What is an Experimental Math Course and Why Should We Care?

Posted on January 23, 2017 by Steven Klee

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What is the first meaningful mathematics problem you remember solving? For me, it was the nine dots, four lines puzzle. When my fourth grade teacher assigned it as an extra credit problem, I spent several days of recess scribbling out attempted solutions in the sandpit, erasing, and trying again until, at last, I found a solution!

I believe this geometric puzzle still sticks out in my memory nearly three decades later because it was one of the first experiences I had with trying to answer a question that didn't simply involve mimicking previous work. For practitioners, informed trial-and-error is a key step in doing mathematics, so the idea of "thinking out of the box" (or in my case, literally thinking in the sandbox…) to build intuition seems natural. However, this is a far stretch from the view of many students who see mathematics as an opportunity to memorize formulas and execute repetitive tasks.

Where do students learn the process of refining mathematical conjectures? Certainly, teaching (via) inquiry in the mathematics classroom has generated much discussion, but often the conversation about inquiry is attached to particular material in the curriculum, with an inquiry-based approach to calculus or statistics, for example. Despite being fundamental to doing mathematics, the majority of the time the inquiry process is a means to an end, rather than a focus of an entire class, and it's rarely addressed directly. In this environment, some students internalize the inquiry process by indirect exposure. Others finish their education without a true sense of how mathematics is actually developed.

Experimental mathematics courses are one answer to the need to celebrate and study inquiry for the sake of inquiry. In particular, an experimental mathematics course is not a course about a particular set of material; it is a course about a particular approach to doing mathematics.

Courses in experimental mathematics have been offered by at least 7 different colleges and universities [1]. Outside of those who have taught or taken these courses, there is not widespread understanding of what "experimental mathematics" means in the undergraduate curriculum. My goal in this post is to give a better idea of what such a course looks like.

Comparing the syllabi of various experimental mathematics courses quickly shows the material isn't standardized (nor does it need to be). However, these courses have some common themes.

First, students gain experience with programming and/or computer algebra systems throughout the course. While computer work is common to most experimental courses, it is not a necessary feature. Experimental mathematics could happen even without a computer in the room. (My solution to the nine dots, four lines problem occurred in a sandbox, rather than in a computer lab, and it still used experimental techniques to hone in on a solution.) However, in many cases, especially involving functions, number theory, combinatorics, and more, the use of a computer accomplishes the same thing one might do by hand in considerably less time. The computer can generate data and help sift through the results, quickly locating an example or counterexample. The goal, then, of using computers in an experimental math class is not just for the sake of using computers. Machine computation is a tool to greatly expand students' reach as they explore.

Further, students build intuition via experimentation and conjecture, and, most importantly, students produce projects where they develop their own solutions to open-ended mathematics questions.

Experiment, conjecture, repeat

It's tempting to say "my students experiment when I introduce a new concept with a group activity." But the key word in that sentence is "introduce". In the traditional syllabus, the focus is on material. Students can learn the material in a variety of ways, but a calculus syllabus is generally less focused on *how* students learn and more on the fact that they should learn about limits, derivatives, integrals, and their applications. Even if it is introduced in an interactive inquiry-based fashion, the star is the content. When preparing for course assessments, students don't study strategies for building intuition; they study the theorems and computations that class activities led them toward.

For example, in a calculus class, students learn the limit definition of the derivative. However, once they learn the conceptual idea that derivatives compute slope or rate of change, they look for speedups. We compute the derivatives of x^0, x^1, x^2, and x^3 using the limit definition and look for a pattern. State the pattern and practice computing with other functions like x^7, x^{42}, $x^{1/2}$, or x^{-5}. This approach is computational. In fact, it can be done in a way that builds intuition and uses active learning pedagogy. But this is still a content-centered lesson. The takeaway: $d/dx(x^n) = nx^{n-1}$.

In experimental mathematics, the star is the *approach* to new information. The content could be different each time the course is taught, but the method of figuring out new information involves a sequence of experiment, conjecture, and repeat.

A "typical" experimental math class meeting might look something like this:

1. The instructor presents a new mathematics problem, leading class discussion long enough to make the problem statement clear.

For example, one topic that lends itself well to exploration with experimental methodology is continued fractions. An entire class can be built on the idea that any real number r can be written as $r = a_0 + 1/(a_1 + 1/(a_2 + 1/\ldots))$, where the number of integers a_i required to write r could be finite or infinite. The instructor presents the definition of continued fraction and computes the continued fraction form of a few well-chosen real numbers by hand. Then students are asked to look for (families of) real numbers whose continued fraction expansions have predictable structure.

2. Students brainstorm in small groups to determine what data might help them better solve the problem and then gather the data, often with computer assistance.

In this case, students will find it useful to write code that inputs a real number r and outputs the first n terms $a_0, a_1, a_2, \ldots$ in the continued fraction expansion for r. They may also find it helpful to input the terms

$a_0, a_1, a_2, \ldots$ in a continued fraction expansion and output the simplified corresponding real number. Students use the data to conjecture a solution or patterns for special cases. At this point, the experimentation begins in earnest.

