

THE EFFECT OF NONREPRODUCTIVE GROUPS ON PERSISTENT SEXUALLY TRANSMITTED DISEASES

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ABSTRACT. We describe several population models exposed to a mild life-long sexually transmitted disease, i.e. without significant increased mortality among infected individuals and providing no immunity/recovery. We then modify these models to include groups isolated from sexual contact and analyze their potential effect on the dynamics of the population. We are interested in how the isolated class may curb the growth of the infected group while keeping the healthy population at acceptable levels.

1. Introduction. The dynamics of a population depend on the relation between reproduction and mortality. Short-term perturbations of these two factors are usually not significant for the long term trend of population growth or decline. One factor that we analyze in this paper is the long-term effect on the population growth caused by the segregation of portions of the general (reproductive) population into a nonreproductive class that really consists of individuals of two very different kinds: *sexually active but non-procreating*, such as infertile individuals, and *sexually inactive*, consisting of individuals who by choice or medical reasons refrain from sexual contact for life. The influence of the nonreproductive group on general population dynamics has been analyzed for several exponential and logistic models in [5]. It has been shown that the nonreproductive group can indeed alter the population trend and may even make an exponentially increasing population stagnate or decline. A similar result holds for logistic models.

In this paper we perform a similar analysis of models, also incorporating a sexually transmitted disease that does not increase mortality. Besides the analogous questions regarding the general trend of the population, it will be especially interesting to see whether the segregation into the nonreproductive group of some fraction of the infected class can change the disease persistence in the reproductive population. Even more important from an epidemiological perspective is to determine whether the segregation can control or even eradicate the disease while keeping a positive trend for the healthy reproductive population. An example of such a disease is herpes simplex type 2 (HSV-2), which is incurable and yet mild.

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Typical real-life behavior of infected individuals is to refrain from sexual relations during outbreaks when they are infectious, thus significantly reducing the transmission of the disease. However, we are not going to propose a model for herpes, since we are ignoring vertical transmission, and the fact that most affected individuals do not choose to stay childless for life. The models proposed here are intended to address the issue of how much and in what form isolation from sexual contact by some proportion of individuals may affect two important aspects of a real population with a sexually transmitted disease: its growth as a population and the spread of the disease. The models will, therefore, be based on the assumption that a proportion ν_1 of uninfected individuals and a proportion ν_2 of infected individuals segregate themselves into a group of sexually inactive individuals in which, of course, there is no disease transmission, since the disease considered is sexually transmitted. The first mathematical study of sexually transmitted diseases was carried out in [1], where the underlying demographic model is the two-sex model of Hader [4]. Homogeneous models for sexually transmitted diseases are analyzed in [3] having underlying exponential demographic models. A survey on mathematical modeling of STDs can be found in [2].

In this paper we shall analyze several exponential and logistic models with and without segregated classes. We will model disease transmission using the proportionate mixing law, and we shall not distinguish between females and males; rather, we usually will consider three groups of individuals: reproductive healthy, reproductive infected, and abstained or segregated (both healthy and infected).

The paper is structured as follows: Section 2 introduces an exponential demographic model without segregation into a nonreproductive group and analyzes the long-term impact of segregation on population growth; Section 3 does the analogous with an exponential model with segregation. Sections 4 and 5 introduce logistic demographic models, respectively without and with segregation into a nonreproductive group. Finally, Section 6 presents some conclusions and describes future research related to the present one.

2. Exponential model without nonreproductive class. We consider the simple Malthusian exponential model for the total population

$$P'(t) = rP(t) \quad \text{with} \quad r = \beta - \mu \quad (1)$$

where β and μ are the natural birth and death rates, respectively. The solution is

$$P(t) = P_0 e^{rt} \quad \text{where} \quad P_0 \quad \text{is the initial total population.} \quad (2)$$

We now model an infectious disease that does not cause any disease-induced mortality. We also assume that there is no recovery from infection and that the newborn individuals are all healthy; that is, there is no vertical transmission of the infection. This is the classical S-I epidemic model. Let $S(t)$ denote the size of the class of healthy people and $I(t)$ that of the infected. We shall denote by λ the per capita infection rate of the disease. The model is then described by the following system:

$$\begin{cases} S' &= \beta(S + I) - \lambda \frac{SI}{S+I} - \mu S, \\ I' &= \lambda \frac{SI}{S+I} - \mu I. \end{cases} \quad (3)$$

Note that $P = S + I$ and by adding the above equations we obtain Malthus' exponential model for the total population.

The case $r < 0$ is not interesting since in that case the entire population declines to zero. If $r > 0$ we know that $P(t) \rightarrow \infty$, and the question is then whether the infected class size is important or negligible with respect to the entire population. To answer this question we derive the dynamics for the proportion of infected individuals in the population, $y = \frac{I}{P}$.

The governing equation,

$$y' = (\lambda - \beta)y - \lambda y^2, \quad (4)$$

is of Bernoulli type and it can be solved explicitly, with solution

$$y(t) = \frac{1}{\lambda t + 1/y_0} \quad \text{if } \lambda = \beta,$$

and

$$y(t) = \frac{e^{(\lambda-\beta)t}}{\frac{\lambda}{\lambda-\beta} [e^{(\lambda-\beta)t} - 1] + 1/y_0} \quad \text{otherwise.}$$

It is clear from this explicit solution that the following holds.

PROPOSITION 2.1. *Assume that the Malthusian rate is positive, $r > 0$. Then, as $t \rightarrow \infty$, we have*

$$\lim_{t \rightarrow \infty} \frac{I(t)}{P(t)} = 1 - \frac{\beta}{\lambda} \quad \text{if } \lambda > \beta,$$

$$\lim_{t \rightarrow \infty} \frac{I(t)}{P(t)} = 0 \quad \text{if } \lambda \leq \beta.$$

To analyze the impact of the nonreproductive group, we shall assume in the next section that the disease is successful in attaining a positive stable ratio in the population; that is, $\lambda > \beta$, and we shall ask whether the introduction of the transition rates into a third class of abstinence from sexual contact can cause disease prevalence to decline without eliminating the healthy group.

