A Permutation Game

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MathPath Plenary
July 22, 2025

Counting Permutations breakout

Day 1: Descents

Day 2: Cycles

Day 3: Fixed Points

Day 4: Patterns

Day 5: Solution to *The Game of the Week*

Counting Permutations breakout

Day 1: Descents
Day 2: Cycles
Day 3: Fixed Points
Day 4: Patterns
Day 5: Solution to *The Game of the Week*

Today: More variations on the Game of the Week!

Definition

A permutation is a list where order matters.

Permutations of 1? 1

Permutations of 1, 2? $12, 21$

Permutations of 1, 2, 3?

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Permutations of 1?	1
Permutations of 1, 2?	$12, 21$
Permutations of 1, 2, 3?	$123, 132, 213, 231, 312, 321$

There are ______________________ permutations of $1, 2, \ldots, n$.

Definition

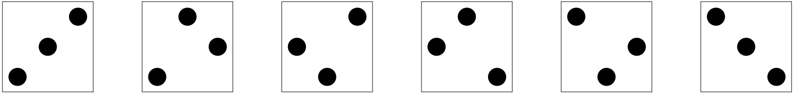
A permutation is a list where order matters.

Permutations of 1?	1
Permutations of 1, 2?	$12, 21$
Permutations of 1, 2, 3?	$123, 132, 213, 231, 312, 321$

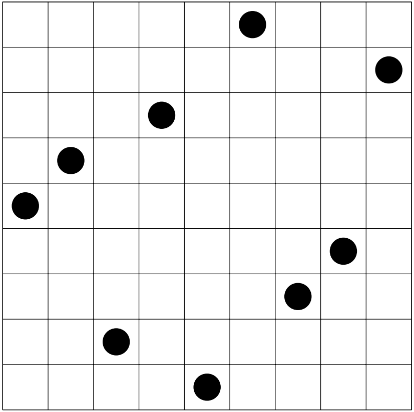
There are $n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$ permutations of $1, 2, \ldots, n$.

Permutation Pictures

123, 132, 213, 231, 312, 321



Prequel: Permutation Patterns



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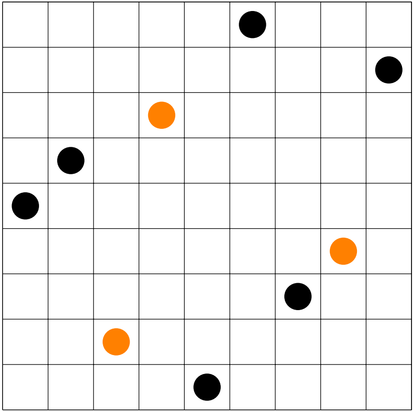

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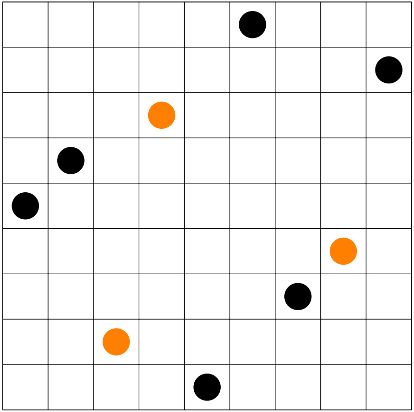
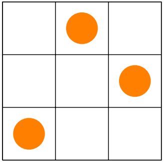
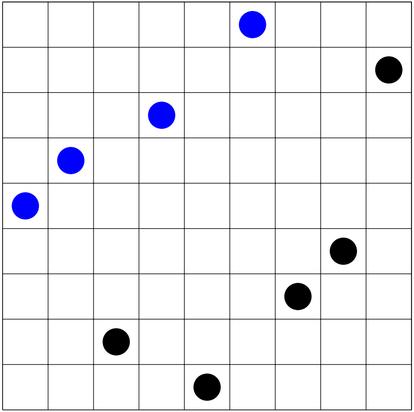
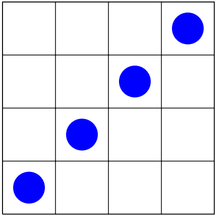




