Packing Patterns in Symmetric Words

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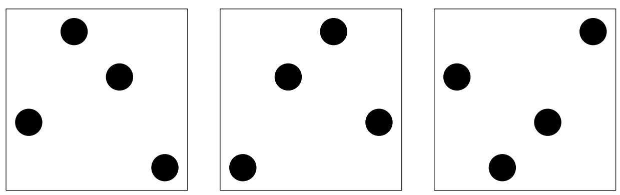
August 2, 2022

Abstract

We discuss packing permutation patterns into two specific subsets of words on $\{1, 1, 2, 2, \ldots, n, n\}$, i.e. those of the form $\pi\pi^r$ and those of the form $\pi\pi$ for some permutation π. In both cases we answer a number of related enumeration questions for packing patterns of length at most 4.

1 Introduction

Let $\mathcal{S}_n$ be the set of all permutations on $[n] = \{1, 2, \ldots, n\}$. A permutation $\pi \in \mathcal{S}_n$ may be viewed as a bijection on $[n]$. When we graph the points (i, π_i) in the Cartesian plane, all points lie in the square $[1, n] \times [1, n]$, and thus we may apply various symmetries of the square to obtain involutions on the set $\mathcal{S}_n$. For $\pi \in \mathcal{S}_n$, let $\pi^r = \pi_n \cdots \pi_1$ and let $\pi^c = (n+1-\pi_1) \cdots (n+1-\pi_n)$, the reverse and complement of π respectively. For example, the graphs of $\pi = 2431$, $\pi^r = 1342$, and $\pi^c = 3124$ are shown in Figure 1. Two common permutations are the increasing permutation of length m, denoted I_m, and the decreasing permutation of length m, denoted J_m. Notice that $I_m^r = I_m^c = J_m$.



$\pi = 2431$ $\quad$ $\pi^r = 1342$ $\quad$ $\pi^c = 3124$

Figure 1: The graphs of $\pi = 2431$, $\pi^r = 1342$, and $\pi^c = 3124$

Two common permutation constructions will be useful throughout this paper. Given $\pi \in \mathcal{S}_a$ and $\tau \in \mathcal{S}_b$, we define $\pi \oplus \tau \in \mathcal{S}_{a+b}$ to be the permutation where

$$(\pi \oplus \tau)_i = \begin{cases} \pi_i & 1 \leq i \leq a \\ a + \tau_{i-a} & a+1 \leq i \leq a+b \end{cases}$$

and $\pi \ominus \tau \in \mathcal{S}_{a+b}$ to be the permutation where

$$(\pi \ominus \tau)_i = \begin{cases} b + \pi_i & 1 \leq i \leq a \\ \tau_{i-a} & a+1 \leq i \leq a+b \end{cases}.$$

The permutation $\pi \oplus \tau$ is known as the *sum* of π and τ, while $\pi \ominus \tau$ is known as the *skew-sum* of π and τ.

We are interested in the notion of patterns in permutations and words. Given a word (i.e. multiset permutation) $w = w_1 \cdots w_n$ and $\rho \in \mathcal{S}_m$ we say that w *contains* ρ as a pattern if there exist $i_1, i_2, \ldots, i_m$ and n such that

$1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $w_{i_a} < w_{i_b}$ if and only if $\rho_a < \rho_b$. In this case we say that $w_{i_1} \cdots w_{i_m}$ is *order-isomorphic* to ρ, and that $w_{i_1} \cdots w_{i_m}$ is an *occurrence* of ρ in w. If w does not contain ρ, then we say that w *avoids* ρ. Of particular interest are the sets $\mathcal{S}_n(\rho) = \{\pi \in \mathcal{S}_n \mid \pi \text{ avoids } \rho\}$. Let $\mathrm{s}_n(\rho) = |\mathcal{S}_n(\rho)|$. It is well known that $\mathrm{s}_n(\rho) = \frac{\binom{2n}{n}}{n+1}$ for $\rho \in \mathcal{S}_3$ (see [11]), while enumeration is much more difficult for patterns ρ of length 4 or more. One of the oldest results in the area of pattern avoidance is the Erdős-Szekeres Theorem:

Theorem 1.1 (Erdős-Szekeres [8])**.** *Every permutation of length at least $(a-1)(b-1)+1$ must contain either the pattern I_a or the pattern J_b.*

Pattern-avoidance has been considered in permutations and in words with specific symmetries. For example, Ferrari [9] studied pattern avoidance in centrosymmetric words. Cratty, Erickson, Negassi, and Pudwell [7] defined the set of *double lists* on n letters to be

$$\mathcal{D}_n = \{\pi\pi \mid \pi \in \mathcal{S}_n\}$$

and completely characterized the members of $\mathcal{D}_n$ that avoid a given permutation pattern of length at most 4. More recently, Anderson, Diepenbroek, Pudwell, and Stoll [2] defined the set of *reverse double lists* on n letters to be

$$\mathcal{R}_n = \{\pi\pi^r \mid \pi \in \mathcal{S}_n\},$$

completely characterized members of $\mathcal{R}_n$ that avoid a given pattern of length at most 4, and analyzed a number of special cases for longer patterns. In each of these situations, the added structure in the words under consideration allows for a more specific analysis than the general case.

In this paper we are concerned with a complementary optimization problem. In particular, let $\nu(\rho, w)$ be the number of occurrences of ρ in word w, let A be an infinite family of words, and let A_n be the members of A of length n. We define

$$\mu_{A_n}(\rho) = \max_{w \in A_n} \nu(\rho, w)$$

and

$$d_A(\rho) = \lim_{n \to \infty} \frac{\max_{w \in A_n} \nu(\rho, w)}{\binom{n}{|\rho|}}.$$

Any permutation (or word) in A_n that achieves $\mu_{A_n}(\rho)$ is said to be *ρ-optimal.*

When we focus on the case of packing patterns into permutations,

$$d(\rho) = \lim_{n\to\infty} \frac{\max_{w\in\mathcal{S}_n} \nu(\rho, w)}{\binom{n}{|\rho|}}$$

is known as the (classical) *packing density* of ρ. It is known that $d(12\cdots m) = 1$ and $d(132) = 2\sqrt{3}-3$. In fact, if ρ is layered (i.e., if $\rho \in \mathcal{S}_n(231, 312)$), then there exists a layered ρ-optimizer in $\mathcal{S}_n$. Price [12] determined the packing densities of 1432 and 2143, while Albert et al. [1] determined the packing density of 1243. There still remain a number of open packing densities for $\rho \in \mathcal{S}_m$, $m \geq 4$ with partial progress [4, 10, 16, 17, 18]. In addition other researchers have considered packing patterns into words [3, 6]. In particular, Burstein, Hästö, and Mansour determined specific packing densities when $\rho \in \mathcal{S}_3$ and $A_n = [k]^n$.

Rather than packing patterns into any word in $[k]^n$, we are concerned with words that have specific substructures. In particular, we will pack patterns into the sets $\mathcal{D}_n$ and $\mathcal{R}_n$ described above. In Section 2, we determine an upper bound on $\mu_{\mathcal{R}_n}(\rho)$ that is independent of ρ and characterize the patterns ρ for which this bound is sharp. We also consider the sequences $\mu_{\mathcal{R}_n}(\rho)$ for specific patterns ρ that fall short of the upper bound. In all cases we also analyze the structure of ρ-optimal members of $\mathcal{R}_n$. In Section 3, we answer analogous questions for packing patterns into $\mathcal{D}_n$. We conclude with a summary of remaining opening questions.

