

Pattern Avoidance in Circular Parking Functions

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Abstract

We extend the notion of patterns in permutations to the context of parking functions arranged around a circle. We completely describe parking functions avoiding any single pattern of length 2 or length 3 as well as those avoiding any pair or triple of patterns of length 3. Our results use a combination of enumerative and algebraic techniques, resulting in both classic combinatorial formulas and new sequences.

1 Introduction

Let \mathcal{S}_n be the set of all permutations on $[n] = \{1, 2, \dots, n\}$. Given $\pi \in \mathcal{S}_n$ and $\rho \in \mathcal{S}_m$ we say that π *contains* ρ as a pattern if there exist $1 \leq i_1 < i_2 < \dots < i_m \leq n$ such that $\pi_{i_a} \leq \pi_{i_b}$ if and only if $\rho_a \leq \rho_b$. In this case we say that $\pi_{i_1} \dots \pi_{i_m}$ is *order-isomorphic* to ρ , and that $\pi_{i_1} \dots \pi_{i_m}$ is an *occurrence* of ρ in π . If π does not contain ρ , then we say that π *avoids* ρ . For example $\pi = 364521$ contains the pattern $\rho = 2341$ because the entries of $\pi_1\pi_3\pi_4\pi_5 = 3452$ have the same relative order as the entries of ρ ; this is one of two instances of 2341 in π . Simion and Schmidt [16] enumerated permutations avoiding a pair of patterns of length 3 and many further enumerative results followed (see Bóna [2] and Kitaev [8] for surveys) for patterns in permutations and more generally for patterns in words. Various authors have also considered many variations on this classic definition of permutation pattern.

One variation considered in the permutation patterns literature involves circular permutations. In a circular permutation, we consider the last number in the permutation to be adjacent to the first, and two permutations are considered the same if they differ by only a rotation. For example, 1234, 2341, 3412, and 4123 are all the same circular permutation. A circular permutation π is said to contain ρ as a pattern if there exists a rotation π' of π that contains ρ . Circular permutations avoiding permutation patterns were studied by Callan [4] and Vella [17], resulting in a number of interesting enumeration sequences.

In another direction, suppose that n cars need to park in n parking spaces along a one-way street. Each car has a favorite parking spot. Each car will traverse the street exactly once and it will take its favorite spot if possible; if its favorite spot is occupied, the car will take the earliest available spot after its favorite. If a car's favorite spot and all subsequent spots are occupied, the car

will leave the street without parking. A collection of preferences that results in all the cars being parked is called a parking function. As extreme examples: if all cars prefer spot 1, then we have a parking function since the i th car will end up in spot i ; however if every car prefers spot 2, we do not have a parking function since all the cars will bypass spot 1 and there are not enough remaining spots to park the cars. It is well known that a collection of preferences is a parking function if and only if for all $1 \leq i \leq n$ at least i cars prefer spot i or earlier. Further, it is known (see [9, 14]) that there are $(n+1)^{n-1}$ possible parking functions for n cars. For a comprehensive survey of results on parking functions, see [18].

There are multiple ways to represent parking functions that associate them with words and permutations. For example, Jelínek and Mansour [7] view a parking function as a word $w \in [n]^n$ where for $1 \leq i \leq n$, w has at least i letters less than or equal to i . In this representation, the three parking functions on 2 letters are represented as 11, 12, and 21. Following the permutations literature, two patterns ρ and ρ' are said to be *Wilf equivalent* if the number of ρ -avoiding parking functions of size n equals the number of ρ' -avoiding parking functions of size n for all n . For example, the *reversal* of $p = p_1 \cdots p_n$ is $p^r = p_n \cdots p_1$. We have that p avoids ρ if and only if p^r avoids ρ^r , so ρ and ρ^r are Wilf equivalent. Jelínek and Mansour apply the classical definition of pattern avoidance to parking functions viewed as words and determine all pairs of patterns ρ and ρ' that are Wilf equivalent when ρ has length at most 5.

In a different direction, Garsia and Haiman [5] represent parking functions as Dyck paths with labeled up-steps. In such a path, we may label each of the n north-steps with a distinct integer from $\{1, \dots, n\}$ such that consecutive north-steps must have their labels in increasing order. In this representation, the labels of north-steps along $x = i$ correspond to the cars who prefer spot $i + 1$. Reading the labels from left to right provides a many-to-one correspondence between parking functions and permutations. In 2023, Adeniran and Pudwell [1] considered this representation and counted parking functions whose associated permutation avoids any set of 2 or more patterns of length 3. This extends work of Qiu and Remmel [13] who enumerated 123-avoiding parking functions in this context.

In this paper we seek to apply the idea of patterns in circular permutations to the context of parking functions. Given a parking function $p = p_1 \cdots p_n$, we say that p avoids pattern ρ if all rotations of p (i.e., $p_1 \cdots p_n$, $p_2 \cdots p_n p_1$, $p_3 \cdots p_n p_1 p_2$, \dots , and $p_n p_1 \cdots p_{n-1}$) avoid ρ , and p contains ρ if some rotation of p contains ρ . We refer to the lexicographically first rotation of p as the *standardization* of p and denote the standardization as $\text{std}(p)$. In Section 2 we review key insights from enumerating parking functions and from enumerating equivalence classes of sets under group actions that will be relevant throughout the paper. In Section 3 we use these tools to enumerate all circular parking functions, and in Sections 4 and 5 we enumerate circular parking functions avoiding one or more patterns. In Section 6 we conclude with questions for future investigation.