562719348 contains the pattern 132

Prequel: Permutation Patterns





562719348 contains the pattern 1234

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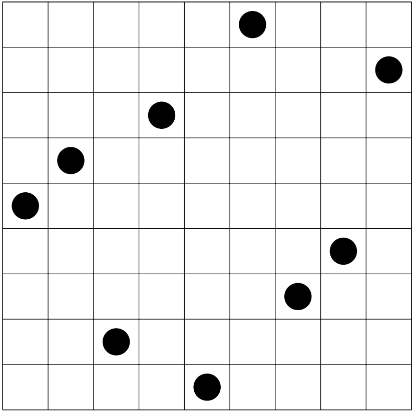

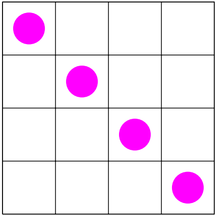

562719348 avoids the pattern 4321

The Game

The (a, b)-permutation game

- Take turns naming *different* positive numbers.
- The game is over when the permutation you made so far has an increasing $1 \cdots a$ pattern or a decreasing $b \cdots 1$ pattern.

Question: How long can you make the (a, b)-permutation game last?

The Goal

Theorem (Erdős-Szekeres, 1935)

Any permutation of length ______________ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

$a = 2,\ b = 2?$

$a = 3,\ b = 2?$

$a = \quad,\ b = 2?$

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$a = 2,\ b = 2?$	2	$a = 3,\ b = 3?$	
$a = 3,\ b = 2?$	3	$a = 4,\ b = 3?$	
$a = \ ,\ b = 2?$	a	$a = \ ,\ b = 3?$	

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$a = 2,\ b = 2?$ 2 $a = 3,\ b = 3?$ 5

$a = 3,\ b = 2?$ 3 $a = 4,\ b = 3?$

$a = \ ,\ b = 2?$ a $a = \ ,\ b = 3?$

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Any permutation of length ________________ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

$a = 2,\ b = 2?$	2	$a = 3,\ b = 3?$	5
$a = 3,\ b = 2?$	3	$a = 4,\ b = 3?$	7
$a = \ ,\ b = 2?$	a	$a = \ ,\ b = 3?$	

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$a = \ ,\ b = 2?$	a	$a = \ ,\ b = 3?$	$2a - 1$	$a = \ ,\ b = 4?$	

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Any permutation of length ______________ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

$a = 2,\ b = 2?$	2	$a = 3,\ b = 3?$	5	$a = 4,\ b = 4?$	10
$a = 3,\ b = 2?$	3	$a = 4,\ b = 3?$	7	$a = 5,\ b = 4?$	
$a = \quad,\ b = 2?$	a	$a = \quad,\ b = 3?$	$2a - 1$	$a = \quad,\ b = 4?$	

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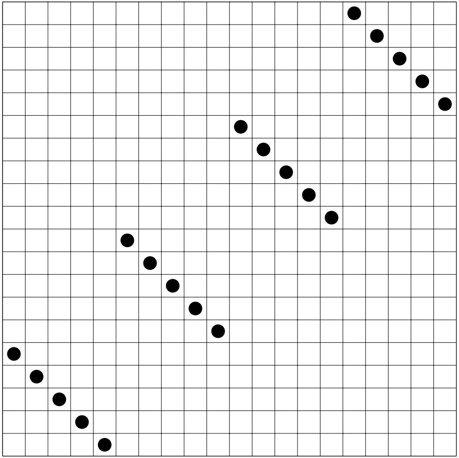
Optimal play intuition



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Theorem (Erdős-Szekeres, 1935)

Any permutation of length $\underline{(a-1)(b-1)+1}$ or more contains either an increasing pattern of length a or a decreasing pattern of length b.

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$a=\quad,\ b=2?$	a	$a=\quad,\ b=3?$	$2a-1$	$a=\quad,\ b=4?$	$3a-2$

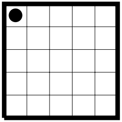
Proof Sketch (Seidenberg, 1959)

For each turn, keep track of:

1. the longest increasing sequence ending with the new number.
2. the longest decreasing sequence ending with the new number.

Example:

Chosen number:	5
increasing:	1
decreasing:	1



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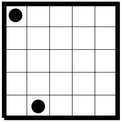

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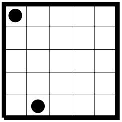

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For each turn, keep track of:

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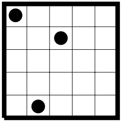
Example:

Chosen number:	5	1	4
increasing:	1	1	
decreasing:	1	2	



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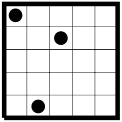

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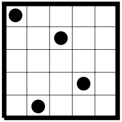

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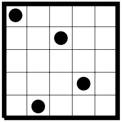

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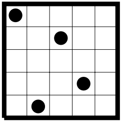

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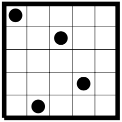

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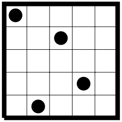

- The corresponding pairs of numbers are different because...
- The game is over when...
- There are $(a-1)(b-1)$ different possible pairs before a game-ending choice must be made.