3. Exponential model with nonreproductive class. We shall assume that there is a constant removal rate ν_1 from the uninfected reproductive group and ν_2 from the infected one. The first accounts for those individuals that will not have sexual relations and thus cannot produce offspring nor spread disease. The second accounts for those previously sexually active, infected individuals who refrain from sexual contact because of their infection. We denote by A the size of the group of individuals who abstain from sexual contact. The model is then described by the following system:

$$\begin{cases} S' &= \beta(S + I) - \lambda \frac{SI}{S+I+A} - \mu S - \nu_1 S, \\ I' &= \lambda \frac{SI}{S+I+A} - \mu I - \nu_2 I, \\ A' &= \nu_1 S + \nu_2 I - \mu A. \end{cases} \quad (5)$$

This model keeps the abstinent all together. Obviously, since the three groups are disjoint, the total population size is now $P(t) = S(t) + I(t) + A(t)$.

This system does not admit positive steady states, except in the singular case

$$\left[\lambda - (\mu + \nu_2) \left(1 + \frac{\nu_1}{\mu} \right) \right] (\beta - \mu - \nu_2) = (\mu + \nu_2) \left(1 + \frac{\nu_2}{\mu} \right) (\mu + \nu_1 - \beta),$$

that leads to the continuum of positive equilibria

$$(S^*, I^*, A^*) = \left(\left[\frac{\beta - \mu - \nu_2}{\mu + \nu_1 - \beta} \right] I^*, I^*, \left[\frac{\nu_1(\beta - \mu - \nu_2)}{\mu(\mu + \nu_1 - \beta)} + \frac{\nu_2}{\mu} \right] I^* \right), \quad I^* > 0,$$

provided $\min\{\nu_1, \nu_2\} < \beta - \mu < \max\{\nu_1, \nu_2\}$ (so that S^* is a positive multiple of I^*).

It is easy to see that if $\beta - \mu > \max\{\nu_1, \nu_2\}$, then $P(t) \rightarrow \infty$ (since $S + I$ then grows exponentially at a rate no smaller than $\beta - \mu - \max\{\nu_1, \nu_2\} > 0$), while for $\beta - \mu < \min\{\nu_1, \nu_2\}$ we have $S(t), I(t) \rightarrow 0$ (since $S + I$ then decreases exponentially at a rate no smaller than $\beta - \mu - \min\{\nu_1, \nu_2\} < 0$). In that case the whole population disappears asymptotically if $\mu > 0$, while all of it abstains from sexual contact but does not go extinct if $\mu = 0$ (which is, of course, impossible in real life).

If the natural growth rate, $r = \beta - \mu$, is between the two removal rates ν_1 and ν_2 , then it can be shown that the total population either grows exponentially or declines to zero. As in the previous section, we derive the equations for the proportions $x = \frac{S(t)}{P(t)}$ and $y = \frac{I(t)}{P(t)}$.

$$\begin{cases} x' &= \beta(1-x)(x+y) - \lambda xy - \nu_1 x, \\ y' &= \lambda xy - \beta y(x+y) - \nu_2 y. \end{cases} \quad (6)$$

This system admits an extinction and a "disease-free" equilibrium,

$$(x_*, y_*) = \left(1 - \frac{\nu_1}{\beta}, 0 \right).$$

The Jacobian of (6) is

$$J(x, y) = \begin{pmatrix} -2\beta x - (\lambda + \beta)y + \beta - \nu_1 & -(\lambda + \beta)x + \beta \\ (\lambda - \beta)y & (\lambda - \beta)x - 2\beta y - \nu_2 \end{pmatrix}.$$

Notice that

$$J(0, 0) = \begin{pmatrix} \beta - \nu_1 & \beta \\ 0 & -\nu_2 \end{pmatrix},$$

with eigenvalues are $\beta - \nu_1$ and $-\nu_2$, which means that if $\beta < \nu_1$ then the steady state $(0, 0)$ is locally stable and both the proportions of healthy and infected decline to zero and, consequently, the entire population $P(t)$ will go extinct.

We assume now that $\beta > \nu_1$ and analyze the asymptotic stability of (x_*, y_*) . We can show that abstinent behavior may eliminate the infected class or the proportion of infected individuals in the total population. Note that

$$J(x_*, y_*) = \begin{pmatrix} -\beta + \nu_1 & -\lambda + \frac{\lambda\nu_1}{\beta} + \nu_1 \\ 0 & (\lambda - \beta)\left(1 - \frac{\nu_1}{\beta}\right) - \nu_2 \end{pmatrix},$$

and its eigenvalues are $-\beta + \nu_1$ and $(\lambda - \beta)\left(1 - \frac{\nu_1}{\beta}\right) - \nu_2$. Hence (x_*, y_*) is locally stable if

$$\lambda < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}.$$

Assuming now that

$$\frac{S(t)}{P(t)} \rightarrow 1 - \frac{\nu_1}{\beta} \quad \text{and} \quad \frac{I(t)}{P(t)} \rightarrow 0,$$

we can use the second equation of the original system (5) to compute the limiting value of $I(t)$: we have

$$\frac{I'(t)}{I(t)} \rightarrow \lambda \left(1 - \frac{\nu_1}{\beta}\right) - \mu - \nu_2,$$

which means that

$$I(t) \rightarrow 0 \quad \text{provided that} \quad \lambda < \frac{\mu + \nu_2}{1 - \frac{\nu_1}{\beta}}.$$

Going back to the equation for $S(t)$, we notice that

$$S'(t) = (\beta - \mu - \nu_1)S(t) + \left(\beta - \frac{\lambda S}{P}\right)I,$$

where the second term is negligible and, therefore, it follows that $S(t) \rightarrow \infty$ if $\beta - \mu > \nu_1$.

By the same reasoning, it is clear that if $\lambda > (\mu + \nu_2)/(1 - \frac{\nu_1}{\beta})$, then $I(t)$ grows exponentially and thus the entire population grows to infinity.