2 Counting techniques

Before we address pattern-avoiding parking functions, we review known techniques that will be relevant in the rest of the paper.

Theorem 1. *The number of parking functions of length n is given by $(n+1)^{n-1}$.*

While Konheim and Weiss [9] derive this enumeration, Riordan [14] gives a particularly elegant argument for Theorem 1, which he credits to Pollack. This latter argument is given below.

Proof. Instead of parking n cars in n spots, imagine that we wish to park n cars in $n+1$ spots, which are arranged around a circle. If car i prefers spot j and spots $j, \dots, n+1$ are all occupied, car i proceeds back to the beginning of the circle. Clearly, there are $(n+1)^n$ possible preference lists. Since there are more spots than cars, eventually every car will park leaving one spot unoccupied. Since all spots are equivalent in this setting, by symmetry exactly $\frac{1}{n+1}$ of the preference lists leave spot $n+1$ unoccupied and thus exactly $\frac{1}{n+1}(n+1)^n = (n+1)^{n-1}$ of the preference lists are parking functions. \square

Throughout this paper, the strategy of considering $n+1$ spots arranged around a circle and then using symmetry to specialize to parking functions will be useful.

In addition, we will make use of more algebraic techniques. In particular, Burnside's Lemma [3] enumerates orbits of a set under a group action. We quote this result using the notation of Chapter 8 of Roberts and Tesman [15].

Theorem 2 (Burnside's Lemma). *Let G be a group of permutations of a set A , and let S be the equivalence relation on A induced by G . Then the number of equivalence classes in S is given by*

$$\frac{1}{|G|} \sum_{\pi \in G} \text{Inv}(\pi),$$

where $\text{Inv}(\pi)$ is the number of elements of A that are invariant under π .

Burnside's Lemma considers any group acting on any set in general, but in our work, we are interested in a particular group and a particular set. To that end, we adapt Theorem 2 to a special case of Pólya's Theorem [11], also phrased in the notation of Roberts and Tesman [15].

Theorem 3 (Pólya's Theorem (special case)). *Let G be a group of permutations of the set D , and let $C(D, R)$ be the set of colorings of elements of D using colors in R , a set of m elements. Then the number of distinct colorings in $C(D, R)$ is given by*

$$\frac{1}{|G|} \sum_{\pi \in G} m^{\text{cyc}(\pi)},$$

where $\text{cyc}(\pi)$ is the number of cycles of π .

Effectively, Theorem 3 makes use of the fact that all elements in the same cycle of permutation π must receive the same color in order for a coloring to be invariant under π .

A canonical example of applying Theorem 3 is to count distinct n -bead necklaces with m colors of beads. If we were to count 4-bead necklaces, considering two necklaces are equivalent if one can be rotated to the other, we have that G is the cyclic group on 4 elements, i.e.,

$$G = C_4 = \{(1)(2)(3)(4), (1234), (13)(24), (1432)\},$$

and thus the number of necklaces is $\frac{1}{4}(m^4 + m^2 + 2m)$. This matches that there is $\frac{1}{4}(1^4 + 1^2 + 2 \cdot 1) = 1$ necklace with $m = 1$ color of bead available, $\frac{1}{4}(2^4 + 2^2 + 2 \cdot 2) = 6$ necklaces with $m = 2$ colors of beads available (two are monochromatic, two have two beads of each color, two have three of one color), and so on.

In the context of circular parking functions, $A = C(D, R)$ is a set of parking preference lists where D represents the positions in the list and R denotes the set of possible parking preferences one car may have. The equivalence relation S is that two parking preference lists are equivalent if one can be obtained from the other via circular rotation, and thus the group of interest is $G = C_n$, the cyclic group on n elements. Since $G = C_n$, we have $|G| = n$. Furthermore, let π_i be the unique permutation in C_n that sends 1 to $i + 1$ where $0 \leq i \leq n - 1$. For all i , we find that $\text{cyc}(\pi_i) = \gcd(i, n)$. This means that if $\text{cyc}(\pi_i) = c$ then $c \mid n$, and the number of permutations for which $\text{cyc}(\pi_i) = c$ is $\phi(\frac{n}{c})$ where ϕ is Euler's totient function. Letting $d = \frac{n}{c}$ means that when $G = C_n$ the quantity in Theorem 3 is $\frac{1}{n} \sum_{d \mid n} \phi(d) (m)^{n/d}$.

Throughout this paper, we require a combination of techniques from Theorems 1, 2, and 3 and their proofs. In particular, the symmetry argument of Theorem 1 will help us ensure we are considering parking functions in particular, rather than general words in $[n]^n$, and Theorems 2 and 3 will account for the symmetry of rotation instead of working with linear parking functions. We are now ready to start counting.

3 Enumerating all circular parking functions

The enumerative context for circular permutations is straightforward. Since permutations have distinct entries and there are $n!$ permutations of length n , there are $\frac{n!}{n} = (n - 1)!$ circular permutations of size n . Similarly, there are $m!$ patterns of length m one can consider in the context of linear permutations while there are $\frac{m!}{m} = (m - 1)!$ distinct patterns of length m after circular rotation.

The general situation for parking functions is made more complicated by the fact that parking functions may have multiple copies of the same entry. Let cpf_n denote the number of circular parking functions of length n .