Clearly any finite continued fraction represents a rational number, but is the converse true? What patterns are there in the continued fractions for irrational numbers? Students play with expansions forπ, e, and powers of π and e; e^n has some nice structure when n is an integer that π^n does not. The continued fraction for $\sqrt{n}$ (where n is a positive integer) has particularly attractive eventually-periodic structure. The instructor's role during this exploration can vary. I circulate around the room and talk with individual students as they work. I also periodically ask students to share interesting observations with the rest of the class. Then, if one person hits on a promising idea, it can quickly be shared across the room, encouraging other students to explore related avenues of inquiry.

3. If possible, students prove their conjectures. If not, students refine or revise their conjectures by iterating steps (2)-(3). Or students try the same process for a related or generalized version of the same problem.

In this case, students may notice the eventually-periodic structure of the continued fraction for $\sqrt{n}$ and conjecture patterns for entire families of continued fractions for square roots. This could lead students to ask a converse question: if I have a periodic continued fraction, is it necessarily the continued fraction for a square root? If so, which one? If not, what other kinds of numbers have periodic continued fractions? Students could also look at generalized continued fraction expansions, where the numerators in the fractions aren't all 1s.

The big idea from this class meeting is not "square roots have periodic continued fractions". The takeaway is: you made a conjecture and confirmed or refined it; what's the next natural conjecture? The instructions given at the beginning of class have some specific mathematical questions to get started, but they also involve reflection. For example, the final part of students' written work from class could be to write a paragraph response to: "if you were to continue this problem, what question would you investigate next and why?" The particular material is less important than the process of asking and revising questions.

Projects

Classrooms require structure. Often, that course structure is dictated by a list of content and computational skills that are required in subsequent courses. Many days of experimental mathematics courses also require structure so that student learning can be assessed. In both the calculus example and the continued fraction example above, one could argue that structure exists because the instructor has a clear end result in mind, whether students take different routes or a prescribed route to get there. However, it is also possible to provide structure without having a pre-determined final content goal in mind.

The beauty of experimental mathematics is that it gives students the tools to conduct open-ended inquiry, which is often described in terms of a "project". These could consist of several weekly projects or could take the form of a single long-term project. Projects involve questions where a student can't just conduct a literature search and find answers to all the questions they generate. On the other hand, it's ok if their work isn't all new to the literature, but has some overlap of re-discovered results. In my experimental mathematics class, I have students complete one large, semester-long project. I make sure that no two students have the same project area on any given iteration of the course. There is a project deadline approximately once per month during the four-month semester.

- Month one: Pick a project topic. Each student selects a broad area they want to investigate, but not necessarily a specific research question yet.
- Month two: Each student presents a question or two they've independently decided to explore and gets feedback from their classmates.
- Month three: Each student submits a preliminary written report on their progress to me for feedback.
- Month four: Each student gives a 15-minute conference-style presentation on the results of their project to the rest of the class.

At the beginning of each semester, I provide a list of suggested project areas, including some resources for finding tractable open problems. Students may choose from the list, propose a twist on a suggested project, or propose their own problem. The delightful thing about these projects is: in the four iterations of the course I've taught at Valparaiso (representing over 40 different student projects), some general topics have been selected more than once, but each time, the students went in completely different directions with their experimentation. For example, several students have chosen to study cellular automata, but one student may look at variations of rules that generate the automata and study entire families of automata, while another student may become very interested in a particular automaton and study iterations of that one automaton over time. Either way, by the end of the course, each student has true ownership of their project and the direction it took over the duration of the semester. While they've had me and their classmates as a sounding board, the final result of the project was never prescribed to them. It is the result of doing what we do each day in class: program, experiment, conjecture, and repeat, run over the course of an entire semester to see what happens.

Conclusion

As long as we only discuss inquiry in the context of standard course material, we're missing half of mathematics! The true joy of doing research is trying a problem that has never been solved before and then experimenting and refining conjectures until you hone in on something that works. Experimental mathematics lets students experience the process of how mathematics is actually discovered. Far beyond generating examples together or following inquiry-based activities to arrive at an expected theorem, experimental mathematics pushes discussion and inquiry past the standard classroom boundaries and owns up to the process of making your own conjectures without an intended final answer. Ultimately, students will be richer when we encourage them to create and answer their own questions instead of only leading them through our own.

[1] Links to experimental math courses:

Dartmouth College (https://math.dartmouth.edu/archive/m56s13/public_html/)
Grinnell College (http://www.math.grin.edu/~chamberl/courses/444/syllabus.html)
Ithaca College (http://www.tandfonline.com/doi/abs/10.1080/10511970.2013.870264)
Lynchburg College (http://lasi.lynchburg.edu/peterson_km/public/Courses/Fall%202016/Math350_f16.htm)
Rutgers University (http://www.math.rutgers.edu/~zeilberg/math611.html)
Share ane University (http://129.81.170.14/~vhm/syllabus.pdf)
paraiso University (http://www.tandfonline.com/doi/abs/10.1080/10511970.2016.1143899?nalCode=upri20

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John Q. Public *says:*

January 23, 2017 at 4:29 pm

Great post! Agree! The Arnol'd book "Experimental Math" is a great book promoting similar ideas.

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