2 Packing into Reverse Double Lists

In this section, we consider the sequence $\mu_{\mathcal{R}_n}(\rho)$ for various patterns ρ. Notice that if $w = \pi\pi^r$ contains k copies of ρ, then by symmetry, $w^r = (\pi\pi^r)^r = \pi\pi^r = w$ contains k copies of ρ^r and $w^c = \pi^c\pi^{rc}$ contains k copies of ρ^c, so $\mu_{\mathcal{R}_n}(\rho) = \mu_{\mathcal{R}_n}(\rho^r) = \mu_{\mathcal{R}_n}(\rho^c)$ for all permutations ρ and all $n \geq 0$. When $\mu_{\mathcal{R}_n}(\rho) = \mu_{\mathcal{R}_n}(\rho')$ for all $n \geq 0$, we say that ρ and ρ' are *Wilf-equivalent*.

Before we consider specific cases, we begin with an upper bound on the number of times any permutation pattern can be packed into a reverse double list.

Theorem 2.1. *Suppose $\rho \in \mathcal{S}_m$. Then*

$$\mu_{\mathcal{R}_n}(\rho) \leq 2\binom{n}{m}.$$

Proof. Suppose $w \in \mathcal{R}_n$. There are exactly $\binom{n}{m}$ ways to choose a collection of m distinct letters in w. We claim that for any collection of m distinct letters in w, there are exactly zero or two copies of ρ in w using that alphabet.

Consider a copy of ρ in w using the alphabet $a_1, \ldots, a_m$. Further, suppose that $w_{i_1} w_{i_2} \cdots w_{i_m}$ is the copy of ρ in w that uses the earliest possible copy of each of the letters $a_1, \ldots, a_m$ that form a ρ pattern and that $i_j = n - e$ is the largest index such that $i_j \leq n$. Then $w_{i_1} \cdots w_{i_{j-1}} w_{n+e+1} w_{i_{j+1}} \cdots w_{i_m}$ is the only other possible copy of ρ using this alphabet. To check, notice that choosing to use the other copy w_{i_ℓ} for $i_\ell > i_j$ violates the idea that $w_{i_1} w_{i_2} \cdots w_{i_m}$ uses the earliest possible copy of each letter that can be used to form a ρ pattern from this alphabet. Choosing to use the other copy of w_{i_ℓ} for $i_\ell < i_j$ forms a subword that is no longer order-isomorphic to ρ since these letters w_{i_ℓ} and w_{i_j} will be transposed in the new subword. Similarly, choosing multiple letters from π to replace with their copies in π^r transposes those letters in the chosen subword and no longer forms a ρ pattern. So, for any alphabet $a_1, \ldots, a_m$, there are either zero copies or two copies of ρ in w using that alphabet. $\square$

Now that we have an upper bound on $\mu_{\mathcal{R}_n}(\rho)$, we consider which patterns ρ achieve this upper bound. We refer to any pattern $\rho \in \mathcal{S}_m$ for which $\mu_{\mathcal{R}_n}(\rho) = 2\binom{n}{m}$ as *maximal*. To characterize maximal patterns, recall that a *peak* of permutation $\rho \in \mathcal{S}_m$ is a position $1 < i < m$ such that $\rho_{i-1} < \rho_i > \rho_{i+1}$, and a *valley* is a position $1 < i < m$ such that $\rho_{i-1} > \rho_i < \rho_{i+1}$. Let $\mathrm{pk}(\rho)$ denote the number of peaks of ρ and let $\mathrm{vl}(\rho)$ denote the number of valleys of ρ. Together, we refer to any position that is a peak or a valley as an *extreme point*.

Theorem 2.2. *Let $\rho \in \mathcal{S}_m$. $\mu_{\mathcal{R}_n}(\rho) = 2\binom{n}{m}$ for all $n \geq m$ if and only if* $\mathrm{pk}(\rho) + \mathrm{vl}(\rho) \leq 1$.

Proof. First, assume that $\mathrm{pk}(\rho) + \mathrm{vl}(\rho) \leq 1$. If $\mathrm{vl}(\rho) = 0$, it is easy to check that there are $2\binom{n}{m}$ copies of ρ in the word $I_n J_n$. If $\mathrm{vl}(\rho) = 1$, there are $2\binom{n}{m}$ copies of ρ in the word $J_n I_n$.

On the other hand, assume that $\mathrm{pk}(\rho) + \mathrm{vl}(\rho) \geq 2$. Since peaks and valleys alternate in a permutation that means there is at least one peak and at least one valley.

Now, suppose $n \geq (m-1)^2+1$. This implies that $\pi = w_1 \cdots w_n$ has either an increasing subsequence of length m or a decreasing subsequence of length m by the Erdős-Szekeres Theorem. If we focus on the digits forming such a monotone sequence of length m in π, then in $\pi\pi^r$ they form a subsequence of the form $I_m J_m$ or $J_m I_m$. In either case, this subsequence has only one extreme point and therefore cannot have a ρ subsequence. Since there exists at least one collection of m letters chosen from $\{1, 2, \ldots, n\}$ that does not form a ρ pattern, it must be the case that $\mu_{\mathcal{R}_n}(\rho) < 2\binom{n}{m}$ and therefore ρ is not maximal.

$\square$

By Theorem 2.2, every permutation of length $m \leq 3$ is *maximal*. This implies that for $\rho \in \mathcal{S}_m$ with $m \leq 3$, $\mu_{\mathcal{R}_n}(\rho) = 2\binom{n}{m}$ and

$$d_{\mathcal{R}_n}(\rho) = \lim_{n\to\infty} \frac{2\binom{n}{m}}{\binom{2n}{m}} = \frac{1}{2^{m-1}}.$$

Notice that ρ-optimal members of $\mathcal{R}_n$ are not unique. In fact, as shown in Theorem 2.3, for $\rho \in \{1, 12, 21\}$, *every* member of $\mathcal{R}_n$ is ρ-optimal. More generally, if $n \geq 2$, there are at least two ρ-optimal members of $\mathcal{R}_n$ for any ρ, as shown in Theorem 2.4.

Theorem 2.3. *Let $\rho \in \{1, 12, 21\}$. Then every member of $\mathcal{R}_n$ is ρ-optimal.*

Proof. For $\rho = 1$, every digit of $w = \pi\pi^r \in \mathcal{R}_n$ is a ρ pattern, and $w = \pi\pi^r$ contains $2n$ copies of ρ, for any underlying permutation π.

For $\rho = 12$, consider two distinct digits x and y of π and assume that $x < y$. Either x and y form an $xyyx$ subword or a $yxxy$ subword in $w = \pi\pi^r$. In the first case, the first occurrence of x together with either y forms a 12 pattern. In the second case, either x together with the second occurrence of y forms a 12 pattern. Therefore, no matter the particular permutation π, for each of the $\binom{n}{2}$ ways to choose two letters of π we get two copies of 12, for a total of $2\binom{n}{2}$ copies of 12.

Since $21 = 12^r$, every word in $\mathcal{R}_n$ is also 21-optimal.