Theorem 4. For $n \geq 1$,

$$\text{cpf}_n = \frac{1}{n(n+1)} \sum_{d|n} \phi(d) (n+1)^{n/d}.$$

Proof. We first consider n -bead necklaces equivalent under cyclic rotation with $n+1$ colors available. Using $G = C_n$ and $m = n+1$ in Theorem 3 we have $\frac{1}{n} \sum_{d|n} \phi(d) (n+1)^{n/d}$ such necklaces.

Given a necklace with colors chosen from $1, \dots, n+1$, we may repeatedly add 1 to each color (and rename $n+2$ as 1) to obtain a family of $n+1$ distinct necklaces because the smallest absent bead color is different for each. Reading the beads of each of these necklaces as a parking preference list, we notice that exactly one of the $n+1$ distinct necklaces in this family will be a parking function, following the same reasoning as in the proof of Theorem 1.

Since we have produced $n+1$ times as many necklaces as there are circular parking functions, we divide by $n+1$ to get that there are

$$\frac{1}{n(n+1)} \sum_{d|n} \phi(d) (n+1)^{n/d}.$$

circular parking functions of size n . □

The quantity in Theorem 4 is OEIS [10] sequence A121774 and the necklace interpretation was first given by Grinberg and Mao [6].

Next, we consider how many circular parking function *patterns* exist. For example, while there are 6 circular parking functions of length 3 (namely 111, 112, 122, 113, 123, and 132), two of them (i.e. 112 and 113) are order-isomorphic to one another. To that end, we call a circular parking function a *circular pattern* if the actual entries of the parking function form a consecutive set of positive integers and we let cp_n be the number of such patterns.

Theorem 5. For $n \geq 1$,

$$\text{cp}_n = \sum_{k=1}^n \frac{k!}{n} \sum_{d|n} \phi(d) S(n/d, k),$$

where $S(n/d, k)$ is a Stirling number of the second kind.

Proof. First, if we use all colors from $1, \dots, k$ as parking preferences, we are guaranteed to have a parking function since, if j cars have repeat preferences, there will be j empty spots at the end of the street that are not preferred by any car, where these cars will end up.

Considering the group $G = C_n$, we know that the number of $\pi \in C_n$ for which $\text{cyc}(\pi) = \frac{n}{d}$ is $\phi(d)$. Thinking of parking preference lists as n spots being colored with colors from $\{1, 2, \dots, n\}$, a coloring will be invariant under π if every element in the same cycle is assigned the same color. Since $S(n/d, k)$ is the number of ways to partition n/d distinct elements into k non-empty sets,

$k! \cdot S(n/d, k)$ is the number of ways to assign a color to each cycle of π where all k colors are used. Summing over all possible values of k completes the computation. \square

This enumeration is given in OEIS sequence A019536, using the necklace interpretation. For comparison, we give the total number of circular parking functions and the total number of circular patterns for small n in Table 1.

n	1	2	3	4	5	6	7	8	9	10
cpf_n	1	2	6	33	260	2812	37450	97965	11111134	235796238
cp_n	1	2	5	20	109	784	6757	68240	787477	10224812

Table 1: Number of circular parking functions and number of circular patterns for small n

Before we begin enumeration in general it is worth pointing out that reversing a pattern $\rho = \rho_1 \cdots \rho_n$ to $\rho^r = \rho_n \cdots \rho_1$ (and then standardizing) produces a Wilf equivalent pattern since if p avoids ρ , p^r avoids ρ^r . For both circular patterns of length 2, $\text{std}(\rho^r) = \rho$. For circular patterns of length 3, the only new Wilf equivalence this yields is $\text{std}(123^r) = \text{std}(321) = 132$. For longer patterns, this reversal accounts for additional Wilf equivalent pairs.

4 Avoiding one pattern

Let ρ be a circular pattern, and let $\text{cpf}_n(\rho)$ be the number of circular parking functions of size n avoiding ρ . In this section we determine $\text{cpf}_n(\rho)$ if ρ has length 2 or length 3. A summary of results is given in Table 2.

Pattern ρ	$(\text{cpf}_n(\rho))$	OEIS	Theorem
12	1, 1, 1, 1, 1, 1, ...	A000012	Theorem 6
11	1, 1, 2, 6, 24, 120, ...	A000142	Theorem 7
112	1, 2, 4, 11, 42, 207, ...	A213937	Theorem 8
123 132	1, 2, 5, 16, 47, 153, ...	A394236	Theorem 9
122	1, 2, 5, 19, 101, 676, ...	A394237	Theorem 10
111	1, 2, 5, 28, 204, 2000, ...	A394238	Theorem 11

Table 2: Enumeration of circular parking functions avoiding a single pattern of length 2 or 3

We first consider the two circular patterns of length 2 and we obtain two distinct sequences.

Theorem 6. For $n \geq 0$,

$$\text{cpf}_n(12) = 1.$$

Proof. In order to avoid the pattern 12, all entries must be identical. The only parking function with identical entries is the all 1s parking function. \square

Theorem 7. For $n \geq 1$,

$$\text{cpf}_n(11) = (n-1)!.$$

Proof. In order to avoid the pattern 11, all entries must be distinct. Any of the $(n-1)!$ circular permutations of length n is such a circular parking function. \square

Next, we consider the five circular patterns of length 3. As we observed in Section 3, 123 is Wilf equivalent to 132 via reversal. This is the only Wilf equivalence among these patterns, though, and we obtain four distinct sequences, three of which are new to the OEIS.

