Catalan Numbers and Permutations

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Occasionally a new angle on an old question gives a new and beautiful perspective. This article shows a new occurrence of the Catalan numbers in a familiar location – among permutations.

1 Catalan numbers

Our main characters are the Catalan numbers. The sequence begins:

$$1, 1, 2, 5, 14, 42, 132, \ldots$$

The On-Line Encyclopedia of Integer Sequences [3], a database with hundreds of thousands of mathematically-interesting sequences, says of the Catalan numbers "This is probably the longest entry in the OEIS, and rightly so." In fact, there exists an entire book by Richard Stanley [8] that gives over 200 counting problems whose answer is the Catalan numbers. For example: "how many ways can we legally arrange n pairs of parentheses?" or "how many ways can a regular $(n+2)$-sided polygon be partitioned into n triangles?" are both answered by "the nth Catalan number".

While the Catalan numbers have a direct formula of $C_n = \frac{\binom{2n}{n}}{n+1}$, they are more easily recognized recursively. The Catalan numbers can be defined as $C_0 = 1$ and for $n \geq 1$,

$$C_n = \sum_{i=1}^{n} C_{i-1} C_{n-i}.$$

To see how this recurrence matches the parentheses problem, consider a legal arrangement of n pairs of parentheses, i.e., an arrangement of n left parentheses and n right parentheses such that there are always at least as many left parentheses as right parentheses when reading from left to right. There is 1 way to arrange 0 pairs of parentheses (write nothing down!), which matches the initial condition that $C_0 = 1$. Now, for $n \geq 1$, consider the first parenthesis. It must be a left parenthesis, which matches with a right parenthesis later in the arrangement. Further, everything between this pair must be a legal arrangement of $i-1$ parentheses pairs and everything after this pair must be a legal arrangement of the remaining $n-i$ parentheses pairs, as shown in Figure 1. Summing over all possible values of i matches the recurrence.

$$\Big(\ \boxed{i-1 \text{ pairs of parentheses}} \ \Big) \ \boxed{n-i \text{ pairs of parentheses}}$$

Figure 1: A generic legal parenthesis arrangement

2 Permutations

One of the many places the Catalan numbers have been sighted is in the context of pattern-avoiding permutations. A *permutation* is a list where order matters. We are interested in permutations of the digits $1, 2, \ldots, n$. For example, the permutations of the digits 1, 2, and 3 are 123, 132, 213, 231, 312, and 321. A *pattern* is a smaller permutation embedded in a larger permutation in a very precise way. Consider, for example, the permutation $p = 18274653$. If, instead of looking at all eight digits, we focus on the digits 8, 4, and 6, we notice that the largest of these three digits comes first, the smallest of these digits comes second, and the middle of the three digits comes last, just as in the permutation 312. The digits 846 are called a 312 pattern inside p. More generally, permutation p contains permutation q as a pattern if we can find $|q|$ digits of p that appear in the same relative order as the digits of q. We have already seen that permutation p, above, contains 312. It also contains 123, 132, 231, and 321 (can you find the examples?). On the other hand, p does not contain 213, so p is said to *avoid* 213.

Pattern-avoiding permutations have provided many delightful counting problems in recent decades. For example, the number of permutations that avoid 213 are counted by... the Catalan numbers! To see why, let the number of 213-avoiding permutations be denoted by a_n. There there is 1 permutation of length 0 and it avoids 213, so $a_0 = 1$. When we consider a 213-avoiding permutation of length $n \geq 1$, the digit 1 appears somewhere. Let's suppose that it's in position i. The $i-1$ digits before 1 must be all larger than the $n-i$ digits after the 1; otherwise we'll have a 213 pattern using the actual digit 1 as the smallest digit in the pattern. As long as we arrange the $i-1$ digits before 1 in one of the a_{i-1} 213-avoiding ways, and we arrange the $n-i$ digits after the 1 in one of the a_{n-i} 213-avoiding ways, we'll produce a length n permutation that itself avoids 213. Summing over all possible locations for the digit 1 gives

$$a_n = \sum_{i=1}^{n} a_{i-1} a_{n-i},$$

which is exactly the Catalan recurrence.