$\square$

Theorem 2.4. *Let $n \geq m \geq 2$, $\rho \in \mathcal{S}_m$, and $\pi = \tau xy \in \mathcal{S}_n$. If $w = \tau xyyx\tau^r$ is ρ-optimal then $w' = \tau yxxy\tau^r$ is ρ-optimal.*

Proof. We wish to show that there is a bijection between copies of ρ in w and copies of ρ in w'. Consider an occurrence of ρ in w.

If neither x nor y is involved in the occurrence, then the occurrence is also in w'. If exactly one of x or y is involved, then the occurrence is also in

w'. If both x and y are involved, they form either a 12 or a 21 pattern. As we saw in the proof of Theorem 2.3 there are exactly two copies of 12 (resp. 21) in $xyyx$ and exactly two copies of 12 (resp. 21) in $yxxy$. In either case, there are two copies of ρ in w using the particular collection of x, y and $m-2$ other letters, and there are two copies of ρ in w' using the same collection of letters. $\square$

As a consequence of Theorem 2.4, we know that for any $\rho \in \mathcal{S}_m$ and $n \geq 2$, the number of ρ-optimal words in $\mathcal{R}_n$ is even. In the case where $\rho = I_m$, we can say even more.

Theorem 2.5. *The word $w = \pi\pi^r \in \mathcal{R}_n$ is I_m-optimal for all $m \leq n$ if and only if π avoids both 213 and 231.*

Proof. By Theorem 2.2, a I_m-optimal word in $\mathcal{R}_n$ contains $2\binom{n}{m}$ copies of $12\cdots m$. In other words, there are two copies of I_m for each collection of m distinct letters in $\{1, 2, \ldots, n\}$.

Suppose that π avoids both 213 and 231. Then for $1 \leq i \leq n-1$, either $\pi_i = \max_{j \geq i} \pi_j$ or $\pi_i = \min_{j \geq i} \pi_j$. Now, consider a collection of m distinct letters in $\{1, 2, \ldots, n\}$ and let π_k be the letter from this collection closest to the end of π. We can form a copy of I_m in the following way: if $\pi_i < \pi_k$ choose its instance in π. If $\pi_i > \pi_k$, choose its instance in π^r. If $\pi_i = \pi_k$, we can choose either instance. Therefore, w has $2\binom{n}{m}$ copies of I_m, and so w is ρ-optimal.

On the other hand, suppose that π contains either a 213 pattern or a 231 pattern using the digits π_i, π_j, and π_k with $i < j < k$. In any collection of m letters that includes π_i, π_j, and π_k, these three letters appear in the order $\pi_i\pi_j\pi_k\pi_k\pi_j\pi_i$. However, if we are to form a copy of I_m, we need π_i to appear between copies of π_j and π_k. Therefore, there is no I_m pattern in w that uses this collection of m letters and w is not ρ-optimal. $\square$

Corollary 2.1. *For $n \geq m \geq 2$, there are 2^{n-1} I_m-optimal members of $\mathcal{R}_n$.*

Proof. We know $w = \pi\pi^r \in \mathcal{R}_n$ is I_m-optimal if and only if π avoids both 213 and 231. This enumeration follows either from the description in the proof of Theorem 2.5 or similarly from Proposition 12 of [14]. $\square$

As an example, there are $2^3 = 8$ members of $\mathcal{R}_4$ that are I_m-optimal for $m \leq 4$. They are 12344321, 12433421, 14233241, 14322341, 41233214, 41322314, 43122134, 43211234. For instance, 14233241 has $2\binom{4}{3} = 8$ copies of 123: two using each possible collection of 3 distinct digits.

At this point, we have completely characterized maximal patterns of any length and enumerated I_m-optimal words. The situation for other patterns is more complex. The smallest non-maximal patterns are of length 4. By symmetry of reverse and complement, we may partition $\mathcal{S}_4$ into eight Wilf classes, where members of the same class are guaranteed to give the same values for $\mu_{\mathcal{R}_n}(\rho)$. One representative of each class is: 1234, 1243, 1324, 1342, 1423, 1432, 2143, and 2413. By Theorem 2.2, four of these patterns (1234, 1243, 1342, and 1432) are maximal and four are not. A summary of values of $\mu_{\mathcal{R}_n}(\rho)$ and $d_{\mathcal{R}_n}(\rho)$ for each of these cases is given in Table 1.

Pattern ρ	$\mu_{\mathcal{R}_n}(\rho)$	$d_{\mathcal{R}_n}(\rho)$	number of ρ-optimal words in $\mathcal{R}_n$
1234	$2\binom{n}{4}$	$\frac{1}{8}$	2^{n-1}
1243	$2\binom{n}{4}$	$\frac{1}{8}$	4
1342	$2\binom{n}{4}$	$\frac{1}{8}$	$2F_n$
1432	$2\binom{n}{4}$	$\frac{1}{8}$	$4(n-2)$
2143	(degree 4 quasi-polynomial)	$\frac{3}{32}$	$2^{\left(\lfloor \frac{n}{2} \rfloor + 1\right)}$
1423	$2, 10, 28, 60, 110, \ldots$ (open)	> 0.071892	
1324	$2, 10, 26, 54, 102, \ldots$ (open)		
2413	$2, 8, 22, 48, 92, \ldots$ (open)		

Table 1: Packing data for patterns of length 4 embedded in $\mathcal{R}_n$

2.1 Non-monotone maximal patterns of length 4

By Theorem 2.2, the non-monotone maximal patterns of length 4 are 1243, 1342, and 1432. By definition of maximal, if ρ is one of these three patterns, there are $2\binom{n}{4}$ copies of ρ in any ρ-optimal word $w \in \mathcal{R}_n$. However, this does not tell us what the ρ-optimal words look like. We consider each of these patterns in turn.

Theorem 2.6. *For $n \geq 5$, $w = \pi\pi^r \in \mathcal{R}_n$ is 1243-optimal if and only if $\pi \in \{I_n, I_{n-2} \oplus J_2, I_{n-3} \oplus 213, I_{n-3} \oplus 231\}$.*

Proof. It is straightforward to check that if

$$\pi \in \{I_n, I_{n-2} \oplus J_2, I_{n-3} \oplus 213, I_{n-3} \oplus 231\},$$

then for any subset of four letters $\{a, b, c, d\} \subseteq \{1, 2, \ldots, n\}$, there is a 1243-pattern in $\pi\pi^r$ using the chosen letters a, b, c, and d.

Now, suppose that $w = \pi\pi^r$ is 1243-optimal. Then any collection of four letters chosen from $\{1, 2, \ldots, n\}$ must produce a 1243 pattern. We see that $\pi_i \neq n$ for $1 \leq i \leq n-3$ since otherwise the subword $n\pi_{n-2}\pi_{n-1}\pi_n\pi_n\pi_{n-1}\pi_{n-2}n$ has no 1243 pattern. We can further rule out the possibility that $\pi_{n-2} = n$. On the one hand, for any $j < n-2$, the subword $\pi_j n\pi_{n-1}\pi_n\pi_n\pi_{n-1}n\pi_j$ must have a 1243 pattern which uses π_j playing the role of 3. This implies that $\{\pi_{n-1}, \pi_n\} = \{1, 2\}$. But now, the subword $\pi_1\pi_2 n\pi_n\pi_n n\pi_2\pi_1$ has no 1243 pattern. We see that n must play the role of 4 since it is the largest digit and π_n must play the role of 1 since it is the smallest digit in the subword, but there is no digit between them to play the role of 2. Therefore either $\pi_n = n$ or $\pi_{n-1} = n$.