Theorem 8. For $n \geq 1$,

$$\text{cpf}_n(112) = 1 + \sum_{k=2}^n \frac{(n-1)!}{(n-k+1)!}.$$

Proof. Suppose the largest entry that appears in a 112-avoiding circular parking function is k . In order to avoid the pattern 112, only the entry k may be repeated, and we have at most 1 of each of the entries $1, 2, \dots, k-1$. However, if we have no copies of some entry from $1, 2, \dots, k-1$, then we must have at least $n+2-k$ copies of k , which violates the definition of parking function. Therefore, a 112-avoiding circular parking function has exactly one copy of each of $1, 2, \dots, k-1$ and $n+1-k$ copies of k .

If $k=1$ there is one parking function, i.e., the all 1s parking function.

If $k > 1$, we standardize by putting 1 in the first position. There are $n-1$ choices for where to place the 2, $n-2$ choices for where to place the 3, and so on, up until there are $n-(k-2)$ choices for where to place the $k-1$. All remaining positions will be filled with k .

Using the identity that $(n-1) \cdots (n-(k-2)) = \frac{(n-1)!}{(n-(k-1))!}$ completes the proof. \square

This enumeration is given in OEIS A213937 with an interpretation in terms of necklaces.

Theorem 9. For $n \geq 1$,

$$\text{cpf}_n(123) = \text{cpf}_n(132) = \left(\frac{1}{2} \sum_{i=1}^n 2^{\text{gcd}(i,n)} \right) - \binom{n+1}{2} + \frac{2 \binom{2n-1}{n-1}}{n+1}.$$

Proof. If we avoid 123, then we have either (a) a parking function consisting of at most two distinct entries, or (b) a parking function with at least three distinct entries. In case (b), the standardized form of the parking function begins with 1, and all larger entries appear in decreasing order. If there are non-consecutive 1s,

i.e., a subsequence of the form $1a1b$, then $a > b$ because of the first 1 and $b > a$ because of the second one, which is a contradiction. Thus, in case (b) we really have a standardized form of all 1s followed by a decreasing arrangement of other entries. Similarly if we avoid 132, the only change is that the entries other than 1 appear in increasing order. In other words 123 and 132 are Wilf-equivalent via reversal. We focus on avoiding 132.

For case (a), by Theorem 3, using $G = C_n$, we have already seen that there are $\frac{1}{n(n+1)} \sum_{d|n} \phi(d) (n+1)^{n/d}$ ways to make an n -bead necklace that is a parking function. However, instead of using any number from $\{1, \dots, n+1\}$ on each cycle, we are interested in limiting ourselves to one or two numbers total out of the $n+1$ options. The expression

$$\frac{1}{n(n+1)} \binom{n+1}{2} \sum_{d|n} \phi(d) 2^{n/d} = \frac{1}{n(n+1)} \binom{n+1}{2} \sum_{i=1}^n 2^{\gcd(i,n)}$$

is the number of ways to choose two members of $\{1, \dots, n+1\}$ and then assign one of those two numbers to each cycle of a given permutation $\pi \in C_n$. This process will perfectly count each of the parking functions with only two distinct entries. However, it overcounts the parking function where all entries are the same. In particular, this parking function is counted twice in each summation term for a total of $2n$ times in the sum. Including the $\frac{\binom{n+1}{2}}{n(n+1)}$ outside the summation, the one all 1s parking function is counted $\frac{2n \binom{n+1}{2}}{n(n+1)} = n$ times, when we only wish to count it once. Therefore, there are

$$\left(\frac{1}{n(n+1)} \binom{n+1}{2} \sum_{i=1}^n 2^{\gcd(i,n)} \right) - (n-1) = \left(\frac{1}{2} \sum_{i=1}^n 2^{\gcd(i,n)} \right) - (n-1)$$

parking functions with at most two distinct entries.

For case (b), we consider necklaces with at least 3 colors whose entries appear in increasing order. Using a stars and bars argument, we see there are $\binom{2n}{n}$ ways to have n beads with colors chosen from $\{1, \dots, n+1\}$. However, $n+1$ of these ways correspond to all beads being the same color, and $\binom{n+1}{2}(n-1)$ of these ways correspond to having two colors (by choosing which two colors from $\{1, \dots, n+1\}$ and then choosing how many of the n are of the first color), both of which were addressed in case (a). Exactly $\frac{1}{n+1}$ of these $\binom{2n}{n} - (n+1) - \binom{n+1}{2}(n-1)$ necklaces are parking functions.

Together we have

$$\left(\frac{1}{2} \sum_{i=1}^n 2^{\gcd(i,n)} \right) - (n-1) + \frac{\left(\binom{2n}{n} - (n+1) - \binom{n+1}{2}(n-1) \right)}{n+1}$$

circular parking functions avoiding 132.

Or, after algebraic simplification, we obtain

$$\text{cpf}_n(132) = \left(\frac{1}{2} \sum_{i=1}^n 2^{\gcd(i,n)} \right) - \binom{n+1}{2} + \frac{2 \binom{2n-1}{n-1}}{n+1}.$$

□

While this sequence is new to the OEIS, it has several interesting components that have shown up in other contexts. For example, the total number of two-color necklaces with colors chosen from $\{1, \dots, n+1\}$ divided by $n+1$ is given by OEIS A034738. This quantity is $n-1$ more than the total number of one-color and two-color circular parking functions of length n and appeared previously in the study of circular compositions. Alternately, the summation $\sum_{i=1}^n 2^{\gcd(i,n)}$, which is double sequence A034738, is given by A053635 and has appeared in the enumeration of polygons with integer side lengths and a prescribed perimeter. Similarly, the number of necklaces with (any number of) increasing colors is given by $\frac{\binom{2n}{n}}{n+1}$, which is the n th Catalan number, A000108.