Even better, suppose q is a permutation of the digits 1, 2, and 3. No matter what permutation you picked as q, it turns out the number of q-avoiding permutations is given by the Catalan numbers! The arguments for avoiding 132, 231, or 312 are quite similar to our 213-avoiding argument above, while the arguments for avoiding 123 and 321 tend to be more complex. However, there are a plethora of bijections explaining this phenomenon. See [1] for a sampling of them.

And yet, another Catalan example lurks. To see it, we need to switch context to one of the oldest theorems that can be phrased in terms of permutation patterns.

Theorem 1 (Erdős-Szekeres Theorem)**.** *Any permutation of length at least* $(a-1)(b-1)+1$ *either contains the increasing pattern* $1\cdots(a-1)a$ *or the decreasing pattern* $b(b-1)\cdots 1$.

This theorem dates back to 1935 [2], long before mathematicians were thinking about pattern-avoiding permutations. There are many beautiful proofs of it, including a particularly friendly and concise one by Seidenberg [7].

While the Erdős-Szekeres Theorem tells us that, for sufficiently long permutations, it's impossible to avoid both an increasing pattern of length a and a decreasing pattern of length b, one might ask: what do the longest $1\cdots(a-1)a$ and $b(b-1)\cdots 1$ avoiders look like? In other words: how many permutations of length $(a-1)(b-1)$ avoid both of these patterns? Others have already asked this question. For example, in the problem section of *American Mathematical Monthly* in 1969 [4] we see the case where $a=b$ answered in detail. The solver, Richard Stanley, author of the aforementioned *Catalan Numbers* text, makes use of a famous bijection known as the Robinson-Schensted correspondence that maps permutations of length n to pairs of standard Young tableaux of the same shape [6]. A delightful expression known as the *hook-length formula* [5] can then be used to enumerate these pairs of tableaux, giving not just the solution to the $a=b$ case, but all the values in Table 1 and more.

$a\backslash b$	2	3	4	5
2	1			
3	1	4		
4	1	25	1764	
5	1	196	213444	577152576

Table 1: Number of permutations of length $(a-1)(b-1)$ avoiding both $1\cdots(a-1)a$ and $b(b-1)\cdots 1$ for small a and b

This is all great mathematical infrastructure if you've delved deep into the theory of pattern-avoiding permutations, but we wish to focus on a specific slice of this larger story. Notice the numbers in the $b=3$ column:

$$4, 25, 196, \ldots$$

While these aren't Catalan numbers, they can be rewritten quite nicely as:

$$2^2, 5^2, 14^2, \ldots$$

In other words, these values are Catalan numbers *squared*! This isn't new information; these values are computed using techniques that have been around for over 60 years. However, it's still a lovely sequence hiding inside a larger technical story. This allows us to ask: is there an explanation for this Catalan appearance that doesn't involve all the technical machinery, an explanation that a newcomer to permutation patterns can enjoy?

It's a tale as simple as pairs of parentheses. Well, make that pairs of parentheses arrangements. We know that there are C_n ways to legally arrange n pairs of parentheses. That means there are $(C_n)^2$ ways to pick two pairs of legal arrangements of n parentheses. We'll use this information to bijectively prove the following result:

Theorem 2. *There are* $(C_{a-1})^2$ *permutations of length* $2(a-1)$ *that avoid both* $12\cdots a$ *and 321.*

Before we begin the proof, we need two friendly definitions.

Definition. Given permutation p, the digit p_i is called a *left-to-right maxima* if $p_i > p_j$ for all $j < i$.

For example, in the permutation, $p = 41256387$, the left-to-right maxima are $p_1 = 4$, $p_4 = 5$, $p_5 = 6$, and $p_7 = 8$. As the name suggests, when we read the digits of p in order from left to right, a digit is a left-to-right maxima if it's the largest digit that's been read so far.