In the case where $\pi_n = n$, any collection of four digits including n draws a 1243 pattern from taking its smaller two digits in increasing order from π and its remaining digit from π^r. This implies that $\pi = I_n$ or $\pi = I_{n-3} \oplus 213$.

In the case where $\pi_n = n-1$, the subword formed by all copies of any collection of four digits including n and π_n should form a 1243 pattern. If $\pi_n \leq n-3$, then the word $\pi_i\pi_j n\pi_n\pi_n n\pi_j\pi_i$ has no 1243 pattern since π_n must play the role of 1, n must play the role of 4, and there is no digit between them to play the role of 2. This implies either $\pi_n = n-1$ or $\pi_n = n-2$. If $\pi_n = n-1$, then in order for $\pi_i\pi_j n\pi_n\pi_n n\pi_j\pi_i$ to have a 1243 pattern for every $1 \leq i < j \leq n-2$, we have $\pi_1 < \pi_2$ so that π_1 and π_2 can play the roles of 1 and 2. By a similar analysis, if $\pi_n = n-2$, then $\pi_{n-2} = n-1$ and $\pi = I_{n-3} \oplus 231$.

We have now exhausted all possible options to form a 1243-optimal word in $\mathcal{R}_n$. □

Corollary 2.2. *For $n \geq 5$, there are four 1243-optimal members of $\mathcal{R}_n$.*

Theorem 2.7. *If $n \geq 5$, then $w = \pi\pi^r \in \mathcal{R}_n$ is 1342-optimal if and only if $\pi \in \mathcal{S}_n(231, 312, 321)$ or $\pi_1 \cdots \pi_{n-2}\pi_n\pi_{n-1} \in \mathcal{S}_n(231, 312, 321)$.*

Proof. First, we show that $\pi \in \mathcal{S}_n(231, 312, 321)$ implies $\pi\pi^r$ is 1342-optimal. It is known that if $\pi \in \mathcal{S}_n(231, 312, 321)$, then $\pi = 1 \oplus \pi'$ where $\pi' \in$

$\mathcal{S}_{n-1}(231, 312, 321)$ or $\pi = J_2 \oplus \pi'$ where $\pi' \in \mathcal{S}_{n-2}(231, 312, 321)$. In other words, $\pi \in \mathcal{S}_n(231, 312, 321)$ if and only if π is a layered permutation with all layers of size 1 or 2. Now consider a collection of four letters $\{a, b, c, d\} \subset \{1, 2, \ldots, n\}$ where $a < b < c < d$. We may always find a 1342 pattern composed of the digits a, b, c, d in $\pi\pi^r$ by taking a and c from π and b from π^r. If c and d are in different layers, then we may take d from π. If d and c are in the same layer, then we may take d from π^r. By Theorem 2.4, if $\pi\pi^r$ is ρ-optimal then $\hat{\pi}\hat{\pi}^r$ is ρ-optimal where $\hat{\pi} = \pi_1 \cdots \pi_{n-2}\pi_n\pi_{n-1}$, so all words described in the theorem statement are indeed 1342-optimal.

Now, for the converse, we suppose that $\pi\pi^r$ is 1342-optimal. First, we know that $\pi_i \neq n$ for $i \leq n-3$ since the word $n\pi_{n-2}\pi_{n-1}\pi_n\pi_n\pi_{n-1}\pi_{n-2}n$ has no 1342 pattern. So it must be the case that if $\pi\pi^r$ is 1342-optimal, then n is among the final three digits of π. In fact, $\pi_{n-2} = n$ is impossible. In order for $\pi_j n\pi_{n-1}\pi_n\pi_n\pi_{n-1}n\pi_j$ to contain a 1342 pattern for all $j < n-2$, it must be the case that the second n plays the role of 4 and π_j plays the role of 2 so $\{\pi_{n-1}, \pi_n\} = \{1, n-1\}$. However, now consider a collection of four digits that include 1 and n but not $n-1$. We must use n to play the role of 4 in 1342, but there is no digit between 1 and n to play the role of 3. Therefore n must be one of the last two digits of π.

Suppose $\pi\pi^r$ is 1342-optimal but π contains 312 or 321. Since $\pi_{n-1} = n$ or $\pi_n = n$, the digits $\pi_i < \pi_j < \pi_k$ in this pattern do not include n. Consider n together with these three digits. We either have a subword of the form $\pi_k\pi_i\pi_j nn\pi_j\pi_i\pi_k$, $\pi_k\pi_j\pi_i nn\pi_i\pi_j\pi_k$, $\pi_k\pi_i n\pi_j\pi_j n\pi_i\pi_k$ or $\pi_k\pi_j n\pi_i\pi_i n\pi_j\pi_k$. In any case, the digits π_i and n must play the roles of 1 and 4 respectively in a 1342 pattern and π_k must play the role of 3. However, π_k is the first (and last) digit in each subword, rather than appearing between a copy of π_i and a copy of n. Therefore if $\pi\pi^r$ is 1342-optimal, π avoids 321 and 321.

Suppose $\pi\pi^r$ is 1342-optimal but π contains 231. If this copy of 231 does not involve the digit n, then the digits that form the 231 pattern together with n (which is either π_{n-1} or π_n) form a subword that avoids 1342, so the 231 must use $\pi_{n-1} = n$ playing the role of 3. Now, suppose $\pi_n < n-2$. In this case, the digits $(n-2)$, $(n-1)$, n, and π_n form a subword that avoids 1342 since the roles of 1 and n must be played by π_n and n respectively and there is no digit between them to play the role of a 3. Therefore, if there is a 231 pattern in π, it is formed with $\pi_{n-1} = n$ and $\pi_n = n-2$. Finally, suppose $\pi_{n-2} \neq n-1$. Then $n-1$, π_{n-2}, n and $n-2$ form a word that is order-isomorphic to 31422413, which has no copy of 1342. Therefore, if $\pi\pi^r$ is 1342-optimal and π contains 231, the sole copy of 231 is formed by the digits

$n-1$, n, and $n-2$ in the final three positions of π. However, transposing the last two digits forms a pattern that avoids 231, 312, and 321.

□

Corollary 2.3. *For $n \geq 5$, there are $2F_n$ 1342-optimal members of $\mathcal{R}_n$ where F_n is the nth Fibonacci number with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.*

Proof. We have seen that if $w = \pi\pi^r$ is 1342-optimal, then either $\pi \in \mathcal{S}_n(231, 312, 321)$ or $\pi_1 \cdots \pi_{n-2}\pi_n\pi_{n-1} \in \mathcal{S}_n(231, 312, 321)$ (or both).

In the first case, $\pi \in \mathcal{S}_n(231, 312, 321)$ if and only if π consists of a direct sum of 1 and 21 permutations, which implies $|\mathcal{S}_n(231, 312, 321)|$ follows the Fibonacci recurrence. We know that $|\mathcal{S}_1(231, 312, 321)| = 1$ and $|\mathcal{S}_2(231, 312, 321)| = 2$, so $|\mathcal{S}_n(231, 312, 321)| = F_{n+1}$.