Theorem 10. For $n \geq 1$,

$$\text{cpf}_n(122) = 1 + \sum_{k=1}^{n-1} \frac{\binom{n-1}{k}^2 \cdot k!}{(n-k)}.$$

Proof. Consider a 122-avoiding circular parking function. By definition the number 1 must be an entry in the parking function. While we may have arbitrarily many 1s, we may have at most one copy of any larger entry.

There is one way to have only 1s.

If there are larger entries, choose k entries from $\{2, 3, \dots, n\}$ in $\binom{n-1}{k}$ ways. Then, select the locations of these entries in $n(n-1)(n-2) \cdots (n-(k-1))$ ways. All remaining positions must be 1s. Using the identity that $n \cdots (n-k+1) = \frac{n!}{(n-k)!}$ gives $\frac{\binom{n-1}{k} n!}{(n-k)!}$ ways that we can arrange $(n-k)$ 1s together with k other entries before standardizing the circular parking function. After accounting for n rotations, there are $\frac{\binom{n-1}{k} n!}{(n-k)! n} = \frac{\binom{n-1}{k}^2 k!}{(n-k)}$ ways to arrange $(n-k)$ 1s together with k distinct other entries. □

While this sequence is new to the literature, if we ignore the all 1s parking function, we get OEIS A350267, which is described computationally, but not combinatorially. This, then, appears to be the first combinatorial interpretation of the sequence.

Before we consider the final pattern of length 3, we recall a connection between parking functions and trees that was known to Riordan [14]. There is a famous bijection of Prüfer [12] between labeled trees on n nodes and words in $[n]^{n-2}$. Given a labeled tree, identify the smallest numbered leaf, record its parent, and delete the leaf. Repeat this until only two vertices remain and a sequence of length $n-2$ has been recorded. To reverse the process, we record a degree for vertex i of the number of times it appears in the Prüfer sequence plus 1. Now, reading across Prüfer sequence $a_1 \cdots a_{n-2}$, when we arrive at a_j , we find the smallest numbered vertex i with a degree of 1, add edge between

i and a_j and reduce the degree of i and a_j each by 1. Finally, after all $n - 2$ terms of the sequence have been read, we will have $n - 2$ edges among n vertices with two vertices with a recorded degree of 1 remaining. Add an edge between these two vertices to complete the tree. Riordan points out that given a parking function $p = p_1 \cdots p_n$, the sequence $q_1 \cdots q_{n-1}$ where $q_i = p_{i+1} - p_i \pmod{(n+1)}$ is a Prüfer sequence for a tree with $n + 1$ nodes.

In our analysis of $\text{cpf}_n(111)$ we make use of a modification of Prüfer codes to study more restricted trees. In particular, define t_n to be the number of n -vertex labeled rooted unordered trees where each vertex has at most 2 children. Let $T(x) = \sum_{k=1}^{\infty} t_n \frac{x^n}{n!}$ be the exponential generating function for t_n . We have that $T(x) = x + xT(x) + \frac{x}{2}T(x)^2$, accounting for the root having 0, 1, or 2 children respectively. The division by 2 in the final case accounts for the trees being unordered. Solving for $T(x)$ gives

$$T(x) = \frac{1 - x - \sqrt{1 - 2x - x^2}}{x} = \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{9}{3!}x^3 + \frac{60}{4!}x^4 + \frac{540}{5!}x^5 \cdots,$$

corresponding to $t_1 = 1$, $t_2 = 2$, $t_3 = 9$, $t_4 = 60$, $t_5 = 540$, and so on as given in OEIS A036774. We associate each such tree with a sequence of length $n - 1$, where we identify the smallest leaf, record its parent, delete the leaf, and repeat. Instead of stopping when two nodes remain, we ultimately record the root vertex as the final entry in our sequence. Note that this process produces a sequence of length $n - 1$, with entries chosen from $\{1, 2, \dots, n\}$. Vertices that are leaves appear 0 times in the sequence; other vertices appear either 1 or 2 times, corresponding to their number of children. With the definition of t_n and this correspondence in mind, we are now ready to address $\text{cpf}_n(111)$.

Theorem 11. For $n \geq 1$,

$$\text{cpf}_n(111) = \begin{cases} \frac{1}{n(n+1)}t_{n+1} & n \text{ is odd,} \\ \frac{1}{n(n+1)} \left(t_{n+1} + \frac{(n+1)!}{(\frac{n}{2}+1)!} \right) & n \text{ is even.} \end{cases}$$

Proof. Avoiding 111 means we have a parking function with at most 2 copies of each entry. In terms of necklaces, we seek necklaces with at most two copies of each bead color. Applying Theorem 2, we consider $\frac{1}{n} \sum_{\pi \in C_n} \text{Inv}(\pi)$, so we are concerned with parking functions invariant under π where $\pi \in C_n$. Entries in the same cycle of π must receive the same color to be invariant under π , so we are not interested in permutations with cycles of size 3 or larger.