In conclusion, we have established two additional thresholds for the infection rate λ (β being the threshold contact rate for the disease to persist in the absence of segregation),

$$\beta < \frac{\mu + \nu_2}{1 - \frac{\nu_1}{\beta}} < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}.$$

If λ is between the first two, then we have a disease free equilibrium; if it is between the second and the third thresholds, then the infected class grows exponentially to infinity but its proportion in the total population tends to zero.

Notice also that the presence of segregation in the infected class is essential in the sense that if $\nu_2 = 0$, then the rightmost threshold is precisely β , meaning that there is no possible value for λ to satisfy the stability of the disease-free steady state of (6). We illustrate this by presenting the phase portrait of the system (6) in Figures 1 and 2. The values of the parameters are chosen for reference only.

Next we show the graphs of S , I and A in Figures 3 and 4 to illustrate the other threshold for λ that causes the infected class to decline to zero.

It is clear from the last figure that the proportion of infected declines to zero although the size of the infected class itself grows exponentially.

The positive steady state of (6) is computed as follows: from the second equation we get

$$x = \frac{\beta y + \nu_2}{\lambda - \beta}.$$

Substituting this relation into the first equation of (6), we obtain the following quadratic that gives the value of y at equilibrium,

$$\lambda^2 \beta y^2 - [\beta^2(\nu_1 - \nu_2) + \lambda^2(\beta - \nu_2) - \lambda\beta(\beta + \nu_1)]y + \nu_2[\beta(\beta - \nu_1 + \nu_2) - \lambda(\beta - \nu_1)] = 0,$$

with solution

$$y = \frac{1}{2\lambda^2\beta} [C \pm (\lambda - \beta)\sqrt{\Delta}],$$

where

$$C = \beta(\lambda - \beta)(\lambda - \nu_1) - \nu_2(\lambda^2 + \beta^2)$$

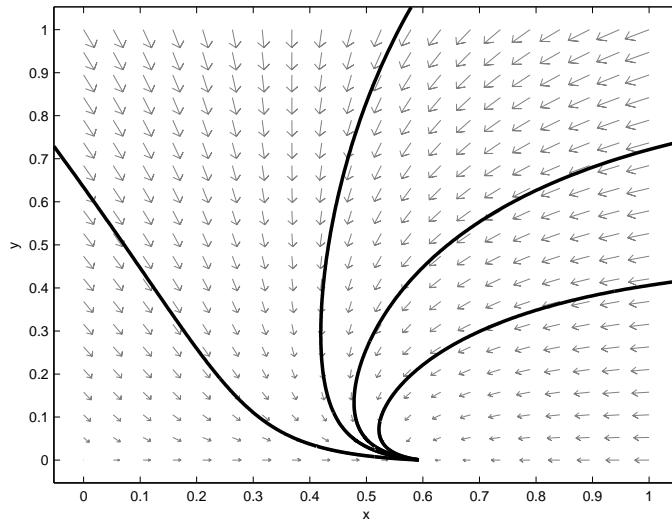


FIGURE 1. Case $\beta < \lambda < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}$. $\beta = 0.05$ $\mu = 0.01$ $\nu_1 = 0.02$ $\nu_2 = 0.06$ $\lambda = 0.1$

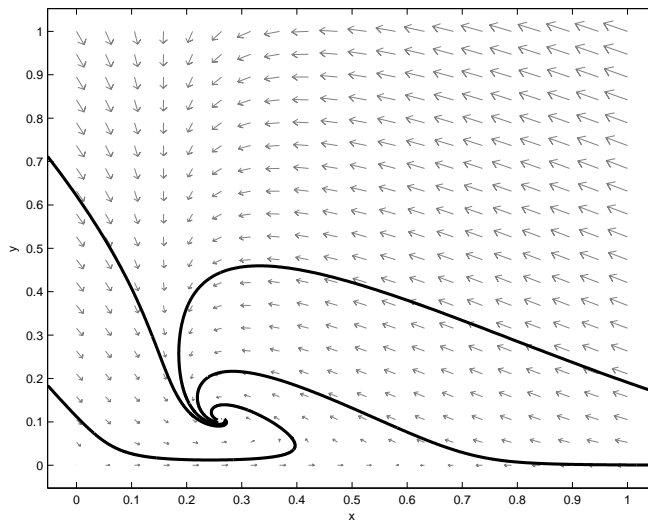


FIGURE 2. Case $\frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}} < \lambda$. $\beta = 0.05$ $\mu = 0.01$ $\nu_1 = 0.02$ $\nu_2 = 0.06$ $\lambda = 0.3$

and

$$\Delta = \lambda^2(\beta - \nu_2)^2 + 2\lambda\beta(\beta + \nu_2)(\nu_2 - \nu_1) + \beta^2(\nu_2 - \nu_1)^2.$$

If

$$\lambda < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}},$$

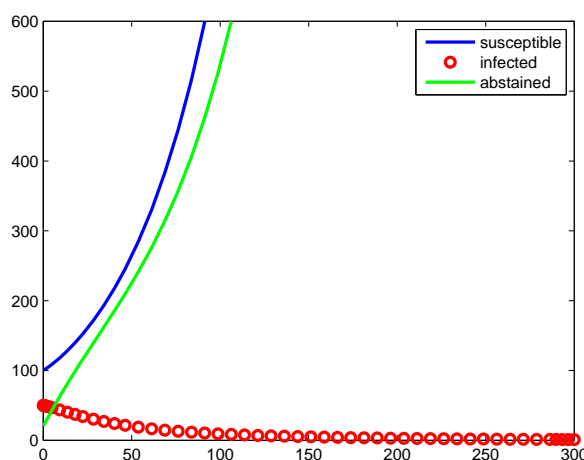


FIGURE 3. Case $\beta < \lambda < \frac{\mu + \nu_2}{1 - \frac{\nu_1}{\beta}}$. $\beta = 0.05$ $\mu = 0.01$ $\nu_1 = 0.02$ $\nu_2 = 0.06$ $\lambda = 0.1$