Counting

Question: How many ways can we play the game optimally?

How many permutations of length $(a-1)(b-1)$ avoid both $1\cdots a$ and $b\cdots 1$?

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$a \backslash b$	2	3	4	5	6	7
2	1	1	1	1	1	1
3	1	4	25	196	1764	17424
4	1	25	1764	213444	36072036	7659050256
5	1	196	213444	577152576	2764917142416	1.96×10^{16}

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Not a new question...

SOLUTIONS OF ADVANCED PROBLEMS

Increasing and Decreasing Subsequences

5641 [1968, 1125]. *Proposed by Stanley Rabinowitz, Far Rockaway, N. Y.*

From the set $\{1, 2, 3, \cdots, n^2\}$ how many arrangements of the n^2 elements are there such that there is no subsequence of $n+1$ elements either monotone increasing or monotone decreasing?

Solution by Richard Stanley, Harvard University. Let A be an arrangement of $\{1, 2, \cdots, n^2\}$ with the desired property. Then the longest increasing and longest decreasing subsequences of A have length n (which is proved in the same way that one proves Erdös problem: every sequence of length n^2+1 has an increasing or a decreasing sequence of length $n+1$). It follows from a theorem of Schensted (*Increasing and decreasing subsequences*, Canadian J. of Math., 13 (1961) 179–191. Thm. 3), using the fact that the only partition of n^2 into n parts with largest part n is $n^2 = n+n+\cdots+n$, that the number of such A is given by

$$\left[\frac{n^2!}{1^1\cdot 2^2\cdot 3^3\cdots n^n\cdot(n+1)^{n-1}(n+2)^{n-2}\cdots(2n-1)^1}\right]^2.$$

Schensted in fact solves the more general problem of finding the number of arrangements of $\{1, 2, \cdots, n\}$ whose longest increasing subsequence has a given length a and whose longest decreasing subsequence has a given length b.

Also solved by Joel Spencer.

Advanced Problem 5641. Amer. Math. Monthly 76.10 (1969), p.1153.

... but a new bijection

Fact

There are $(C_{a-1})^2$ maximum length permutations with no $1 \cdots a$ and no 321.

Questions:

- What is C_{a-1}?
- Can we write a visual proof of this fact?

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In general: (☐) ☐

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In general: ($\boxed{i \text{ pairs}}$) $\boxed{n-i-1 \text{ pairs}}$ So, $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$

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In general: (i pairs) $n-i-1$ pairs So, $C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} = \frac{\binom{2n}{n}}{n+1}$

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Strategy:

- C_{a-1} is the number of ways to arrange $a-1$ pairs of parentheses
- So, $(C_{a-1})^2$ is...

... but a new bijection

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There are $(C_{a-1})^2$ maximum length permutations with no $1 \cdots a$ and no 321.

Questions:

- What is C_{a-1}?
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Strategy:

- C_{a-1} is the number of ways to arrange $a-1$ pairs of parentheses
- So, $(C_{a-1})^2$ is the number of *pairs* of parentheses arrangements
  Example: $(C_2)^2 = 4$ matches

(( )) , (( ))   (( )) , ( ) ( )
( ) ( ) , (( ))   ( ) ( ) , ( ) ( )

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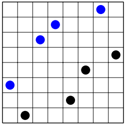

Observations about maximum length permutations avoiding $12 \cdots a$ and 321:

- The permutation has length $2(a-1)$.

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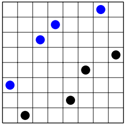

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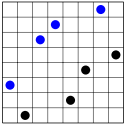
- The permutation has length $2(a-1)$.
- Every digit is a left-to-right max or not.

Fact

There are $(C_{a-1})^2$ maximum length permutations with no $1 \cdots a$ and no 321.