If n is odd, there is exactly one relevant permutation in C_n , i.e., the identity permutation $(1)(2) \cdots (n)$. We only need to count necklaces with at most two copies of each entry, with entries chosen from $\{1, \dots, n + 1\}$ and then as in the proof of Theorem 1, divide by $n + 1$ to address that $\frac{1}{n+1}$ of these necklaces are actually parking functions. We claim the number of necklaces is t_{n+1} , i.e., the number of labeled rooted unordered trees with $n + 1$ nodes where each vertex has at most 2 children. Indeed, using the modified Prüfer correspondence above, each $(n + 1)$ -node tree corresponds to a distinct sequence of length n with entries chosen from $\{1, 2, \dots, n + 1\}$ with at most 2 copies of each entry, as desired.

If n is even, there are two relevant permutations in C_n , i.e., $(1)(2)\cdots(n)$ and $(1, \frac{n}{2} + 1)(2, \frac{n}{2} + 2)\cdots(\frac{n}{2}, n)$. In addition to the necklaces counted in the odd case, we also count necklaces left invariant by half rotation. There are $n + 1$ choices for the color of the first cycle, n choices for the color of the second cycle, and so on, until there are $\frac{n}{2} + 2$ choices for the color of the final cycle. In this case, knowing the first half of the parking function, encoded as a necklace, tells the second half and there are exactly two copies of each entry that appears. \square

5 Avoiding multiple patterns

Next, we consider the enumeration of circular parking functions avoiding multiple patterns of length 3 simultaneously. Let $\text{cpf}_n(\rho, \sigma)$ be the number of circular parking functions of size n avoiding both ρ and σ simultaneously. In this section we first determine $\text{cpf}_n(\rho, \sigma)$ if ρ and σ are both of length 3. A summary of results for all $\binom{5}{2} = 10$ pairs of patterns is given in Table 3. Notice that the only Wilf equivalent pairs follow from the reversal symmetry, analyzed earlier. Similarly, let $\text{cpf}_n(\rho, \sigma, \tau)$ be the number of circular parking functions of size n avoiding all of ρ, σ , and τ simultaneously. A summary of results for all $\binom{5}{3} = 10$ triples of patterns of length 3 is given in Table 4.

Patterns ρ, σ	$(\text{cpf}_n(\rho, \sigma))$	OEIS	Theorem
112, 123 112, 132	1, 2, 3, 4, 5, 6, ...	A000027	Theorem 12
112, 122	1, 2, 3, 7, 25, 121, ...	A038507 (for $n \geq 2$)	Theorem 13
111, 112	1, 2, 3, 9, 36, 180, ...	A070960	Theorem 14
122, 123 122, 132	1, 2, 4, 8, 16, 32, ...	A000079	Theorem 15
123, 132	1, 2, 4, 9, 16, 37, ...	A393352	Theorem 16
111, 123 111, 132	1, 2, 4, 11, 21, 51, ...	A001006 (for $n \neq 4$)	Theorem 17
111, 122	1, 2, 4, 15, 72, 420, ...	A171005	Theorem 18

Table 3: Enumeration of circular parking functions avoiding two patterns of length 3

Theorem 12. For $n \geq 1$,

$$\text{cpf}_n(112, 123) = \text{cpf}_n(112, 132) = n.$$

Proof. If we avoid 112, we either have one distinct entry total (the all 1s parking function) or we have at most one copy of each entry except the largest. If we avoid 132, all entries after 1 are in increasing order. There are n circular parking functions of the form $1 \cdots \ell$ followed by $n - \ell$ copies of ℓ . Reverse the entries after the 1 to obtain the analogous circular parking functions that avoid 123. \square

Theorem 13. For $n \geq 2$,

$$\text{cpf}_n(112, 122) = 1 + (n - 1)!$$

Proof. We may either have the all 1s parking function, or we may have multiple distinct entries. If we have multiple entries, we have at most one copy of each, yielding $(n - 1)!$ circular permutations. \square

Theorem 14. For $n \geq 1$,

$$\text{cpf}_n(111, 112) = \frac{3}{2}(n - 1)!$$

Proof. We have at most 1 copy of each entry other than the largest entry, and we have two copies of the largest entry. There are $(n - 1)!$ ways to make a circular permutation out of $1, 2, \dots, n$. Similarly, there are $\frac{(n-1)!}{2}$ ways to make a circular permutation of $1, 2, \dots, n - 1, n - 1$. \square

The results of these last two theorems match OEIS A038507 and A070960 respectively, and both formulas have appeared in number theory and combinatorics contexts.

Theorem 15. For $n \geq 1$,

$$\text{cpf}_n(122, 123) = \text{cpf}_n(122, 132) = 2^{n-1}.$$

Proof. If we avoid both 122 and 123 (resp. 132), while we can have as many 1s as we like, we have at most one copy of every other entry, and the entries larger than 1 are in decreasing (resp. increasing) order. Further, all 1s appear at the beginning of the standardization since if we have a subsequence of the form $1a1b$, while $1ab$ avoids 123 using the first 1, $1ba$ contains 123, using the second 1. We can pick k non-1 entries to appear in decreasing (resp. increasing) order at the end of the standardized parking function in $\binom{n-1}{k}$ ways. Summing over all possible values of k gives

$$\sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}.$$

\square

Theorem 16. For $n \geq 1$,

$$\text{cpf}_n(123, 132) = \left(\frac{1}{2} \sum_{k=1}^n 2^{\text{gcd}(k,n)} \right) - (n - 1).$$