In the second case, if $\pi_1 \cdots \pi_{n-2}\pi_n\pi_{n-1}$ ends with one layer of size 2, $\pi \in \mathcal{S}_n(231, 312, 321)$ ends in two layers of size 1, so it has already been counted in the first case. If $\pi_1 \cdots \pi_{n-2}\pi_n\pi_{n-1}$ ends with two layers of size 1, then $\pi \in \mathcal{S}_n(231, 312, 321)$ ends in a layer of size 2, so it has also already been counted in the first case. However, we may also have the situation that $\pi_1 \cdots \pi_{n-2}\pi_n\pi_{n-1}$ ends with a layer of size 2 followed by a layer of size 1. In this case $\pi_{n-2}\pi_{n-1}\pi_n$ forms 231 pattern using the largest three digits, but $\pi_1 \cdots \pi_{n-3} \in \mathcal{S}_{n-3}(231, 312, 321)$, and so the number of such permutations in case 2 but not case 1 is given by F_{n-2}.

Using the Fibonacci recurrence, we have a total of $F_{n+1} + F_{n-2} = (F_n + F_{n-1}) + F_{n-2} = F_n + (F_{n-1} + F_{n-2}) = 2F_n$ 1342-optimal words in $\mathcal{R}_n$. □

Theorem 2.8. *If $n \geq 5$, then $w = \pi\pi^r \in \mathcal{R}_n$ is 1432-optimal if and only if π has one of the following forms:*

- $23 \cdots n1$
- $23 \cdots (n-2)1(n-1)n$
- $23 \cdots (n-1)1n$
- $23 \cdots (n-2)1n(n-1)$
- $\sigma \oplus \tau$ *where* $\tau \in \{123, 132, 312, 321\}$ *and* σ *is any of the* $n-3$ *permutations that reduces to* I_{n-4} *when the digit* 1 *is removed.*

Proof. One can check by brute force that if π has one of the forms in the theorem statement, then $\pi\pi^r$ has a 1432 pattern using any selection of four digits. Therefore each of these permutations forms the first half of a 1432-optimal reverse double list.

Now, suppose that $w = \pi\pi^r$ is ρ-optimal. First consider the digits $n-2$, $n-1$ and n. In a copy of 1432 that uses all three of these digits, these three letters must play the role of 4, 3, and 2; however, if $n-1$ appears first these three digits form a subword order-isomorphic to 213312 or 231132 which has no 321 pattern. Therefore, of the digits $n-2$, $n-1$, and n, either the digit $n-2$ or the digit n appears first in π.

Now, consider the placement of 1 relative to $n-2$, $n-1$, and n in π.

If 1 appears after n, then any copy of 1432 in w involving both 1 and n must use the copy of n from π^r, and so all other digits must appear in decreasing order after n in π^r, which means they appear in increasing order before n in π. This results in the situation where $\pi = 23\cdots n1$.

If 1 appears before n but after $n-2$, then the digits $n-2$, $n-1$, and n may be in the order $(n-2)(n-1)n$ or $(n-2)n(n-1)$. There are three possible arrangements of the four digits 1, $n-2$, $n-1$, and n, namely $(n-2)1(n-1)n$, $(n-2)(n-1)1n$, and $(n-2)1n(n-1)$. In all three arrangements, any copy of 1432 that uses 1 as its smallest digit and $n-2$ as its largest digit must take $n-2$ from π^r. This implies that the digits $2, 3, \ldots, n-3$ appear in decreasing order after $n-2$ in π^r, which means they must appear in increasing order before $n-2$ in π. This results in $\pi = 23\cdots(n-2)1(n-1)n$, $\pi = 23\cdots(n-1)1n$, or $\pi = 23\cdots(n-2)1n(n-1)$.

If 1 appears before both n and $n-2$, again, let $b = n-2$ if $n-2$ appears before n in π, and let $b = n$ if n appears before $n-2$ in π. Consider the subword formed by the digits 1, $n-2$, $n-1$, and n. Any copy of 1432 using 1 as the smallest digit and b as the largest digit must use the copy of b in π^r which implies all smaller digits appear in decreasing order after b in π^r, or equivalently they appear in increasing order before b in π^r. This implies that $\pi = \sigma \oplus \tau$ where $\tau \in \{123, 132, 312, 321\}$ and σ is any of the $n-3$ permutations that reduces to I_{n-4} when the digit 1 is removed. $\square$

Corollary 2.4. *For $n \geq 5$, there are $4(n-2)$ 1432-optimal members of $\mathcal{R}_n$.*

Proof. When $w = \pi\pi^r$ is 1432-optimal, we have 4 possible permutations where 1 appears among the last three digits. Otherwise, the location of 1 uniquely determines the first $n-3$ digits of the permutation, and there are four options for the order of the last three digits.

This yields a total of $4 + 4(n-3) = 4(n-2)$ 1432-optimal members of $\mathcal{R}_n$. $\square$

2.2 Non-maximal patterns of length 4

Recall that there are four non-maximal patterns of length 4. They are 2143, 1423, 1324, and 2413. In this section we completely characterize 2143-optimal members of $\mathcal{R}_n$ and give a lower bound on $d_{\mathcal{R}_n}(1423)$.

In Theorem 2.9, we will determine the maximum number of copies of 2143 in a reverse double list, but first we we need an additional definition. Given a permutation $\pi' \in \mathcal{S}_{n-1}$ define $\mathrm{ins}_i(\pi')$ as the permutation π such that

$$\pi_j = \begin{cases} i & j = 1, \\ \pi'_{j-1} & j > 1 \text{ and } \pi'_{j-1} < i, \\ \pi'_{j-1} + 1 & j > 1 \text{ and } \pi'_{j-1} \geq i. \end{cases}$$

In other words, $\mathrm{ins}_i(\pi')$ inserts the number i at the beginning of π', incrementing numbers larger than i accordingly.

Consider the word $w^{\diamond}(n) = \pi(n)\pi(n)^r \in \mathcal{R}_n$ defined recursively as follows: $\pi(1) = 1$. Otherwise, if n is even, $\pi(n) = \mathrm{ins}_{\frac{n}{2}}(\pi(n-1))$ and if n is odd, $\pi(n) = \mathrm{ins}_{\frac{n+1}{2}}(\pi(n-1))$

The graph of $w^{\diamond}(11)$ is given in Figure 2. In general this construction results in $\pi(n)$ being a permutation of length n that alternates between an increasing sequence formed by the largest digits and a decreasing sequence formed by the smallest digits, so the word $w^{\diamond}(n)$ has a diamond shape.

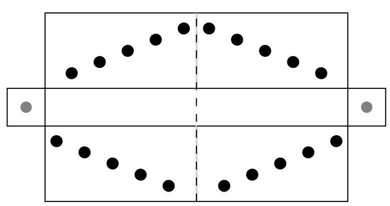


Figure 2: A 2143-optimal member of $\mathcal{R}_n$

Theorem 2.9. *For $n \geq 4$,*

$$\mu_{\mathcal{R}_n}(2143) = \begin{cases} \dfrac{(3n^2 - 8n - 4)(n-2)n}{48} & n \text{ even}, \\ \dfrac{(3n-5)(n-3)(n^2-1)}{48} & n \text{ odd}, \end{cases}$$

and $w^{\diamond}(n) \in \mathcal{R}_n$ is one 2143-optimal word that achieves this number of copies.