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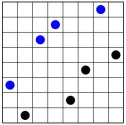

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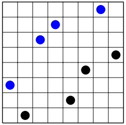

Observations about maximum length permutations avoiding $12 \cdots a$ and 321:

- The permutation has length $2(a-1)$.
- Every digit is a left-to-right max or not.
- Left-to-right maxes are in increasing order.
- Non left-to-right maxes are in increasing order.
- Exactly half the digits are left-to-right maxes.

… but a new bijection

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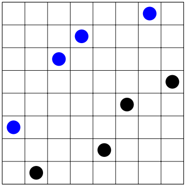

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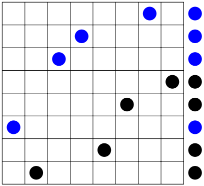

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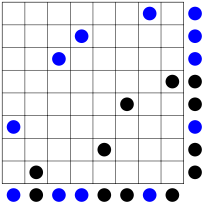

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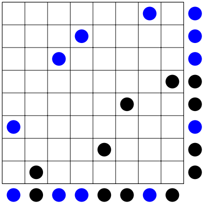
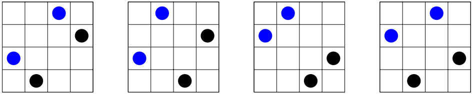


vertical sequence $\rightarrow$ (()(()))

horizontal sequence $\rightarrow$ ()(())()

Small Examples



()() ()() (()) (())

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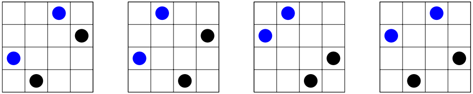

()() ()() (()) (())

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Full details at: Catalan Numbers and Permutations, *Mathematics Magazine* 97.3 (2024), 279-283.
https://doi.org/10.1080/0025570X.2024.2336423

What happens when you play this competitively?

The (a, b)-permutation game

- Take turns naming *different* positive numbers.
- The game is over when the permutation you made so far has an increasing $1 \cdots a$ pattern or a decreasing $b \cdots 1$ pattern.

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  When you complete an increasing $1 \cdots a$ pattern or a decreasing $b \cdots 1$ pattern, *you lose*.

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Question: Who has the winning strategy?

$a \geq 2,\ b = 2$

The $(a, 2)$-permutation game

- Take turns naming *different* positive numbers.
- When you complete an increasing $1 \cdots a$ pattern or a decreasing 21 pattern, *you lose*.

Minimum length game? 2 turns
Maximum length game? a turns

$a \geq 2,\ b = 2$

The $(a, 2)$-permutation game

- Take turns naming *different* positive numbers.
- When you complete an increasing $1 \cdots a$ pattern or a decreasing 21 pattern, *you lose*.

Minimum length game? 2 turns
Maximum length game? a turns

Strategy:

- Generate an increasing permutation
- If a is even player 1 wins; if a is odd player 2 wins.

Example:

$$\epsilon \to_1 1 \to_2 12 \to_1 123 \to_2 1234 \to_1 12345 \to_2 \cdots$$

$a \geq 3,\ b = 3$

The $(a, 3)$-permutation game

- Take turns naming *different* positive numbers.
- When you complete an increasing $1 \cdots a$ pattern or a decreasing 321 pattern, *you lose*.

Minimum length game? 3 turns

$a \geq 3,\ b = 3$

The $(a, 3)$-permutation game

- Take turns naming *different* positive numbers.
- When you complete an increasing $1 \cdots a$ pattern or a decreasing 321 pattern, *you lose*.

Minimum length game? 3 turns
Maximum length game? $2(a-1)+1$ turns

$a \geq 3,\ b = 3$

The $(a, 3)$-permutation game

- Take turns naming *different* positive numbers.
- When you complete an increasing $1 \cdots a$ pattern or a decreasing 321 pattern, *you lose*.

Minimum length game? 3 turns
Maximum length game? $2(a-1)+1$ turns

You try: play against a neighbor

- Is there a winning strategy... for the first player? ... for the second player?
- How long does the game last with your strategy?

Representing moves visually

Draw a grid, and label cells by ordered pairs as follows:

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	$\cdots$
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	
		$\vdots$			$\ddots$

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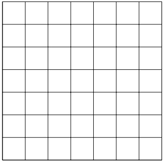
		$\vdots$			$\ddots$

Shade a cell when the most recent move has this ordered pair in the Seidenberg proof of Erdős-Szekeres Theorem.