Proof. Parking functions avoiding 123 and 132 have just two distinct entries with no restriction on frequency other than there must be at least one 1. Using Theorem 3, we consider two-color necklaces with colors chosen from $\{1, 2, \dots, n+1\}$

that are invariant under rotation. After choosing which two colors to use in one of $\binom{n+1}{2}$ ways, the number of such necklaces is

$$\frac{1}{n} \sum_{k=1}^n 2^{\gcd(k,n)}.$$

As we have seen in the proof of Theorem 1, exactly $\frac{1}{(n+1)}$ of these necklaces are actually parking functions. However, we have overcounted. The monochromatic necklace shows up twice in each summation term for a total of $2n$ times. This means it is accounted for $\frac{2n\binom{n+1}{2}}{n(n+1)} = n$ times in

$$\frac{1}{n(n+1)} \binom{n+1}{2} \sum_{k=1}^n 2^{\gcd(k,n)},$$

so we need to subtract off the extra $n - 1$ times it is counted.

We have a total of $(\frac{1}{2} \sum_{k=1}^n 2^{\gcd(k,n)}) - (n - 1)$ parking functions. \square

This enumeration matches OEIS A393352 and is new to the literature; it also appeared in the casework of Theorem 9 above.

Theorem 17. For $n \geq 1$,

$$\text{cpf}_n(111, 123) = \text{cpf}_n(111, 132) = \begin{cases} M_n & n \neq 4, \\ M_n + 2 & n = 4, \end{cases}$$

where M_n is the n th Motzkin number.

Proof. A parking function avoiding 111 and 123 (resp. 132) has at most two copies of each entry, and all entries other than 1 are in decreasing (resp. increasing) order.

We consider two cases, focused on those avoiding 111 and 132: either the circular parking function contains n or it has no n .

If the circular parking function contains n , we may delete the n to obtain a pattern-avoiding circular parking function of size $n - 1$.

If the circular parking function has no n , let $a_1 \cdots a_n$ be the entries of the parking function which are already arranged in non-decreasing order as a consequence of avoiding 132. Identify the largest value where $a_i = i$. Since this is the rightmost position where $a_i = i$, we must have that $a_{i+1} = i$ as well. Delete both copies of i . The substring $a_1 \cdots a_{i-1}$ is a pattern-avoiding parking function of size $i - 1$. For the entries $a_{i+2} \cdots a_n$, subtract $a_{i+2} - 1$ from each entry to obtain a pattern-avoiding parking function of size $n - i - 1$.

We have

$$\text{cpf}_n(111, 132) = \text{cpf}_{n-1}(111, 132) + \sum_{i=1}^{n-1} \text{cpf}_{i-1}(111, 132) \text{cpf}_{n-i-1}(111, 132),$$

which is the recurrence for the Motzkin numbers.

The only two exceptions to this structure are 1212 and 1313 of length 4. While each of these has a position where $a_i = i$, the entries are not already in non-decreasing order, and it is not true that $a_{i+1} = i$ to continue with the decomposition. These are an artifact of having fewer than three distinct entries, and no exceptions happen with larger parking functions. \square

The Motzkin numbers, given in A001006, appear in a variety of contexts, including lattice path enumeration and pattern-avoidance in other combinatorial structures.

Theorem 18. For $n \geq 1$,

$$\text{cpf}_n(111, 122) = \frac{(n+1)(n-1)!}{2}.$$

Proof. A parking function avoiding 111 and 122 has at most 1 copy of each entry larger than 1 and at most two copies of 1.

If there is one 1, we are really considering a circular permutation on n entries, and there are $(n-1)!$ such circular permutations.

If we have two 1s, we have $(n-1)$ choices for which member of $\{1, 2, \dots, n\}$ does not appear in the parking function. We may form a circular permutation of the n entries (which include two copies of 1) in $\frac{(n-1)!}{2}$ ways.

Together, we have

$$(n-1)! + \frac{(n-1)(n-1)!}{2} = \left(1 + \frac{n-1}{2}\right) (n-1)! = \frac{(n+1)(n-1)!}{2}$$

possible circular parking functions. \square

We now consider those circular parking functions that simultaneously avoid three patterns of length 3. While the sequences involved are relatively simple, they reveal our first Wilf equivalences that do not follow from reversing patterns.

Patterns ρ, σ, τ	$(\text{cpf}_n(\rho, \sigma, \tau))$	Theorem
111, 123, 132	1, 2, 3, 4, 0, 0, ...	Theorem 19
111, 112, 123 111, 112, 132 112, 122, 123 112, 122, 132 112, 123, 132	1, 2, 2, 2, 2, 2, ...	Theorem 20
111, 122, 123 111, 122, 132 122, 123, 132	1, 2, 3, 4, 5, 6, ...	Theorem 21
111, 112, 122	1, 2, 2, 6, 24, 120, ...	Theorem 22

Table 4: Enumeration of circular parking functions avoiding three patterns of length 3

Theorem 19. For $n \geq 1$,

$$\text{cpf}_n(111, 123, 132) = \begin{cases} n & n \leq 4 \\ 0 & n \geq 5. \end{cases}$$

Proof. Suppose a circular parking function avoids all of 111, 123, and 132. It is impossible to avoid both 123 and 132 with three distinct entries. Therefore, a circular parking function avoiding these three patterns consists only of two distinct entries. Further, since it avoids 111, it has at most two copies of each entry. We have the following:

$$\text{cpf}_1(111, 123, 132) = \text{cpf}_1 = \{1\}.$$

$$\text{cpf}_2(111, 123, 132) = \text{cpf}_2 = \{11, 12\}.$$

$$\text{cpf}_3(111, 123, 132) = \text{cpf}_3 \setminus \{111, 123, 132\} = \{112, 113, 122\}.$$

$$\text{cpf}_4(111, 123, 132) = \{1122, 1133, 1212, 1313\}.$$

For $n \geq 5$, when we are restricted to two distinct entries by avoiding 123 and 132, we must have at least three copies of some entry, which would force a 111 pattern. \square