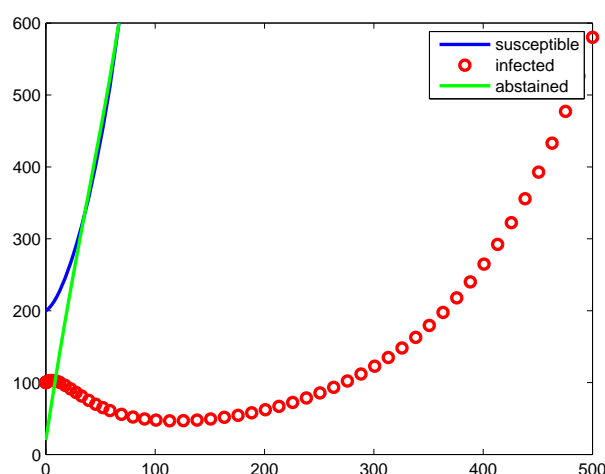


FIGURE 4. Case $\frac{\mu + \nu_2}{1 - \frac{\nu_1}{\beta}} < \lambda < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}$. $\beta = 0.05$ $\mu = 0.01$ $\nu_1 = 0.02$ $\nu_2 = 0.06$ $\lambda = 0.13$

then the product of the roots of this quadratic is positive and their real parts are either both negative or both positive. If they are negative, then there is no interior steady state and (x_*, y_*) is the only nontrivial equilibrium. If they are positive, then the system either admits complex conjugate roots and again the only nontrivial equilibrium is (x_*, y_*) , or it has two positive equilibria, one of which is stable and the other one unstable. In the last case, depending on the initial conditions, the positive semi-orbits approach either the disease-free or the stable interior steady

state. The next proposition establishes the conditions on the parameters that ensure the uniqueness of the disease-free proportions.

PROPOSITION 3.1. *If $\lambda < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}$ and at least one of the following conditions is met:*

$$(i) \nu_1 + \sqrt{\beta\nu_2} - \nu_2 < \beta < \nu_2,$$

$$(ii) \frac{\beta(\nu_1 - \nu_2)(\sqrt{\beta} - \sqrt{\nu_2})^2}{(\beta - \nu_2)^2} < \lambda < \frac{\beta(\nu_1 - \nu_2)(\sqrt{\beta} + \sqrt{\nu_2})^2}{(\beta - \nu_2)^2},$$

then (x_*, y_*) is the only nontrivial steady state of (6).

Proof. We can rewrite C as

$$C = \lambda^2(\beta - \nu_2) - \lambda\beta(\beta + \nu_1) + \beta^2(\nu_1 - \nu_2).$$

Since we know that $\nu_1 < \beta$, then if $\beta < \nu_2$, it is also clear that $C < 0$. Suppose now $\nu_2 < \beta$. If we consider C as a quadratic in λ , we notice that the product and the sum of its roots are positive and that

$$C(\beta) = -2\nu_2\beta^2 < 0,$$

which implies that both of its roots are real and positive. Since we are in the case

$$\beta < \lambda < \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}} \quad \text{and we know } C(\beta) < 0,$$

it follows that we need $C\left(\frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}\right) < 0$ in order to ensure that $C(\lambda) < 0$. We can show by a straightforward calculation that the last condition is equivalent to

$$\nu_1 < \beta - \sqrt{\beta\nu_2} + \nu_2.$$

The last case that needs to be analyzed occurs when $C > 0$ but the roots are complex conjugate. In other words, we need to establish a condition for Δ to be negative. First notice that if $\nu_1 < \nu_2$ then $\Delta > 0$. Hence it is necessary to assume $\nu_2 < \nu_1$. Just as before, we consider Δ a quadratic function of λ . It is easy to see that the discriminant of this quadratic is

$$4\beta^3\nu_2(\nu_2 - \nu_1)^2 > 0.$$

This, together with the fact that the sum and the product of the roots of Δ are positive, implies that both roots are real and positive. Hence λ must lie between these roots in order to have $\Delta < 0$; that is,

$$\frac{\beta(\nu_1 - \nu_2)(\sqrt{\beta} - \sqrt{\nu_2})^2}{(\beta - \nu_2)^2} < \lambda < \frac{\beta(\nu_1 - \nu_2)(\sqrt{\beta} + \sqrt{\nu_2})^2}{(\beta - \nu_2)^2}.$$

□

In the next picture we present the phase portrait of (6) when the conditions in the above proposition are not satisfied. One can see the existence of two positive steady states, of which one is unstable. We clearly see the unstable equilibrium on the separatrix that divides the basin of attraction of the stable positive equilibrium from that of the disease-free.

Getting back to our original equation in y , we have that the product of the roots is negative if and only if

$$\lambda > \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}.$$

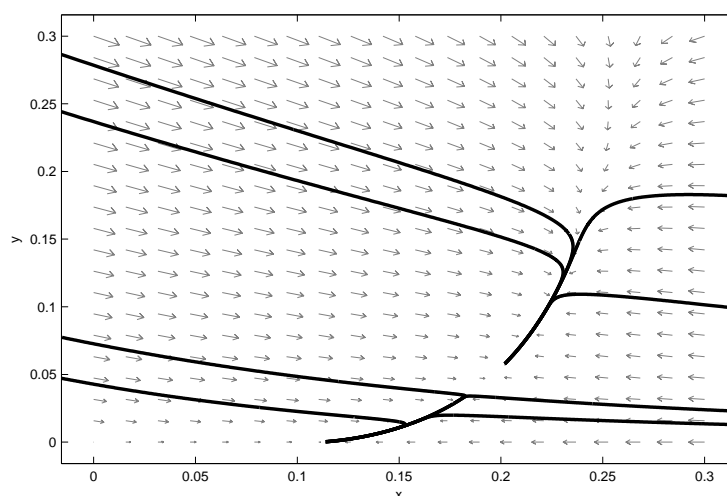


FIGURE 5. Case of two positive steady states. $\beta = 0.09$ $\mu = 0.01$ $\nu_1 = 0.08$ $\nu_2 = 0.02$ $\lambda = 0.2148$

In other words, if the disease-free steady state of (6) is unstable, the system admits a unique interior equilibrium,

$$x^* = \frac{\beta y^* + \nu_2}{\lambda - \beta}, \quad y^* = \frac{1}{2\lambda^2\beta} [C + (\lambda - \beta)\sqrt{\Delta}].$$

