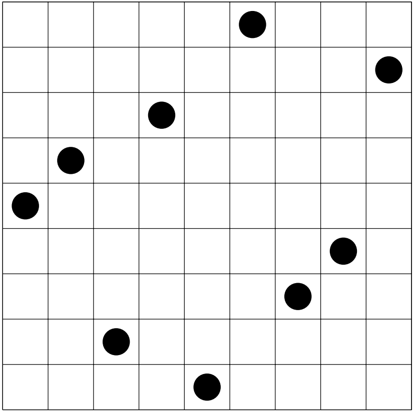


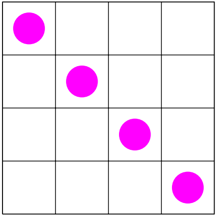
562719348 contains the pattern 132





562719348 contains the pattern 1234





562719348 avoids the pattern 4321

Big question

How many permutations of length n contain the permutation π?

Or, alternatively...

Big question

How many permutations of length n avoid the permutation π?

(depends on what π is!)

Question

How many permutations of length n avoid the permutation ?

Length 1?

Question

How many permutations of length n avoid the permutation ?

Length 1? (1)

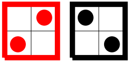
Length 2?

Question

How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



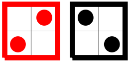
Length 3?

Question

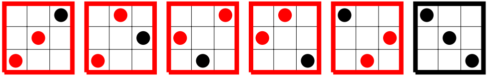
How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



Length 3? (1)

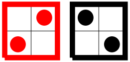


Question

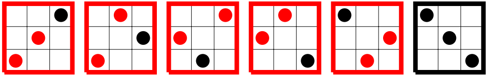
How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



Length 3? (1)



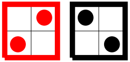
The decreasing permutation is the only permutation of length n that avoids 12.

Question

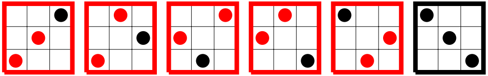
How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



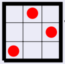
Length 3? (1)



The decreasing permutation is the only permutation of length n that avoids 12.

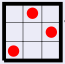
Similar: the increasing permutation is the only permutation of length n that avoids 21.

Question

How many permutations of length n avoid the permutation ?

Length 1?

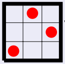
Question

How many permutations of length n avoid the permutation ?

Length 1? (1)

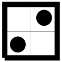
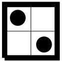
Length 2?

Question

How many permutations of length n avoid the permutation ?

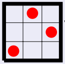
Length 1? (1)

Length 2? (2)

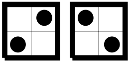
Length 3?

Question

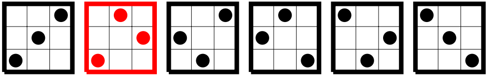
How many permutations of length n avoid the permutation ?

Length 1? (1)

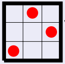
Length 2? (2)



Length 3? (5)

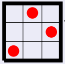


Question

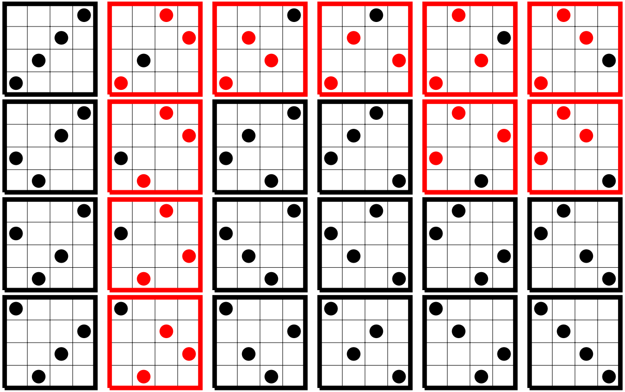
How many permutations of length n avoid the permutation ?

Length 4?

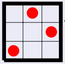
Question

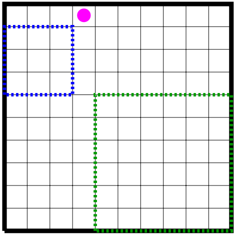
How many permutations of length n avoid the permutation ?

