Pattern avoidance in cyclic parking functions

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Special Session on Parking Functions
AMS Fall Central Sectional Meeting
Saint Louis University
October 18, 2025

Definition

A permutation of length n is an ordered list of the numbers $1, 2, \ldots, n$.
$\mathcal{S}_n$ is the set of all permutations of length n.

$$\mathcal{S}_1 = \{1\}$$

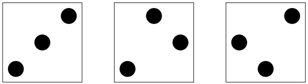
$$\mathcal{S}_2 = \{12, 21\}$$

$$\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$$

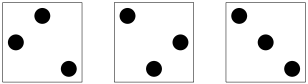
$$|\mathcal{S}_n| = n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$$

Note

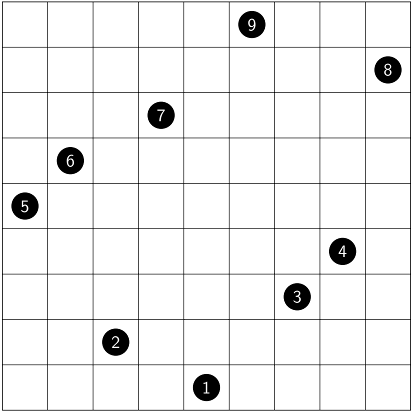
Permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is often visualized by plotting the points (i, π_i) in the Cartesian plane.



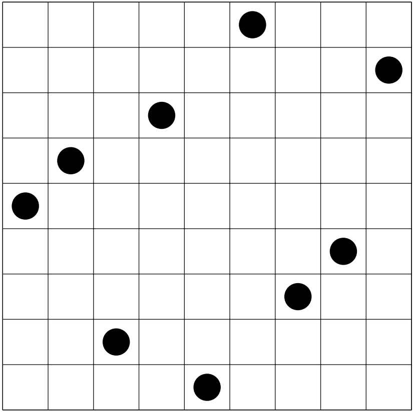
123 132 213

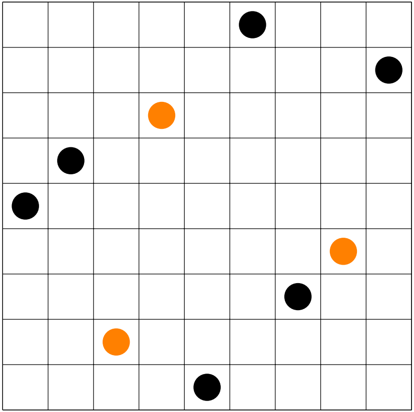


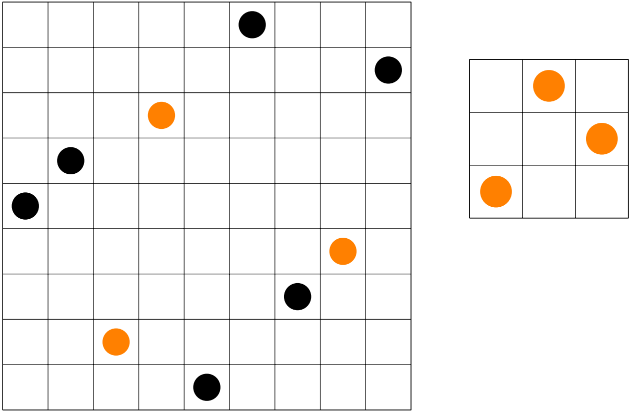
231 312 321



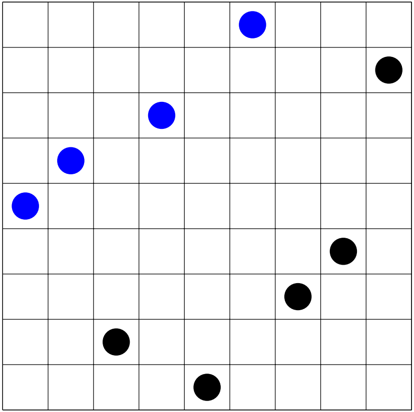
$$\pi = 562719348$$

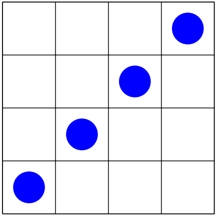




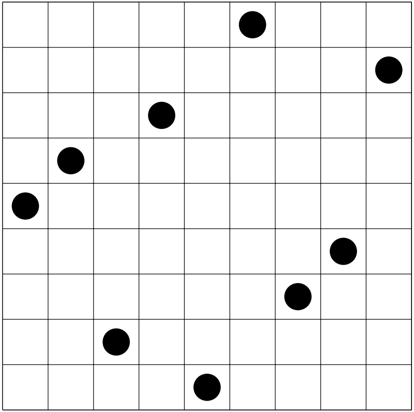


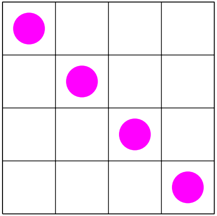
562719348 contains the pattern 132





562719348 contains the pattern 1234





562719348 avoids the pattern 4321

Big question

How many permutations of length n avoid the pattern ρ?