Concerning the stability of the interior equilibrium we have the following:

PROPOSITION 3.2. *If the transmission rate is high enough,*

$$\lambda > \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}},$$

then (x^, y^*) is globally stable and the disease is endemic.*

Proof. First note that all solutions of the system (6) that start positive stay positive for all time. Then it follows that $x(t) \leq 1$ for all time and thus y is bounded for all time by $(\lambda - \beta + \nu_2)/2\beta$. Second, we can rule out the possibility of periodic solutions by applying Dulac's criterion:

$$\nabla \left(\frac{1}{xy} F(x, y) \right) = -\frac{\beta}{xy} \left(x + y + \frac{y}{x} \right) < 0,$$

where $F(x, y)$ is the vector field of (6). In conclusion, the unique steady state is globally stable. \square

Remark 3.1. *If we rewrite (5) in the form*

$$\begin{cases} S' &= (\beta - \mu - \nu_1 - \lambda \frac{I}{P})S + \beta I, \\ I' &= (\lambda \frac{S}{P} - \mu - \nu_2)I, \\ A' &= \nu_1 S + \nu_2 I - \mu A, \end{cases} \quad (7)$$

we can see that the entire population declines to zero if

$$\beta - \mu - \nu_1 - \lambda y^* < 0 \quad \text{and} \quad \lambda x^* - \mu - \nu_2 < 0.$$

Obviously, this can happen if the mortality μ is sufficiently large. However, the infection rate λ can cause the same phenomenon, as is shown in Figure 6, where we plotted the total population for different values of λ . This is quite surprising, since there is no disease-induced mortality. The reason is that when there are enough transfers from S to I , then the exponential growth rates may tend to a negative value for both susceptible and infected groups if there is a sufficiently large proportion of infected that abstain from sexual contact.

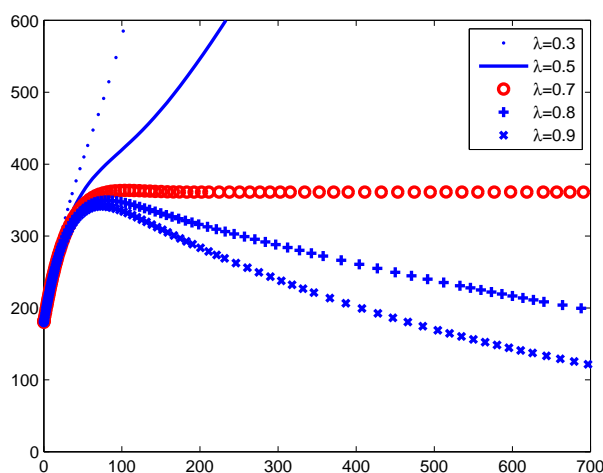


FIGURE 6. Case $\lambda > \frac{\beta - \nu_1 + \nu_2}{1 - \frac{\nu_1}{\beta}}$. $\beta = 0.05$ $\mu = 0.01$ $\nu_1 = 0.02$ $\nu_2 = 0.06$

4. The logistic model without the nonreproductive class. If we are really interested in the long-term behavior of the model, an exponential model for the healthy population is not going to provide a good prediction. A more realistic approach is to consider the classical logistic model [6] as the reference for the total population $P(t)$:

$$P'(t) = r \left[1 - \frac{P}{K} \right] P,$$

with carrying capacity K . We can describe our dynamical system using this model as a basis, but including also an infectious disease, as follows:

$$\begin{cases} S' &= \beta(S + I) - \lambda SI - [\mu + b(S + I)]S, \\ I' &= \lambda SI - [\mu + b(S + I)]I. \end{cases} \quad (8)$$

If we sum the two equations in (8) we obtain the logistic equation for the total population $P = S + I$:

$$P' = (\beta - \mu - bP)P, \quad P(0) = P_0 = S(0) + I(0),$$

where the carrying capacity is $K = \frac{\beta - \mu}{b}$ with $\beta > \mu \gg b$. The explicit solution for P is

$$P(t) = \frac{KP_0e^{bKt}}{P_0e^{bKt} + K - P_0} = \frac{K}{(K/P_0 - 1)e^{-bKt} + 1}.$$

Of course,

$$\lim_{t \rightarrow \infty} P(t) = K.$$

One can now replace S by $P - I$ in the second equation of (8) obtaining another Bernoulli-type ODE that gives I explicitly:

$$I' = [(\lambda - b)P - \mu]I - \lambda I^2, \quad I(0) = I_0 > 0.$$

Denote by ϕ the integrating factor of the above ODE. It follows that

$$\phi(t) = \int_0^t [(\lambda - b)P(\theta) - \mu] d\theta = \frac{\lambda - b}{b} \ln \frac{P_0e^{bKt} + K - P_0}{K} - \mu t.$$

The solution $I(t)$ can then be written in the form

$$I(t) = \frac{e^{\phi(t)}}{\lambda \int_0^t e^{\phi(\theta)} d\theta + 1/I_0}, \quad (9)$$

where

$$e^{\phi(t)} = \left(\frac{P_0e^{bKt} + K - P_0}{Ke^{\frac{\mu b}{\lambda - b}t}} \right)^{\frac{\lambda - b}{b}}.$$

The following result summarizes the behavior of the logistic, nonsegregated, epidemic model: let $\mathcal{R}_0 = \frac{\lambda K}{\beta}$ be the reproductive number of the epidemic.