Length 4? (14)

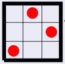


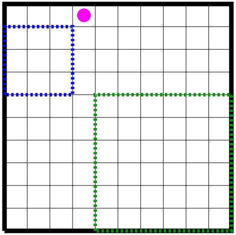
Question

How many permutations of length n avoid the permutation ? (C_n)



Question

How many permutations of length n avoid the permutation ? (C_n)



Answer: $C_0 = 1$, $C_1 = 1$, and for larger n:

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0 = \sum_{i=1}^{n} C_{i-1} C_{n-i}$$

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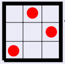
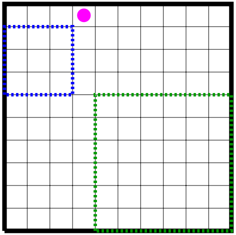
How many permutations of length n avoid the permutation ? (C_n)



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$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0 = \sum_{i=1}^{n} C_{i-1} C_{n-i}$$

$1, 1, 2, 5, 14, 42, 132, \ldots$ (Catalan numbers!)

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

catalan Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **catalan**

Displaying 1-10 of 3978 results found. page 1 2 3 4 5 6 7 8 9 10 ... 398

Sort: relevance | references | number | modified | created Format: long | short | data

A000108 **Catalan** numbers: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). +20 3438

(Formerly M1459 N0577)

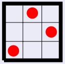
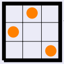
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368, 3814986502092304 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,3

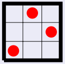
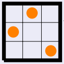
COMMENTS Also called Segner numbers.

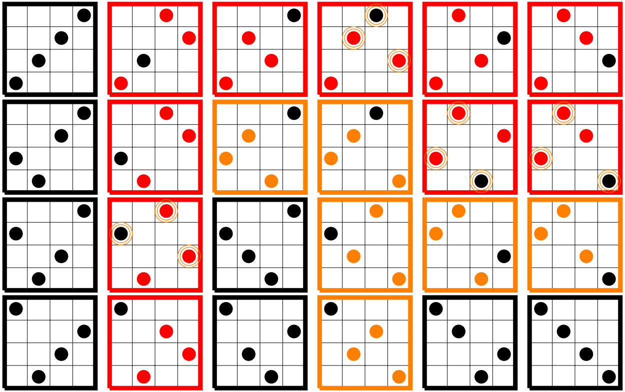
The solution to Schröder's first problem. A very large number of combinatorial interpretations are known - see references, esp. Stanley, Enumerative Combinatorics, Volume 2. This is probably the longest entry in the OEIS, and rightly so.

Question

How many permutations of length n avoid the permutations  and ?

Question

How many permutations of length n avoid the permutations  and ?



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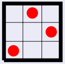
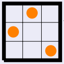
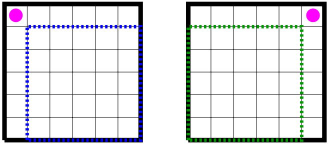
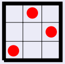
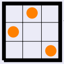
How many permutations of length n avoid the permutations  and ? (T_n)

or



Question

How many permutations of length n avoid the permutations

 and ? (T_n)

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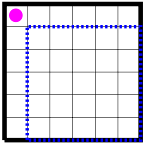
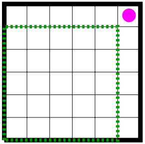
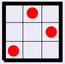
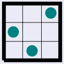
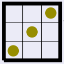
 

Answer: $T_1 = 1$ and $T_n = T_{n-1} + T_{n-1} = 2T_{n-1}$, so...

$$T_n = 2^{n-1}.$$

Question

How many permutations of length n avoid the permutations

 and  and ?