ρ	number of permutations avoiding ρ
$\rho \in \mathcal{S}_2$	1
$\rho \in \mathcal{S}_3$	C_n (Catalan)
1234	$1, 1, 2, 6, 23, 103, 513, 2761, \ldots$ (Gessel, 1990)
1342	$1, 1, 2, 6, 23, 103, 512, 2740, \ldots$ (Bóna, 1997)
1324	$1, 1, 2, 6, 23, 103, 513, 2762, \ldots$ (open question)

Patterns in Cyclic Permutations

Big question

How many permutations of length n *cyclicly* avoid the pattern ρ?
(i.e. there is no i for which $\pi_{i+1}\cdots\pi_n\pi_1\cdots\pi_i$ contains ρ.)

Studied by Callan (2002), Vella (2003)

Total number of cyclic permutations of size n: $(n-1)!$

ρ	number of permutations cyclicly avoiding ρ
$\rho \in \mathcal{S}_3$	1
1234	$2^n + 1 - 2n - \binom{n}{3}$
1342	$2^{n-1} - (n-1)$
1324	F_{2n-3} (Fibonacci)

History

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- Each Dyck path is associated with a permutation (many-to-one correspondence)
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Adeniran and Pudwell (2023)

- Extend Remmel and Qiu's work
- Count parking functions avoiding a subset of $\mathcal{S}_3$

Current Project

- Think of parking function as a word w on $[n]^n$
  $\mathcal{PF}_1 = \{1\}$
  $\mathcal{PF}_2 = \{11, 12, 21\}$
  $\mathcal{PF}_3 = \{111, 112, 121, 211, 122, 212, 221, 113,$
  $131, 311, 123, 132, 213, 231, 312, 321\}$
- Count parking functions such that there is no i where $w_{i+1} \cdots w_n w_1 \cdots w_i$ contains ρ.

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Equivalence classes:

$\mathbf{12} \sim 21$
$\mathbf{112} \sim 121 \sim 211$
$\mathbf{122} \sim 212 \sim 221$

$\mathbf{113} \sim 131 \sim 311$
$\mathbf{123} \sim 231 \sim 312$
$\mathbf{132} \sim 213 \sim 321$

Counting Cyclic Parking Functions

Known: $|\mathcal{PF}_n| = (n+1)^{n-1}$.

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- OEIS A121774: Number of n-bead necklaces with $n+1$ colors, divided by $(n+1)$.
- Necklace to parking function intuition:
  - Equivalence classes of necklaces:

    e.g. $11 \sim 22 \sim 33$ and $12 \sim 23 \sim 31$
  - $n+1$ necklaces in each class, and exactly one is a parking function.

Counting Cyclic Patterns

Known:

- $|\mathcal{PF}_n| = (n+1)^{n-1}$ $(1, 3, 16, 125, 1296, 16807, \dots)$
- Number of cyclic parking functions is given by A121774. $(1, 2, 6, 33, 260, 2812, \dots)$

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How many patterns after rotation?

- OEIS A019536: Number of length n necklaces with integer entries that cover an initial interval of positive integers
- $1, 2, 5, 20, 109, 784, 6757, 68240, \dots$

Known:

- $|\mathcal{PF}_n| = (n+1)^{n-1}$ $(1, 3, 16, 125, 1296, 16807, \dots)$
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Patterns of length 2: 11, 12
Patterns of length 3: 111, 112, 122, 123, 132

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Patterns of length 2: 11, 12
Patterns of length 3: 111, 112, 122, 123, 132

Notation: $cpf_n(\rho)$ is the number of cyclic parking functions of size n avoiding ρ.

Warmup: length 2 patterns

$cpf_n(12) = 1$

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$cpf_n(11) = (n-1)!$

Only *cyclic* parking functions with distinct digits.