PROPOSITION 4.1. *If $\mathcal{R}_0 > 1$, then*

$$\lim_{t \rightarrow \infty} I(t) = K - \frac{\beta}{\lambda} \quad \text{otherwise} \quad \lim_{t \rightarrow \infty} I(t) = 0.$$

Proof. Note that $\mathcal{R}_0 > 1 \Leftrightarrow \lambda > \beta/K$. Also, $\beta/K = \frac{\beta}{\beta - \mu} b > b$ gives $\lambda > b$, and then $\mathcal{R}_0 > 1 \Leftrightarrow bK = \beta - \mu > \frac{\mu b}{\lambda - b}$. This shows that

$$e^{\phi(t)} \quad \text{and also} \quad \frac{1}{I_0} + \lambda \int_0^t e^{\phi(\theta)} d\theta$$

are bounded below by an exponentially increasing function. Hence, using l'Hôpital's rule, (9) yields the limit

$$\lim_{t \rightarrow \infty} I(t) = K - \frac{\beta}{\lambda}.$$

When $\lambda \leq \frac{\beta}{K}$, we notice that the only nontrivial steady state of (8) in the positive quadrant is $(S^*, I^*) = (K, 0)$. Moreover, the solutions of the system are bounded for all time t . Also, if $F(S, I)$ is the direction field of (8), then

$$\nabla \cdot \left[\frac{1}{SI} F(S, I) \right] = \frac{1}{SI} \left(-bS - bI - \beta \frac{I}{S} \right) < 0,$$

which by Dulac's criterion eliminates the possibility of periodic solutions. Hence the disease-free steady state is globally stable, which concludes the proof. \square

5. The logistic model with a nonreproductive class. The corresponding model including nonreproductive groups is given by the following system:

$$\begin{cases} S' &= \beta(S + I) - \lambda SI - [\mu + b(S + I + A)]S - \nu_1 S, \\ I' &= \lambda SI - [\mu + b(S + I + A)]I - \nu_2 I, \\ A' &= \nu_1 S + \nu_2 I - [\mu + b(S + I + A)]A. \end{cases} \quad (10)$$

The total population is now $P = S + I + A$, which satisfies

$$P' = (\beta - \mu - bP)P - \beta A, \quad P(0) = P_0 = S(0) + I(0) + A(0),$$

and the system has the following three equilibrium points:

- *trivial*, $(\bar{S}, \bar{I}, \bar{A}) = (0, 0, 0)$;
- *disease-free*, $(S_*, I_*, A_*) = \left(\bar{K} \left(1 - \frac{\nu_1}{\beta}\right), 0, \bar{K} \frac{\nu_1}{\beta} \right)$;
- *endemic*, $(S^*, I^*, A^*) = \left(\frac{bP^* + \nu_2 + \mu}{\lambda}, \frac{(\beta - bP^* - \nu_1 - \mu)(bP^* + \mu + \nu_2)}{\lambda(bP^* + \mu + \nu_2 - \beta)}, \frac{P^*(\beta - \mu - bP^*)}{\beta} \right)$,

where $\bar{K} = \frac{\beta - \mu - \nu_1}{b} = K - \frac{\nu_1}{b} > 0 \Leftrightarrow \beta > \mu + \nu_1$. Here P^* is a root of the following cubic equation:

$$(bP + \mu + \nu_2)(\nu_2 - \nu_1) = \frac{\lambda}{\beta}(bP + \mu + \nu_2 - \beta)(\mu + bP)P.$$

Obviously, a sufficient condition for the existence of a positive root is $\nu_2 > \nu_1$, since then the leading and constant coefficients of this cubic have opposite signs. However, this condition is not necessary. For the special case $\nu_1 = \nu_2$, we shall find simple conditions that ensure the existence of a positive steady state P^* . In general, for a meaningful endemic equilibrium to exist, $I^* > 0$, it is necessary that $\beta - \mu - \nu_{max} < bP^* < \beta - \mu - \nu_{min}$ (which also gives $P^* \leq K = \frac{\beta - \mu}{b}$, i.e. $A^* > 0$). We can determine a condition to ensure the existence of such a root as follows. Consider the polynomial

$$h(P) = \frac{\lambda}{\beta}P(\mu + bP)(bP + \mu + \nu_2 - \beta) - (bP + \mu + \nu_2)(\nu_2 - \nu_1),$$

and then let us impose the condition that

$$h\left(\frac{\beta - \mu - \nu_1}{b}\right) \times h\left(\frac{\beta - \mu - \nu_2}{b}\right) < 0.$$

This product is equal to

$$-\frac{(\nu_1 - \nu_2)^2[\lambda(\beta - \nu_1)(\beta - \mu - \nu_1) - b\beta(\beta - \nu_1 + \nu_2)]}{b},$$

and we can see that this is negative provided that λ satisfies the condition

$$\lambda > \frac{\beta - \nu_1 + \nu_2}{\left(K - \frac{\nu_1}{b}\right)\left(1 - \frac{\nu_2}{b}\right)}.$$

Let us consider now the dynamics of the model as the Malthusian rate $r = \beta - \mu$ increases from 0 to large values. First we remark that there is a threshold for the Malthusian rate $r = \beta - \mu$ to ensure that the total population does not go extinct for lack of enough reproduction.

PROPOSITION 5.1. *If $bK = \beta - \mu < \nu_{min} = \min\{\nu_1, \nu_2\}$, then $(\bar{S}, \bar{I}, \bar{A})$ is globally stable. If $bK < \nu_1$, then $(\bar{S}, \bar{I}, \bar{A})$ is locally asymptotically stable, and if $bK > \nu_1$, then it is unstable.*

Proof. Summing the first two equations of (10) we obtain the following upper bound for $S + I$:

$$S(t) + I(t) < e^{(bK - \nu_{min})t},$$

which implies that $P(t) \rightarrow 0$ if $bK = \beta - \mu < \nu_{min} = \min\{\nu_1, \nu_2\}$ giving global stability in this case.