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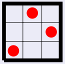
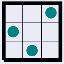
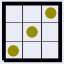
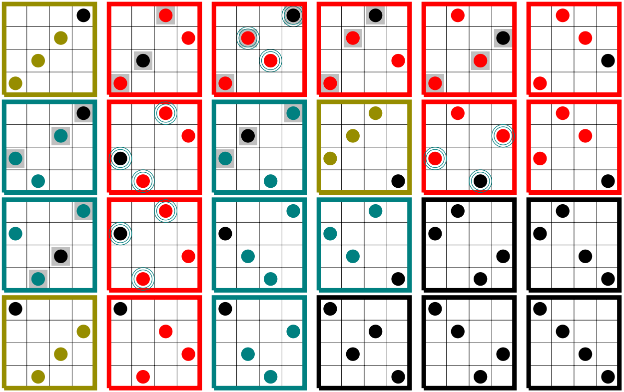
How many permutations of length n avoid the permutations  and  and ?



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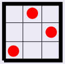
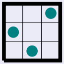
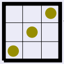
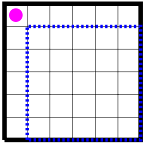
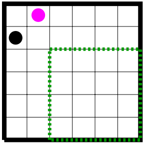
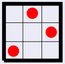
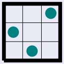
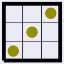
How many permutations of length n avoid the permutations  and  and ? (F_n)

or

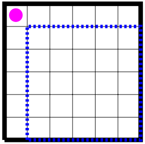
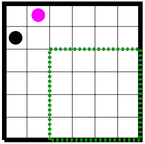




Question

How many permutations of length n avoid the permutations  and  and ? (F_n)

or

Answer: $F_0 = F_1 = 1$ and for larger n,

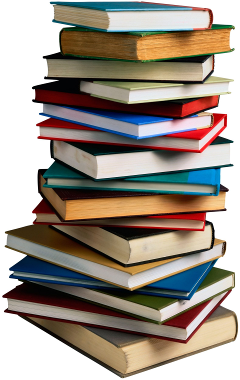
$$F_n = F_{n-1} + F_{n-2}.$$

$1, 1, 2, 3, 5, 8, 13, \ldots$ (Fibonacci numbers!)

How many permutations of length n avoid the pattern(s)...

- 12? 1
- 132? (Catalan)
- 132 and 231? 2^{n-1}
- 132 and 213 and 123? (Fibonacci)
- 1234? $1, 1, 2, 6, 23, 103, 513, 2761, \ldots$ (Gessel, 1990)
- 1342? $1, 1, 2, 6, 23, 103, 512, 2740, \ldots$ (Bóna, 1997)
- 1324? $1, 1, 2, 6, 23, 103, 513, 2762, \ldots$ (open question!)

Why *avoid* patterns?



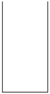
Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Stack Operations

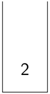
- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534



Output:

Input: 1534



Output:

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534

Output:

Input: 1534

2

Output:

Input: 534

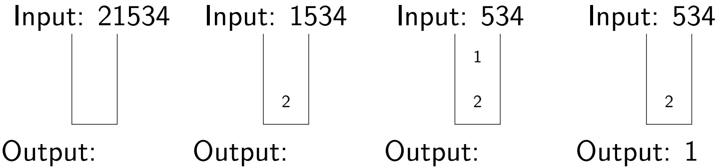
1

2

Output:

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output



Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534

Output:

Input: 1534

2

Output:

Input: 534

1

2

Output:

Input: 534

2

Output: 1

Input: 534

Output: 12

Input: 34

5

Output: 12

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534

Output:

Input: 1534

2

Output:

Input: 534

1

2

Output:

Input: 534

2

Output: 1

Input: 534

Output: 12

Input: 34

5

Output: 12

Input: 4

3

5

Output: 12

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534

Output:

Input: 1534

2

Output:

Input: 534

1
2

Output:

Input: 534

2

Output: 1

Input: 534

Output: 12

Input: 34

5

Output: 12

Input: 4

3
5

Output: 12

Input: 4

5

Output: 123

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534	Input: 1534	Input: 534	Input: 534	Input: 534
	2	1 2	2	
Output:	Output:	Output:	Output: 1	Output: 12

Input: 34	Input: 4	Input: 4	Input:	Input:
5	3 5	5	4 5	
Output: 12	Output: 12	Output: 123	Output: 123	Output: 12345

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

21534 can be sorted after one pass through a stack.

