The Las Vegas Sort is O(n!)

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Dick Botting described the "Las Vegas Sort" as follows:

Throw the cards on the table.
Shuffle at random.
Pick up into deck.
If in order stop otherwise repeat.

He claimed that this "algorithm" was ***O(n!)***, and was good for a laugh in class.

As described, the algorithm is not "effective"; that is, it is not guaranteed to terminate in a solution. However, we can change the specifications to allow ***k*** iterations, and terminate when the probability that the deck has been sorted at least once in those ***k*** iterations is greater than some certainty, ***C***. If we choose ***k*** so large that we will be sure the deck is sorted with probability at least ***C***, then we can ask whether ***k*** is indeed ***O(n!)*** where ***n*** is the number of cards in the deck. It is not clear that this is a ***O(n!)*** situation, so the following proof is presented.

Let ***p*** be the probability that the deck is sorted at the end of any shuffle. Then for n cards, $p = \frac{1}{n!}$. Letting ***q = 1 – p***, then the probability of ***k*** consecutive failures is q^k. We want to find ***k*** such that the probability of having at least one success in ***k*** shuffles is at least ***C*** where ***0 < C < 1***. This is equivalent to saying that we seek ***k*** such that the probability that all ***k*** shuffles result in failure is less than ***1 – C***. Therefore we want ***k*** so large that $q^k < 1 - C$. Taking logs of both sides and using the fact that ***q < 1*** means ***ln(q) < 0***, we obtain

$$k > \frac{\ln(1-C)}{\ln(q)} = \frac{\ln(1-C)}{\ln\left(\frac{n!-1}{n!}\right)} = O\left(\frac{1}{\ln\left(\frac{n!}{n!-1}\right)}\right)$$

(using the fact that ***ln(1 – C)*** is negative and the choice of ***C*** is independent of ***n***). To determine whether this is ***O(n!)***, we have to examine the behavior of $f(x) = \frac{1}{\ln(x)}$ near ***x = 1***.

Consider the function $g(x) = \frac{1}{x-1}$. Not only do both ***f(x)*** and ***g(x)*** approach infinity as ***x*** approaches 1, but also $\lim_{x\to 1}\frac{f(x)}{g(x)} = 1$. Hence ***f*** and ***g*** are asymptotically equivalent near 1.

Replacing ***x*** by $\frac{n!}{n!-1}$ we obtain $O\left(\frac{1}{\ln\left(\frac{n!}{n!-1}\right)}\right) = O\left(\frac{1}{\frac{n!}{n!-1} - 1}\right) = O(n!-1) = O(n!)$.