Theorem 20. For $n \geq 1$ and $\rho \in \{123, 132\}$,

$$\text{cpf}_n(111, 112, \rho) = \text{cpf}_n(112, 122, \rho) = \text{cpf}_n(112, 123, 132)$$

Furthermore,

$$\text{cpf}_n(111, 112, 123) = \begin{cases} 1 & n = 1 \\ 2 & n \geq 2. \end{cases}$$

Proof. If a parking function avoids both 111 and 112, and k is the largest entry, then we have at most two copies of k and at most one copy of all other entries. If we avoid 123, all entries after 1 must appear in decreasing order, so the two circular parking functions are $1n(n-1)\cdots 2$ and $1(n-1)(n-1)(n-2)\cdots 2$. If we avoid 132, all entries after 1 must appear in increasing order, so the two circular parking functions are $12\cdots(n-1)n$ and $12\cdots(n-2)(n-1)(n-1)$.

If we avoid 112 and 122 we either use one entry or we have at most one of each entry. The two such circular parking functions that avoid 123 are $1n(n-1)\cdots 2$ and the all 1s parking function, and similarly, the two such parking functions that avoid 132 are $12\cdots n$ and the all 1s parking function.

Finally, if we avoid both 123 and 132 we have at most two distinct entries, and avoiding 112 ensures that only the largest entry can be repeated. The two relevant parking functions are $1\cdots 1$ and $12\cdots 2$. \square

Theorem 21. For $n \geq 1$,

$$\text{cpf}_n(111, 122, 123) = \text{cpf}_n(111, 122, 132) = \text{cpf}_n(122, 123, 132) = n.$$

Proof. If a circular parking function avoids both 111 and 122, we have at most two copies of 1 and only one copy of any larger entry. If we also avoid 123, we get $n - 1$ circular parking functions of the form $11D^i$ where D^i is the entries of $\{2, \dots, n\} \setminus i$ in decreasing order. We also get one parking function of the form $1n \cdots 2$, for a total of n circular parking functions. The case for avoiding 111, 122, and 132 is analogous, except the decreasing run of entries larger than 1 should be reversed to increasing order.

If we avoid 123 and 132 we are limited to using two distinct entries. Avoiding 122 ensures that only the the entry 1 may be repeated, so we get $n - 1$ circular parking functions of the form $1 \cdots 1i$ where $i > 1$ and we get the all 1s parking function. \square

Theorem 22. For $n \geq 1$,

$$\text{cpf}_n(111, 112, 122) = \begin{cases} n! & n \leq 2 \\ (n - 1)! & n \geq 3. \end{cases}$$

Proof. If $n \leq 2$, all parking functions avoid these patterns.

If $n \geq 3$, avoiding 111 ensures that there is at most two copies of each entry, while avoiding 112 and 122 ensure that there are at most one copy of each entry when there are multiple distinct entries. These are the same parking functions as in Theorem 13 except the all 1s parking function is removed. The relevant parking functions are the $(n - 1)!$ circular permutations. \square

6 Future directions

There are 20 circular patterns of length 4. When we consider $\text{cpf}_n(\rho)$ for these patterns, brute force data indicates we get 14 distinct counting sequences, all of which are new to OEIS and are shown in Table 5.

Recall that ρ and σ are Wilf equivalent if $\text{cpf}_n(\rho) = \text{cpf}_n(\sigma)$ for all n . From this data if ρ and σ are circular patterns of length 4, ρ and σ are Wilf equivalent if and only if $\sigma = \rho^r$.

This initial data leads to a number of possible future directions, including:

1. What can be said about the sequences $(\text{cpf}_n(\rho))$ when ρ is length 4 (or larger)?
2. Are there Wilf equivalences for single circular patterns other than reversal?

Further, the rich literature for permutations can be adapted to this circular context. For example, researchers have studied a number of variations of patterns in permutations, or have studied the distributions of various combinatorial statistics over sets of pattern-avoiding permutations. What additional features of circular parking functions lead to further connections with other combinatorial objects?

Pattern ρ	$(\text{cpf}_n(\rho))$
1123 1132	1, 2, 6, 30, 180, 1243, ...
1213	1, 2, 6, 30, 188, 1491, ...
1223 1322	1, 2, 6, 31, 203, 1610, ...
1232	1, 2, 6, 31, 203, 1666, ...
1243 1342	1, 2, 6, 32, 218, 1711, ...
1324 1423	1, 2, 6, 32, 219, 1748, ...
1432 1234	1, 2, 6, 32, 219, 1750, ...
1122	1, 2, 6, 31, 210, 1812, ...
1233 1332	1, 2, 6, 32, 227, 2028, ...
1112	1, 2, 6, 30, 214, 2083, ...
1323	1, 2, 6, 32, 229, 2120, ...
1212	1, 2, 6, 31, 235, 2286, ...
1222	1, 2, 6, 32, 239, 2418, ...
1111	1, 2, 6, 32, 254, 2712, ...

Table 5: Enumeration of circular parking functions avoiding a single pattern of length 4

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