$a = 6, b = 3$ Game Example

ϵ

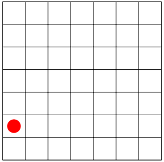
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6$, $b = 3$ Game Example

$\epsilon \rightarrow_1 1$

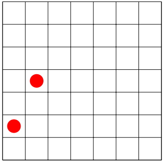
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6, b = 3$ Game Example

$\epsilon \rightarrow_1 1 \rightarrow_2 12$

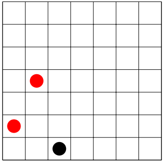
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6, b = 3$ Game Example

$\epsilon \to_1 1 \to_2 12 \to_1 231$

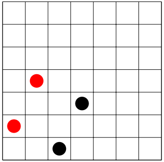
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6, b = 3$ Game Example

$\epsilon \to_1 1 \to_2 12 \to_1 231 \to_2 2413$

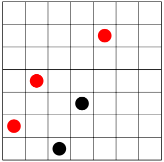
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6, b = 3$ Game Example

$$\epsilon \to_1 1 \to_2 12 \to_1 231 \to_2 2413 \to_1 24135$$

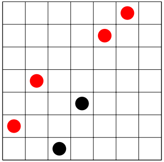
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6, b = 3$ Game Example

$$\epsilon \to_1 1 \to_2 12 \to_1 231 \to_2 2413 \to_1 24135 \to_2 241356$$

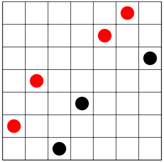
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



$a = 6, b = 3$ Game Example

$$\epsilon \to_1 1 \to_2 12 \to_1 231 \to_2 2413 \to_1 24135 \to_2 241356 \to_1 2413675$$

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)



The permutation game, re-framed

Play on a $(b-1) \times (a-1)$ grid.

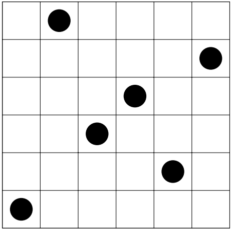
- First move: Player 1 shades the $(1,1)$ cell.
- Players take turns shading *legal* cells.
- Last move: player who takes the $(a-1, b-1)$ cell wins.

What makes a cell *legal*?

The permutation game, re-framed

What makes a cell *legal*?

Example:



(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
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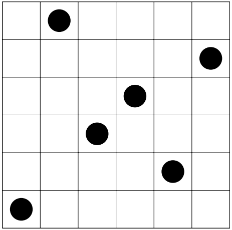
The permutation game, re-framed

What makes a cell *legal*?

- Either the row number or column number is strictly larger each previously-shaded cell.

Example:



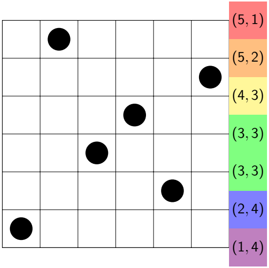
(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)

The permutation game, re-framed

What makes a cell *legal*?

- Either the row number or column number is strictly larger each previously-shaded cell.
- Must share an edge with some previously eliminated cell.

Example:



(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
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The permutation game, re-framed

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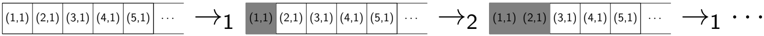
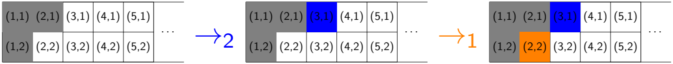
$a \geq 2,\ b = 2$

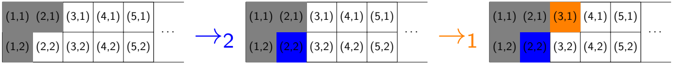


The $(a, 3)$-permutation game, re-framed

Case 1:



Case 2:



The $(a, 3)$-permutation game, re-framed

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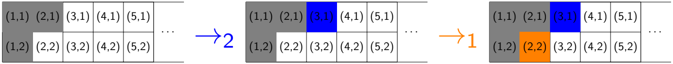
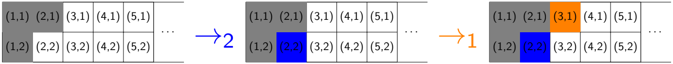


Case 2:



Endgame:

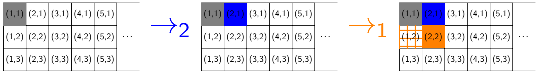
The $(a, 4)$-permutation game

You try: play against a neighbor

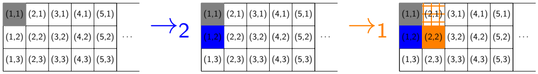
- Is there a winning strategy... for the first player? ... for the second player?
- How long does the game last with your strategy?