Next, consider the Jacobian matrix associated with (10),

$$\begin{pmatrix} a_{11} & \beta - (\lambda + b)S & -bS \\ (\lambda - b)I & a_{22} & -bI \\ \nu_1 - bA & \nu_2 - bA & -\mu - b(S + I + 2A) \end{pmatrix}.$$

Where $a_{11} = \beta - \mu - \nu_1 - \lambda I - b(2S + I + A)$ and $a_{22} = -\mu - \nu_2 + \lambda S - b(S + 2I + A)$. Evaluating this Jacobian at $(\bar{S}, \bar{I}, \bar{A}) = (0, 0, 0)$ we obtain

$$J = \begin{pmatrix} -\mu + \beta - \nu_1 & \beta & 0 \\ 0 & -\mu - \nu_2 & 0 \\ \nu_1 & \nu_2 & -\mu \end{pmatrix},$$

whose characteristic polynomial is

$$-(\mu + \xi)(\mu - \beta + \nu_1 + \xi)(\mu + \nu_2 + \xi).$$

Therefore, if $\beta - \mu < \nu_1$ (equivalently, $bK < \nu_1$), the disease-free steady state is locally asymptotically stable, and it is unstable when $\beta - \mu > \nu_1$. \square

We see that, to ensure that the population remains at positive levels regardless of the initial conditions, we must assume $bK = \beta - \mu > \nu_1$. We have

PROPOSITION 5.2. *If $bK = \beta - \mu > \nu_1$, then (10) admits the disease-free equilibrium (S_*, I_*, A_*) . It is locally asymptotically stable if*

$$\beta/K < \lambda < \frac{\beta - \nu_1 + \nu_2}{(1 - \frac{\nu_1}{\beta})(K - \frac{\nu_1}{b})}. \quad (11)$$

Proof. To show the local asymptotic stability of the disease-free steady state, we evaluate the Jacobian matrix at $(S, I, A) = (S_*, 0, A_*)$:

$$\begin{pmatrix} \beta - \mu - \nu_1 - b(A_* + 2S_*) & \beta - (\lambda + b)S_* & -bS_* \\ 0 & \sigma & 0 \\ \nu_1 - bA_* & \nu_2 - bA_* & -\mu - b(S_* + 2A_*) \end{pmatrix}.$$

Its eigenvalues are $\sigma \equiv -\mu - \nu_2 + \lambda S_* - b(S_* + A_*)$ and those of the matrix

$$M = \begin{pmatrix} \beta - \mu - \nu_1 - b(A_* + 2S_*) & -bS_* \\ \nu_1 - bA_* & -\mu - b(S_* + 2A_*) \end{pmatrix}.$$

Since $\text{tr}(M) = -\mu - 2(\beta - \mu - \nu_1) < 0$ and $\det(M) > 0$ if $\beta/K < \lambda$, it follows that the disease-free equilibrium is locally asymptotically stable if the third eigenvalue is negative, that is

$$\lambda S_* < \beta + \nu_2 - \nu_1.$$

\square

Note that the right-hand side of condition (11) is greater than β/K if $\beta - \mu > \nu_1$. Thus, this local stability condition suggests that the nonreproductive groups can cause the infected class to decline to zero while the healthy population remains stable at a positive level. The left-hand side condition on λ ensures disease persistence in the absence of segregation.

Concerning the endemic equilibrium, we do not have a proof of its stability. However, our simulations seem to indicate that if $bK = \beta - \mu > \nu_1$, it is also globally stable whenever it exists. In the case of equal nonreproductive rates, $\nu_1 = \nu_2 = \nu$, we can prove this fact.

PROPOSITION 5.3. *If $\nu_1 = \nu_2 = \nu < \beta - \mu$, the system (10) has a unique interior steady state which is globally stable if $\lambda > \frac{\beta}{(K - \frac{\nu}{b})(1 - \frac{\nu}{\beta})}$. Moreover, if*

$$\lambda < \frac{\beta}{(K - \frac{\nu}{b})(1 - \frac{\nu}{\beta})}, \text{ then } \lim_{t \rightarrow \infty} I(t) = 0.$$

Proof. Notice that by adding the first two equations in (10) and then adding all three of them, one obtains a closed two-dimensional system in $x = S + I$ and $y = P = S + I + A$:

$$\begin{cases} x' &= (\beta - \mu - \nu - by)x, \\ y' &= \beta x - (\mu + by)y. \end{cases} \tag{12}$$

First notice that the solution of (12) is bounded for all time t . This is true because the total population in the nonsegregated case is just the solution of the logistic model (8)—therefore bounded—and it is a super-solution for the total population of (10). Since the nonreproductive groups can only have a negative contribution to the population growth in time, it follows that the solution of (12) is also bounded. It is easy to see that Dulac’s criterion works as well for the system above:

$$\nabla \left(\frac{1}{xy} F(x, y) \right) = -\frac{1}{xy} \left(2\mu + 3by + \frac{\beta x}{y} \right) < 0,$$

where $F(x, y)$ is the vector field of (12). Therefore, the steady state

$$x^* = S^* + I^* = \left(K - \frac{\nu}{b} \right) \left(1 - \frac{\nu}{\beta} \right) \quad \text{and} \quad y^* = P^* = \bar{K} = K - \frac{\nu}{b}$$

is the unique positive one and it is globally stable. Thus, knowing that

$$\lim_{t \rightarrow \infty} [S(t) + I(t)] = x^* \quad \text{and} \quad \lim_{t \rightarrow \infty} P(t) = y^*,$$

we turn to the system (10) and use the second equation to show that both $S(t)$ and $I(t)$ tend to a positive limit. To accomplish this, we again shall use a sub-solution and super-solution of I . For every $\epsilon > 0$, there exists a time t_0 such that for every $t > t_0$ we have

$$x^* - I(t) - \epsilon < S(t) < x^* - I(t) + \epsilon \quad \text{and} \quad y^* - \epsilon < P(t) < y^* + \epsilon.$$

This leads to

$$I[\lambda x^* - by^* - \mu - \nu - (\lambda + b)\epsilon] - \lambda I^2 < I' < I[\lambda x^* - by^* - \mu - \nu + (\lambda + b)\epsilon] - \lambda I^2,$$

which is equivalent to

$$I[D - O(\epsilon)] - \lambda I^2 < I' < I[D + O(\epsilon)] - \lambda I^2,$$

where

$$D = \lambda \left(K - \frac{\nu}{b} \right) \left(1 - \frac{\nu}{\beta} \right) - \beta \quad \text{and} \quad O(\epsilon) = (\lambda + b)\epsilon.$$