Proof. We claim that for $n \geq 4$, $w^\diamond(n)$ is 2143-optimal among words in $\mathcal{R}_n$.

We can check computationally that this is true for $n = 4$ and $n = 5$. $w^\diamond(4) = 23144132$ has two copies of 2143, as desired, and $w^\diamond(5) = 3241551423$ has ten copies of 2143. These are indeed 2143-optimal words since every collection of four digits produces two copies of 2143.

We proceed by induction, assuming that $w^\diamond(n-1)$ is 2143-optimal and showing that $w^\diamond(n)$ is also 2143-optimal.

Now consider a 2143-optimal word $w = \pi\pi^r$ by focusing on its first digit. There are two cases: either the first digit is involved in the 2143 pattern or it is not.

We first seek to maximize the number of copies of 2143 using the first digit of w. Notice that this digit can only appear in a 2143 pattern as the first (or last) digit of the pattern, so it can only play the role of 2 or 3.

In $w^\diamond(n)$ we see that the first digit is involved in a 2143 pattern using *every* combination of letters where it is not the smallest or the largest (take the smaller two digits from $\pi(n)$ and the larger two from $\pi(n)^r$). We further know that the number of copies of 2143 not involving the first digit is maximized because these digits form $w^\diamond(n-1)$.

It remains to show that the first digit has been chosen so that the number of size 4 subsets of $\{1, 2, \ldots, n\}$ using the first digit as it smallest or largest has been minimized. If the first digit of w is a, there are $\binom{a-1}{3}$ size 4 subsets where a is largest and there are $\binom{n-a}{3}$ size 4 subsets where a is smallest. The quantity $\binom{a-1}{3} + \binom{n-a}{3}$ is minimized when $n = 2a - 1$, or $a = \frac{n+1}{2}$, which is exactly what we have chosen our first digit to be in the case when n is odd. When n is even, a is chosen to be the nearest integer to $\frac{n+1}{2}$.

Now that we know $w^\diamond(n)$ is 2143-optimal, we consider the number of copies of 2143 in $w^\diamond(n)$. There are $\mu_{\mathcal{R}_{n-1}}(2143)$ copies of 2143 not involving the first digit of the word. While there are $\binom{n-1}{3}$ possible collections of four digits including the first digit of the word, we have seen that when n is even, $\binom{\frac{n}{2}-1}{3} + \binom{n-\frac{n}{2}}{3}$ of them do not produce a 2143 pattern. Therefore $\binom{n-1}{3} - \left(\binom{\frac{n}{2}-1}{3} + \binom{n-\frac{n}{2}}{3}\right)$ collections do, and each of them produce two copies, which simplifies to $\frac{n(n-2)(n-3)}{4}$ copies of 2143 involving $w^\diamond(n)_1$. When n is odd, $\binom{\frac{n+1}{2}-1}{3} + \binom{n-\frac{n+1}{2}}{3}$ collections of four digits involving the first digit do not produce a 2143 pattern. Therefore $\binom{n-1}{3} - \left(\binom{\frac{n+1}{2}-1}{3} + \binom{n-\frac{n+1}{2}}{3}\right)$ collections do, and each of them produce two copies, which simplifies to $\frac{(n-1)^2(n-3)}{4}$ copies of 2143 involving $w^\diamond(n)_1$.

Together, we have:

$$\mu_{\mathcal{R}_n}(2143) = \begin{cases} 2\binom{n}{4} & n \le 5, \\ \mu_{\mathcal{R}_{n-1}}(2143) + \frac{n(n-2)(n-3)}{4} & n \text{ even and } n > 5, \\ \mu_{\mathcal{R}_{n-1}}(2143) + \frac{(n-1)^2(n-3)}{4} & n \text{ odd and } n > 5. \end{cases}$$

It can be verified algebraically that the quasi-polynomial in the theorem statement uniquely satisfies this recurrence.

□

Corollary 2.5. *For $n \ge 4$, there are $2^{\left(\lfloor \frac{n}{2} \rfloor + 1\right)}$ 2143-optimal members of $\mathcal{R}_n$.*

Proof. We proceed by induction.

For $n = 4$, by brute force, there are $2^3 = 8$ reverse double lists with two copies of 2143. They are 21344312, 21433412, 23144132, 23411432, 32144123, 32411423, 34122143, and 34211243.

Further, we will say a reverse double list $w = \pi\pi^r$ on n letters has the *diamond property* if its smallest $\lceil \frac{n}{2} \rceil$ digits appear in decreasing order in π and its largest $\lceil \frac{n}{2} \rceil$ digits appear in increasing order in π. By Theorem 2.4, if $\tau xyyx\tau^r$ is ρ-optimal, so is $\tau yxxy\tau^r$. If $\tau xyyx\tau^r$ has the diamond property but $\tau yxxy\tau^r$ does not, we will say $\tau yxxy\tau^r$ has the *near-diamond property.* Notice that each of the 2143-optimal words on four letters either has the diamond property or the near-diamond property.

Now suppose there are $2^{\left(\lfloor \frac{n}{2} \rfloor + 1\right)}$ 2143-optimal members of $\mathcal{R}_n$. Suppose further that each of these words has the diamond property or the near-diamond property. We are ready to consider the 2143-optimal members of $\mathcal{R}_{n+1}$.

Notice that any copy of 2143 in $w = \pi\pi^r \in \mathcal{R}_{n+1}$ either uses the digit π_1 or it does not. The maximum number of copies without using the first (or last) digit is realized by adding a new first (and last) digit to a 2143-optimal word in $\mathcal{R}_n$. As we saw in the proof of Theorem 2.9, the maximum number of copies including π_1 is realized when π_1 is as close to the median of $\{1, 2, \dots, n+1\}$ as possible, where every collection of four digits including π_1 (where π_1 is not smallest or largest) contains a 2143 pattern. We are able to find 2143-optimal members of $\mathcal{R}_{n+1}$ by optimizing both of these types of copies simultaneously, and so we focus on this new first (and last) digit.

If $n+1$ is odd, then there is exactly one choice for the new first/last digit: it must be $\frac{n+2}{2}$ so that there are $\frac{n}{2}$ smaller digits and $\frac{n}{2}$ larger digits. Since we are adding a new first/last digit to a word with the diamond property or the near-diamond property, we have maximized copies of 2143 using the first/last digit. In this situation the number of 2143-optimal reverse double lists on n letters is the same as the number of 2143-optimal reverse double lists on $n+1$ letters.

On the other hand, if $n+1$ is even, then there are two choices for the first digit. It can be $\frac{n+1}{2}$ or $\frac{n+3}{2}$ so that there are $\frac{n-1}{2}$ smaller digits and $\frac{n+1}{2}$ larger digits or vice versa. Since we are adding a new first/last digit to a word with the diamond property or the near-diamond property, we have maximized copies of 2143 using the first/last digit. In this case, the number of 2143-optimal members of $\mathcal{R}_{n+1}$ is double the number of 2143-optimal members of $\mathcal{R}_n$, which matches the given enumeration.

In both cases, the 2143-optimal words $w = \pi\pi^r \in \mathcal{R}_{n+1}$ still either have the diamond property or the near-diamond property.

Since we have maximized copies of 2143 with and without the first digit simultaneously, we have constructed all possible 2143-optimal words.