The $(a, 4)$-permutation game

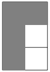
Opening moves:

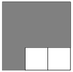


or

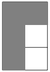


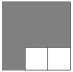
The $(a, 4)$-permutation game

State 1: 

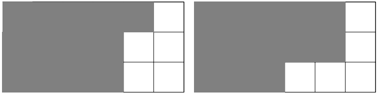
State 2: 

The $(a, 4)$-permutation game

State 1: 

State 2: 

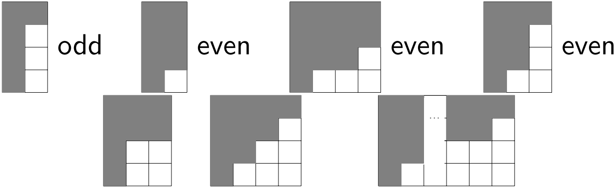
Endgame:



The $(a, 5)$-permutation game

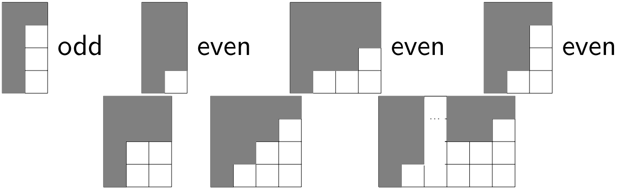
The $(a, 5)$-permutation game

Seven(ish) states:

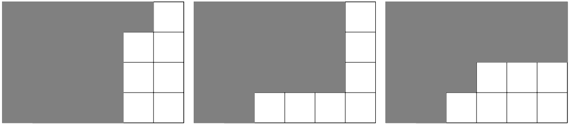


The $(a, 5)$-permutation game

Seven(ish) states:



Endgame:



Moves for $a \geq b$ game

b	minimum moves	maximum moves	actual moves using strategy
2	2	a	a
3	3	$2a-1$	$2a-2$
4	4	$3a-2$	$2a-3$ or $2a-1$
5	5	$4a-3$	between $2a-C$ and $4a-6$

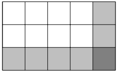
What if... ?

- First player to complete $1 \cdots a$ or $b \cdots 1$ *wins*?

What if... ?

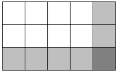
- First player to complete $1 \cdots a$ or $b \cdots 1$ *wins*?

  Same strategy on smaller board



What if... ?

- First player to complete $1 \cdots a$ or $b \cdots 1$ *wins*?
  Same strategy on smaller board



- Players pick specific numbers from $\{1, \ldots, (a-1)(b-1)+1\}$, rather than forming a pattern?

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  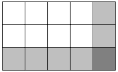
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  Results for $(a-1)(b-1)+1 \leq 15$ (Harary, Sagan, and West); general case open

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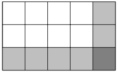
  

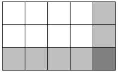
- Players pick specific numbers from $\{1, \ldots, (a-1)(b-1)+1\}$, rather than forming a pattern?
  Results for $(a-1)(b-1)+1 \leq 15$ (Harary, Sagan, and West); general case open

- More than two players?

What if... ?

- First player to complete $1 \cdots a$ or $b \cdots 1$ *wins*?

  Same strategy on smaller board

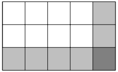
- Players pick specific numbers from $\{1, \ldots, (a-1)(b-1)+1\}$, rather than forming a pattern?

  Results for $(a-1)(b-1)+1 \leq 15$ (Harary, Sagan, and West); general case open

- More than two players?

  Open

What if... ?

- First player to complete $1 \cdots a$ or $b \cdots 1$ *wins*?
  Same strategy on smaller board
  
- Players pick specific numbers from $\{1, \ldots, (a-1)(b-1)+1\}$, rather than forming a pattern?
  Results for $(a-1)(b-1)+1 \leq 15$ (Harary, Sagan, and West); general case open
- More than two players?
  Open
- $b \geq 6$?
  Open, but conjectured first player winning strategy exists.

For more details...

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- Abraham Seidenberg, A Simple Proof of a Theorem of Erdős and Szekeres, *J. Lond. Math. Soc.* **34.3** (1959), 352.

Thanks for listening!

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