Definition. Given permutation p, the *inverse* of p, denoted p^{-1}, is the permutation where $p_i = j$ if and only if $\left(p^{-1}\right)_j = i$.

For example, the inverse of $p = 41256387$ is $p^{-1} = 23614578$. Notice that in the inverse, the permutation values and the positions exchange roles. For a visual interpretation, plot the points (i, p_i) in the Cartesian plane and call this the *graph* of p. Reflecting over the line $y = x$ produces the graph of p^{-1}. If p avoids q, then by reflecting both the graphs of p and q over the line $y = x$, we see that p^{-1} avoids q^{-1}. For our particular proof, notice that $12\cdots a$ and 321 are their own inverses, and so, conveniently, if p avoids $12\cdots a$ and 321, then so does p^{-1}.

Proof. Consider a permutation p of length $2(a-1)$ that avoids both $12\cdots a$ and 321. Let A be the set of all values that are left-to-right maxima of p, and let B be the set of all positions that are locations of left-to-right maxima of p.

We make the following observations. First, left-to-right maxima of p appear in increasing order by definition of left-to-right maxima. Further, the members of $\{1, 2, \ldots, 2(a-1)\} \setminus A$ appear in increasing order in p, because otherwise the two decreasing digits, together with the right-most previous left-to-right maxima would form a 321 pattern in p. As a consequence, $|A| =$

$|\{1,2,\ldots,2(a-1)\} \setminus A| = a-1$, since otherwise one of the two sets would contain a $12\cdots a$ pattern. This also implies that $|B| = |\{1,2,\ldots,2(a-1)\} \setminus B| = a-1$. In summary, exactly half the digits of p are left-to-right maxima, and exactly half are not. Each of these halves appears in increasing order within p.

We'll now use the sets A and B to construct a pair of legal parentheses arrangements. For the first set, construct a legal parentheses arrangement by recording a right parenthesis in position i when $i \in A$, and a left parenthesis otherwise. For the second set, construct a legal parentheses arrangement by recording a left parenthesis in position i when $i \in B$, and a right parenthesis otherwise.

To show these are legal parentheses arrangements, we need only show that in each arrangement for any $1 \leq i \leq 2(a-1)$, there are at least as many left parentheses in the first i entries as there are right parentheses. Suppose, to the contrary that there exists some point j where there are more right parentheses than left in the second arrangement, coming from B and its complement. Then, take the non-left-to-right maxima in the first j positions (encoded by right parentheses) together with the left-to-right maxima after position j to get an increasing subsequence of length longer than $a-1$, which contradicts that p avoids $12\cdots a$.

Similarly, notice that if p avoids $12\cdots a$ and 321, then so does p^{-1}. Since the second parentheses arrangement for p^{-1} must be a legal arrangement, by taking inverses, we see that the first parentheses arrangement for p must also be legal, giving our bijection.

□

A quick example of this bijection in action is in order. According to our theorem there are $(C_2)^2$ permutations of length 4 avoiding both 123 and 321. They are 2143, 2413, 3142, and 3412. Table 2 shows each of these permutations, followed by their left-to-right-maxima values and left-to-right-maxima positions. The final column shows the corresponding parentheses arrangement pair for each permutation.

permutation	left-to-right maxima	positions	parentheses pairs
2143	{2,4}	{1,3}	()(), ()()
2413	{2,4}	{1,2}	()(), (())
3412	{3,4}	{1,2}	(()), (())
3142	{3,4}	{1,3}	(()), ()()

Table 2: Small examples of the permutation to parentheses bijection

As an even larger example, the permutation $p = 31672485$ avoids both 12345 and 321. Its left-to-right maxima are $\{3,6,7,8\}$ and they appear in positions $\{1,3,4,7\}$, so this permutation corresponds to the following pair of parentheses arrangements:

$$(()(())),\quad ()(())().$$

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