□

We next consider the pattern 1423, which is much more challenging to analyze. Based on experimental data, we know that $\pi\pi^r$ where $\pi = 1 \oplus J_{n-1} \oplus 1$ is 1423-optimal for small n (i.e. $n \leq 9$), but for sufficiently large n, there are layered permutations with more than than three layers and with longer first and last layers that have more copies. We conjecture that for all n there exists a layered π such that $\pi\pi^r$ is 1423-optimal, and we conjecture further that as n increases, the number of layers required in a 1423-optimal reverse double list should also increase. In lieu of exact enumeration, we provide a construction to give a lower bound on its packing density in reverse double lists.

Theorem 2.10. *For $n \geq 4$,*

$$d_{\mathcal{R}_n}(1423) > 0.071892$$

Proof. We will count copies of 1423 in $\pi\pi^r$ for $\pi = J_b \oplus J_a \oplus J_b$ where $b = \frac{n-a}{2}$. While analyzing this construction guarantees there are at least this many copies of 1423 in some reverse double list, we do not prove that this is the optimal construction.

Notice that a copy of 1423 can be obtained by choosing the 1 and 4 from different layers in π and the 23 from the same layer of π^r. If 1, 2, and 3 all come from the initial J_b, then there are $\binom{b}{3}(a+b)$ ways to choose four digits that form a 1423 pattern. If the 1, 2, and 3 all come from the J_a layers, then there are $\binom{a}{3}b$ ways to choose the four digits that form a 1423 pattern. Similarly there are $\binom{b}{3}(a+b)+\binom{a}{3}b$ ways to choose the four digits if the 2, 3, and 4 all come from the same decreasing layer. Finally, if the 2 and 3 come from J_a while the 1 and 4 come from J_b layers, there are $\binom{a}{2}b^2$ ways to choose the digits for a total of $\binom{a}{2}b^2+2\binom{b}{3}(a+b)+2\binom{a}{3}b$ ways to choose four digits that form a 1423 pattern in $\pi\pi^r$. Each of these combinations of four digits yields two copies of 1423, since we may select the "4" from either π or π^r to get the 1423 pattern.

Using calculus (and a CAS), this number of copies is optimized when

$$a=\left(\frac{\left(6i\sqrt{23}-37\right)^{2/3}-\left(6i\sqrt{23}-37\right)^{1/3}+13}{6(6i\sqrt{23}-37)^{1/3}}\right)n\approx 0.647209n$$

and yields a packing density of ≈ 0.0718921066 after plugging a into the exact count of 1423 patterns in the previous paragraph, dividing by $\binom{2n}{4}$ and taking the limit as n approaches infinity. $\square$

We have now considered the packing densities of six of the eight patterns in Table 1. The cases of 1324 and 2413 are more complicated, and it remains open to find a construction that illustrates a positive packing density in the limit.

3 Packing into Double Lists

In this section, we will consider packing patterns into words in $\mathcal{D}_n$.

3.1 Monotone patterns

We first consider packing $\rho=I_m$. We also calculate the packing densities of these patterns for any $m\in\mathbb{N}$.

Theorem 3.1. *The only I_m-optimal word in $\mathcal{D}_n$ is $\pi\pi=I_nI_n$. Further, $\mu_{\mathcal{D}_n}(I_m)=\binom{n}{m}(m+1)$ and $d_{\mathcal{D}_n}(I_m)=\frac{m+1}{2^m}$.*

Proof. Consider any combination of m distinct letters in $w=\pi\pi$. If they are arranged in ascending order within π, then there are $(m+1)$ possible ways for these letters to form I_m in the entire word, since there are $(m+1)$ ways to choose how many letters come from the first π. If, however, these m letters are not in ascending relative order in π, then they either form 1 or 0 instances of I_m, since, if any were possible at all, there would only be one option for which letters must come from the first π and which must come from the second π. Since the word where $\pi = I_n$ is the only word where all combinations of m letters are in increasing order in π, it must be the only I_m-optimal word, and must contain $\binom{n}{m}(m+1)$ instances of I_m.

From this characterization, it is straightforward to compute

$$d_{\mathcal{D}_n}(I_m) = \lim_{n\to\infty} \frac{\binom{n}{m}(m+1)}{\binom{2n}{m}} = \frac{m+1}{2^m}.$$

$\square$

Table 2 gives the packing densities of I_m for when $m \leq 10$. As m increases, this density approaches 0.

m	$d_{\mathcal{D}_n}(I_m)$
2	0.75
3	0.5
4	0.3125
5	0.1875
6	0.109375
7	0.0625
8	0.035156
9	0.019531
10	0.01074

Table 2: Packing densities of I_m for small values of m

3.2 Layered patterns

Next, we consider layered patterns, i.e., patterns avoiding both 231 and 312.

Theorem 3.2. *For $n \geq 3$, there exists a 132-optimal word $\pi\pi \in \mathcal{D}_n$ where π is layered.*

Proof. Consider any three letters in π. These letters can have any of the six length-three permutations as their relative order. Table 3 shows how many total instances of 132 a set of three letters will make in $\pi\pi$ based on their order in π.

Arrangement	Instances of 132
123 123	1
132 132	4
213 213	1
231 231	0
312 312	1
321 321	1

Table 3: Number of instances of 132 any given collection of three letters make in $\pi\pi$ based on their relative order in π

Based on these counts, if we want to pack 132 into the entire $\pi\pi$ word, it suffices to pack 132 into π alone if π then also avoids 231. Stromquist (unpublished) and later Barton [3] proved that there exists a layered permutation that optimally packs 132. Since layered permutations necessarily avoid 231, this fulfills our requirements. $\square$

Now that we know that a layered 132-optimal word exists, we can enumerate the number of instances of 132 in layered words and use this enumeration to compute $d_{\mathcal{D}_n}(132)$.

Theorem 3.3. $d_{\mathcal{D}_n}(132) = \dfrac{3\sqrt{3}-4}{4}$.

Proof. The 132-optimal permutation π of length n contains $\mu_{\mathcal{S}_n}(132)$ copies of 132, each of which yields four copies of 132 in $\pi\pi$. The other $\left(\binom{n}{3} - \mu_{\mathcal{S}_n}(132)\right)$ subsequences of length 3 in π are each of the form 123, 213, or 321 and each yields one copy of 132 in $\pi\pi$. Therefore, we have

$$\mu_{\mathcal{D}_n}(132) = 4\mu_{\mathcal{S}_n}(132) + \left(\binom{n}{3} - \mu_{\mathcal{S}_n}(132)\right).$$

Dividing both sides by $\binom{2n}{3}$ and taking the limit as n approaches infinity yields:

$$d_{\mathcal{D}_n}(132) = \lim_{n\to\infty} \frac{\binom{n}{3} + 3\mu_{\mathcal{S}_n}(132)}{\binom{2n}{3}}.$$

On the right, we divide both the numerator and denominator by $\binom{n}{3}$ to obtain:

$$d_{\mathcal{D}_n}(132) = \lim_{n\to\infty} \frac{1 + 3d_{\mathcal{S}_n}(132)}{\binom{2n}{3}/\binom{n}{3}} = \frac{1+3(2\sqrt{3}-3)}{8} = \frac{3\sqrt{3}-4}{4} \approx 0.299.$$

□

In Table 3 we listed all the possible arrangements of three distinct letters in a double list to show that 132 had a layered optimizer. We now present a similar table, with all the length 4 layered pattern classes. For conciseness, we have only included layered arrangements of the four digits rather than all arrangements.