Can you find a permutation that *can't* be sorted after one pass through a stack?

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

What about 231?

Input: 231

Output:

Input: 31

2

Output:

Input: 231

Output:

Input: 31

2

Output:

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

What about 231?

Input: 231

Output:

Input: 31

2

Output:

Input: 1

3

2

Output:

Input: 231

Output:

Input: 31

2

Output:

Input: 31

Output: 2

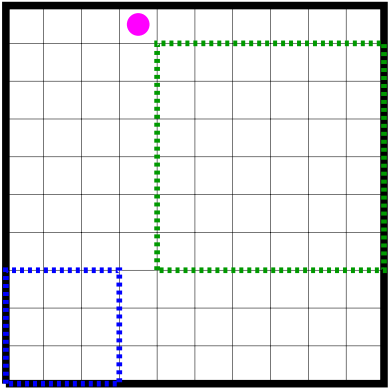
Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

Proof sketch: (by induction)



Output? 

Pattern Containment

How many permutations of $\mathcal{S}_3$ contain 12?

Pattern Containment

How many permutations of $\mathcal{S}_3$ contain 12?
How many times?

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How many permutations of $\mathcal{S}_3$ contain 12?
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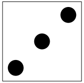
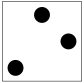

3 copies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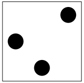
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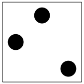
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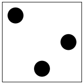
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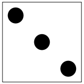
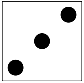
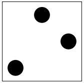

0 copies

Pattern Containment

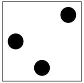
How many permutations of $\mathcal{S}_3$ contain 12?
How many times?



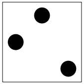
3 copies



2 copies



2 copies



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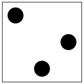

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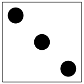


0 copies

The maximum number of copies of 12 in a member of $\mathcal{S}_3$ is 3.

Alternating Permutations

A permutation $\pi = \pi_1 \cdots \pi_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 \cdots$.

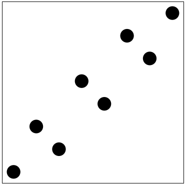
Examples:

1324 1423 2314 2413 3412

Interesting(?) Counting Question

What is the largest possible number of copies of 123 in π if π is alternating?

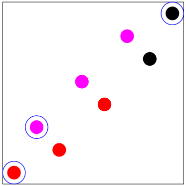
Packing 123

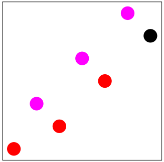


$1\ 32\ 54\ 76 \cdots$ is the alternating permutation of length n with the most copies of 123.

Packing 123

n even vs. n odd





$\frac{n}{2} - 1$ layers of size 2

$\frac{n-1}{2}$ layers of size 2

Copies of 123 can use:

n even	n odd
three layers of size 2	three layers of size 2
two layers of size 2	two layers of size 2
one layer of size 2	

Counting Sequences

Let $a(n)$ be the number of copies of 123 in 1 32 54 76 $\cdots$.

$$a(n) = \begin{cases} 2(\frac{n}{2}-1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$

Counting Sequences

Let $a(n)$ be the number of copies of 123 in $1\ 32\ 54\ 76\cdots$.

$$a(n) = \begin{cases} 2\left(\frac{n}{2}-1\right) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$

A099956 Atomic numbers of the alkaline earth metals. 9

4, 12, 20, 38, 56, 88 (list; graph; refs; listen; history; text; internal format)

OFFSET	1,1
LINKS	Table of n, a(n) for n=1..6.
EXAMPLE	12 is the atomic number of magnesium.
CROSSREFS	Cf. A099955, alkali metals; A101648, metalloids; A101647, nonmetals (except halogens and noble gases); A097478, halogens; A018227, noble gases; A101649, poor metals. Sequence in context: A057317 A008068 A008183 * A301066 A008092 A316299 Adjacent sequences: A099953 A099954 A099955 * A099957 A099958 A099959
KEYWORD	nonn,fini,full
AUTHOR	Parthasarathy Nambi, Nov 12 2004
STATUS	approved

Counting Sequences

Let $a(n)$ be the number of copies of 123 in 1 32 54 76 $\cdots$.