Length 3 Patterns: Data

ρ	$cpf_n(\rho)$	OEIS
112	$1, 2, 4, 11, 42, 207, \ldots$	A213937
123	$1, 2, 5, 16, 47, 153, \ldots$	new
132	$1, 2, 5, 16, 47, 153, \ldots$	new
122	$1, 2, 5, 19, 101, 676, \ldots$	new (but related to A350267)
111	$1, 2, 5, 28, 204, 2000, \ldots$	new

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Generally explained in terms of Polya's Theorem and bijections with necklaces

Pairs of Length 3 Patterns: Data

ρ, σ	$cpf_n(\rho, \sigma)$	formula
112, 123	$1, 2, 3, 4, 5, 6, \ldots$	n
112, 132	$1, 2, 3, 4, 5, 6, \ldots$	n
112, 122	$1, 2, 3, 7, 25, 121, \ldots$	$(n-1)! + 1$
111, 112	$1, 2, 3, 9, 36, 180, \ldots$	$\frac{3(n-1)!}{2}$ for $n \geq 3$
122, 123	$1, 2, 4, 8, 16, 32, \ldots$	2^{n-1}
122, 132	$1, 2, 4, 8, 16, 32, \ldots$	2^{n-1}
123, 132	$1, 2, 4, 9, 16, 37, \ldots$	new
111, 123	$1, 2, 4, 11, 21, 51, \ldots$	Motzkin (except $n = 4$)
111, 132	$1, 2, 4, 11, 21, 51, \ldots$	Motzkin (except $n = 4$)
111, 122	$1, 2, 4, 15, 72, 420, \ldots$	$\frac{(n+1)(n-1)!}{2}$ for $n \geq 3$

Avoiding 111 and 112

Examples of length 4:

1234, 1243, 1324, 1342, 1423, 1432, 1233, 1323, 1332

Avoiding 111 and 112

Examples of length 4:

1234, 1243, 1324, 1342, 1423, 1432, 1233, 1323, 1332

Enumerate:

- $(n-1)!$ ways to make a cyclic permutation of $1, \ldots, n$
- $\frac{(n-1)!}{2}$ ways to make a cyclic permutation of $1, 2, \ldots, n-3, n-2, \mathbf{n\text{-}1}, \mathbf{n\text{-}1}$

$$cpf_n(111, 112) = (n-1)! + \frac{(n-1)!}{2} = \frac{3(n-1)!}{2}$$

Avoiding 122 and 123

Examples of length 4:

1111, 1112, 1113, 1114, 1132, 1142, 1143, 1432

Avoiding 122 and 123

Examples of length 4:

1111, 1112, 1113, 1114, 1132, 1142, 1143, 1432

Enumerate:

- Pick k of the digits $\{2, \ldots, n\}$ in $\binom{n-1}{k}$ ways
- Sum over $0 \leq k \leq n-1$

$$cpf_n(122, 123) = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}$$

Avoiding 111 and 132

Case 1: contains one n

Examples of length 5:

112**2**5, 1133**5**, 1123**5**, 1124**5**, 1134**5**, 1223**5**, 1224**5**, 1334**5**, 1234**5**

Avoiding 111 and 132

Case 1: contains one n

Examples of length 5:

112**2**5, 113**3**5, 112**3**5, 112**4**5, 113**4**5, 122**3**5, 122**4**5, 133**4**5, 123**4**5

Case 2: no n; find the rightmost i where $a(i) = i$.

Examples of length 5:

11223, **1**1224, **1**1233, **1**1234, 1**2**233, 1**2**234,
11**3**34, 12**3**34, 112**4**4, 113**4**4, 122**4**4, 123**4**4

Usually, rightmost $a(i) = i$ means $a(i+1) = i$.

(Only exceptions are 1212 and 1313 of length 4.)

Avoiding 111 and 132

Case 1: contains one n

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11**3**34, 12**3**34, 112**4**4, 113**4**4, 122**4**4, 123**4**4

Usually, rightmost $a(i) = i$ means $a(i+1) = i$.

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$$cpf_n(111,132) = cpf_{n-1}(111,132) + \sum_{k=0}^{n-2} cpf_k(111,132)cpf_{n-k-2}(111,132)$$

(Motzkin recurrence)

Pairs of Length 3 Patterns: Data

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Future directions

- Avoiding more patterns simultaneously?
- Avoiding longer patterns?
  - 20 patterns of length 4
  - Avoiding one at a time produces 14 distinct sequences, all are new to OEIS
- Connect length 3 avoidance results to other combinatorial objects.

References

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Thanks for listening!

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