Arrangement	1432	2143	1243	1324
1234 1234	0	0	1	1
1243 1243	1	1	5	1
1324 1324	1	0	1	5
1432 1432	5	1	1	1
2134 2134	0	1	0	1
2143 2143	1	5	1	1
3214 3214	1	1	0	1
4321 4321	1	1	0	0

Table 4: Number of copies of length 4 layered patterns in $\pi\pi$ based on the relative order of four digits in π

Table 4 allows us to compute one more density with ease by building on previous results. The 2143-optimal permutation given in [12] is of the form $J_{n/2} \oplus J_{n/2}$ for even n and with one layer larger than the other by a single letter when n is odd. This 2143-optimal permutation is layered and it avoids 1234 and 1324, and so using the same methodology as in Theorem 3.3 we have the following:

Theorem 3.4. $d_{\mathcal{D}_n}(2143) = \frac{5}{32}$.

Proof. The 2143-optimal permutation of length n contains $\mu_{\mathcal{S}_n}(2143)$ copies of 2143, each of which yields five copies of 2143 in $\pi\pi$. All other subsequences of length 4 in π are of the form 1432, 3214, or 4321 and each yields one copy of 2143 in $\pi\pi$. Therefore, we have

$$\mu_{\mathcal{D}_n}(2143) = 5\mu_{\mathcal{S}_n}(2143) + \left(\binom{n}{4} - \mu_{\mathcal{S}_n}(2143)\right).$$

Dividing both sides by $\binom{2n}{4}$ and taking the limit as n approaches infinity yields:

$$d_{\mathcal{D}_n}(2143) = \lim_{n\to\infty} \frac{\binom{n}{4} + 4\mu_{\mathcal{S}_n}(132)}{\binom{2n}{4}}.$$

On the right, we divide both the numerator and denominator by $\binom{n}{3}$ to obtain:

$$d_{\mathcal{D}_n}(2143) = \lim_{n\to\infty} \frac{1 + 4d_{\mathcal{S}_n}(2143)}{\binom{2n}{4}/\binom{n}{4}} = \frac{1 + 4\left(\frac{3}{8}\right)}{16} = \frac{5}{32} = 0.15625.$$

□

Unfortunately this strategy does not work directly for other layered patterns ρ. Note that the optimal permutation for packing 1432, given in [12] contains both 1234 and 2134 for sufficiently large n. Similarly the 1243-optimal permutation given in [1] contains 4321 for sufficiently large n and the 1324-optimal permutation is known to be layered, but the exact packing density in permutations remains open.

However, we can use previous work to give a bound on one more packing density. In [1] the 1243-optimal permutation of length n is shown to be $I_{n/2} \oplus J_{n/2}$ when n is even and where the lengths of the increasing and decreasing subpermutations differ by one letter if n is odd. We use this permutation to give a lower bound on $d_{\mathcal{D}_n}(1243)$.

Theorem 3.5. $d_{\mathcal{D}_n}(1243) \geq \frac{576\sqrt{3}}{12167} + \frac{13751}{194672}$.

Proof. Consider the word $\pi\pi$ where $\pi = I_a \oplus J_{n-a}$. We wish to count copies of 1243 in this word.

Any choice of two digits from the smallest a digits and two digits from the largest $n-a$ digits produces five copies of 1243 in $\pi\pi$, while any choice of three digits from the smallest a digits and one digit from the largest $n-a$ digits produces one copy of 1243 and any choice of one digit from the smallest a digits and three digits from the largest $n-a$ digits produces one copy of 1243 in $\pi\pi$. While choosing four digits from the largest $n-a$ does not produce any copies of 1243, choosing four digits from the smallest a digits produces one copy. This enumeration gives a total of

$$\binom{a}{2}\binom{n-a}{2}+\binom{a}{3}(n-a)+a\binom{n-a}{3}+\binom{a}{4}$$

copies, which (using calculus and a CAS) is optimized when $a=\left(\frac{5+4\sqrt{3}}{23}\right)n$. This computation yields a lower bound of

$$d_{\mathcal{D}_n}(1243) \geq \frac{576\sqrt{3}}{12167}+\frac{13751}{194672} \approx 0.152634.$$

□

Two interesting pieces are at work here. In the context of permutations, it is known that $d_{\mathcal{S}_n}(2143) = d_{\mathcal{S}_n}(1243) = \frac{3}{8}$. In the context of double lists, we have shown that $d_{\mathcal{D}_n}(2143) = \frac{5}{32}$ and the maximum is achieved by concatenating two copies of the 2143-optimal permutation. On the other hand, the lower bound we obtain for $d_{\mathcal{D}_n}(1243)$ has more copies of 1243 than concatenating two copies of the 1243-optimal permutation, and our lower bound is still below $\frac{5}{32}$. It still remains to determine the exact packing density of 1243 in double lists, and to determine it the packing density for other patterns ρ.

4 Future Work

The question of packing permutations into words of the form $\pi\pi^r$ and $\pi\pi$ is the natural packing analogue to the pattern avoidance work done in [2] and [7]. In the case of packing into $\pi\pi^r$ words, we note that packing a copy of ρ into $\pi\pi^r$ is equivalent to packing a copy of some shuffle of $\rho_1\cdots\rho_i$ and $(\rho_{i+1}\cdots\rho_m)^r$ into π, so these packing problems are a special case of the packing definitions for sets given in [1]. Nonetheless, even with this machinery, the sets of patterns to be packed are complicated (i.e. generally not all layered), and are challenging to study. The following problems would be interesting extensions of this paper, requiring tools and techniques beyond the scope of the work here.

1. Is $\mu_{\mathcal{R}_n}(1423)$ always attained by a layered π? If so, what can be said about the layers?

2. Determine $d_{\mathcal{R}_n}(1423)$, $d_{\mathcal{R}_n}(1324)$ and $d_{\mathcal{R}_n}(2413)$.

3. We showed that $\mu_{\mathcal{D}_n}(\rho)$ is optimized by using the permutation that optimizes $\mu_{\mathcal{S}_n}(\rho)$ concatenated with itself when $\rho = I_m$, $\rho = 132$, or $\rho = 2143$. However, we know this is not the case for $\rho = 1243$. For what permutations is it true that $\mu_{\mathcal{D}_n}(\rho)$ is optimized by concatenating two copies of a ρ-optimal permutation?

4. Is it true that if ρ is layered then the word $\mu_{\mathcal{D}_n}(\rho)$ is obtained by some word $\pi\pi$ where π is layered?

In another direction, a Gray code is a list of all members of a set where two consecutive members differ in a pre-determined small way. While the canonical use of a Gray code is to list all binary strings of a specific length so that two consecutive strings differ by only one digit, analogous ideas have been used by various authors to systematically list all permutations or all permutations with a specific property (such as avoidance of a given pattern) [5, 13, 15]. Any Gray code listing of permutations can be extended naturally to a listing of the members of $\mathcal{D}_n$ or $\mathcal{R}_n$. It is worth investigating how the techniques for systematically generating permutations could be used to further the packing results of this paper.

Acknowledgments

This research was supported by the National Science Foundation (NSF DMS-1559912). We also are grateful to an anonymous referee for helpful comments about clarity and for suggesting the possible avenue of investigation with Gray codes.

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