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$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$

A168380 Row sums of A168281. +20 14

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, 1140, 1340, 1540, 1782, 2024, 2312, 2600, 2938, 3276, 3668, 4060, 4510, 4960, 5472, 5984, 6562, 7140, 7788, 8436, 9158, 9880, 10680, 11480, 12362, 13244, 14212, 15180, 16238, 17296, 18448, 19600, 20850, 22100 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS The atomic numbers of the augmented alkaline earth group in Charles Janet's spiral periodic table are 0 and the first eight terms of this sequence (see Stewart reference). - Alonso del Arte, May 13 2011

LINKS Vincenzo Librandi, Table of n, a(n) for n = 1..10000
Stewart, Philip, Charles Janet: unrecognized genius of the Periodic System. Foundations of Chemistry (2010), p. 9.
Index entries for linear recurrences with constant coefficients, signature (2,1,-4,1,2,-1).

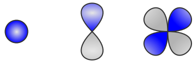
FORMULA a(n) = 2*A005993(n-1).
a(n) = (n+1)*(3 + 2*n^2 + 4*n - 3*(-1)^n)/12.
a(n+1) - a(n) = A093907(n) = A137583(n+1).
a(2n+1) = A035597(n+1) a(2n)=A002492(n).

Alkaline Earth Metals (Group 2)

Group → / Period ↓	1	2	3		4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc		22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y		40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	**	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				*	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				**	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

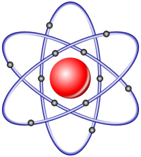
A little chemistry...

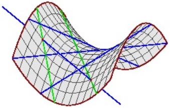
- *Quantum numbers* describe trajectories of electrons.
  - n (principal number) determines the electron shell
    $$n = 1, 2, 3, \dots$$
  - ℓ (orbital angular momentum) determines the shape of the orbital
    $$0 \leq \ell \leq n - 1$$

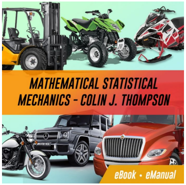


$\ell = 0$ $\ell = 1$ $\ell = 2$

  - m (magnetic number) determines number of orbitals and orientation within shell
    $$-\ell \leq m \leq \ell$$
  - Two possible spin numbers for each choice of (n, ℓ, m)



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There are **365 statistics** on **Permutations** in the database. There are possibly some more waiting for verification.

St000001 The number of reduced words for a permutation.
St000002 The number of occurrences of the pattern 123 in a permutation.
St000004 The major index of a permutation.
St000007 The number of saliances of the permutation.
St000018 The number of inversions of a permutation.
St000019 The cardinality of the support of a permutation.
St000020 The rank of the permutation.
St000021 The number of descents of a permutation.
St000022 The number of fixed points of a permutation.
St000023 The number of inner peaks of a permutation.
St000028 The number of stack-sorts needed to sort a permutation.
St000029 The depth of a permutation.
St000030 The sum of the descent differences of a permutations.
St000031 The number of cycles in the cycle decomposition of a permutation.
St000033 The number of permutations greater than or equal to the given permutation in (strong) Bruhat order.

For further reading...

- Miklos Bóna, *Combinatorics of Permutations*, Chapman & Hall, 2004.
- Donald Knuth, *The Art of Computer Programming: Volume 1*, Addison Wesley, 1968.
- Lara Pudwell, From permutation patterns to the periodic table, *Notices of the American Mathematical Society*. **67.7** (2020), 994–1001.
- Lara Pudwell, The hidden and surprising structure of ordered lists, *Math Horizons*. **29.3** (February 2022), 5–7.
- The On-Line Encyclopedia of Integer Sequences at `oeis.org`.
- FindStat at `findstat.org`

Thanks for listening!

slides at `faculty.valpo.edu/lpudwell`

email: `Lara.Pudwell@valpo.edu`