Solving these two ODEs we obtain

$$\frac{e^{[D-O(\epsilon)]t}}{\frac{\lambda}{D-O(\epsilon)}e^{[D-O(\epsilon)]t} + C_1} < I(t) < \frac{e^{[D+O(\epsilon)]t}}{\frac{\lambda}{D+O(\epsilon)}e^{[D+O(\epsilon)]t} + C_2}.$$

First notice that if $D > 0$, then

$$\lim_{t \rightarrow \infty} I(t) = \left(K - \frac{\nu}{b} \right) \left(1 - \frac{\nu}{\beta} \right) - \frac{\beta}{\lambda} \quad \text{and, consequently,} \quad \lim_{t \rightarrow \infty} S(t) = \frac{\beta}{\lambda}.$$

That is, the endemic equilibrium is globally stable. On the other hand, if

$$D < 0 \quad \text{i.e.} \quad \lambda < \frac{\beta}{\left(K - \frac{\nu}{b} \right) \left(1 - \frac{\nu}{\beta} \right)}, \quad \text{then} \quad \lim_{t \rightarrow \infty} I(t) = 0;$$

that is, the disease-free equilibrium is globally stable. \square

We present in Figures 7 and 8 below results from two numerical simulations for the logistic model with and without segregation into a nonreproductive group. The first example shows a logistic model in which the infection rate is sufficient to guarantee the persistence of the disease while there is no segregation to diminish the impact of infection. The second example, a segregated logistic model with the nonreproductive rates chosen so that the infection rate λ satisfies the condition (13), shows the extinction of the infected class with the persistence of the segregated. In both examples, we took the birth rate corresponding to the US 2000 census $\beta = 0.01442$. The other parameter values are chosen for reference only to illustrate our result.

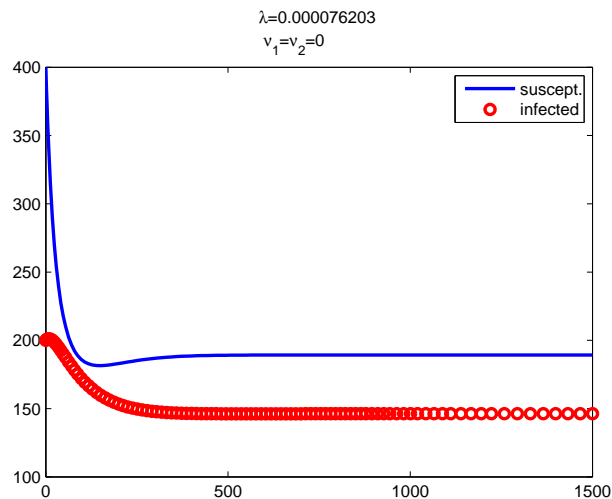


FIGURE 7. Case $\beta/K < \lambda$, no segregation.

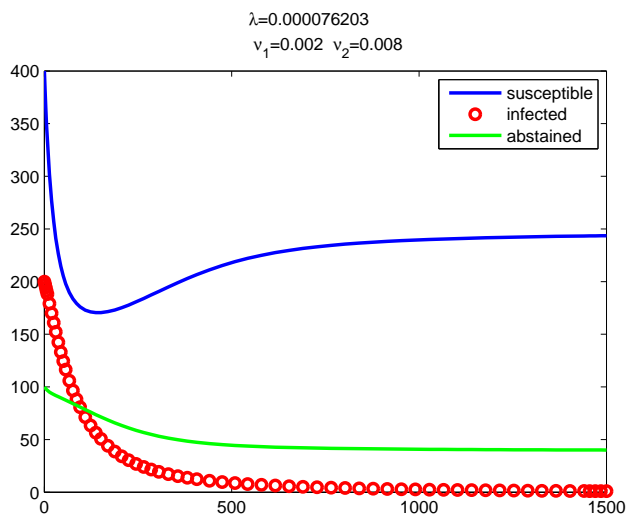


FIGURE 8. Case $\beta/K < \lambda < \frac{\beta - \nu_1 + \nu_2}{(K - \frac{\nu_1}{b})(1 - \frac{\nu_1}{\beta})}$.

Remark: If $\nu_1 \neq \nu_2$, then the interior steady state is in general impossible to compute explicitly. We conjecture that the system (10) has a similar behavior if we impose the same conditions on λ ; namely

$$\frac{\beta}{K} < \lambda < \frac{\beta - \nu_1 + \nu_2}{(K - \frac{\nu_1}{b})(1 - \frac{\nu_1}{\beta})}. \quad (13)$$

6. Conclusions. We modified two basic models (the exponential Malthusian growth and the logistic demographic models) to include an infected and a nonreproductive class. We aimed to study the effect of the abstinence (nonreproductive) class on the general evolution of the population and on the evolution of the disease in particular. Therefore, we considered just a simple S-I type epidemic model, without recovery and with no disease-induced increase in mortality, and we added to it a third class of individuals, A, who abstain from sexual contact (thus being nonreproductive and separated from the disease dynamics). Just as we might expect, the nonreproductive group can slow down or reverse the exponential growth of the total population. We also found that under some conditions on the parameters, the nonreproductive group can eliminate the disease while keeping the healthy population at positive levels. One limitation of our models is the assumption that the nonreproductive groups do not contact the disease which in turn would imply that they refrain from sexual activity. This is, of course, not true in general.

When $\nu_1 = 0$ our $S - I - A$ model resembles the classical $S - I - R$ epidemic model with recovery rate ν_2 but it has a substantial difference: individuals in R do not reproduce. For $\nu_1 > 0$ our model resembles the $S - I - R$ epidemic model with vaccination, where ν_1 is the vaccination rate, but again with the same essential difference concerning births.

For the exponential model we found first that when the Malthusian rate is less than both isolation rates, the population goes asymptotically extinct for lack of sufficient reproduction. We also found that when it is larger than both of them,

the total population grows unboundedly, with the proportions of susceptible (uninfected) and infected individuals stabilizing. When the Malthusian growth rate lies between the two isolation rates, then total population may grow or decline exponentially to extinction. Assuming that the population grows exponentially in the absence of disease ($r > \nu_1$), there are two thresholds for the infection rate: being below the lower one assures disease extinction, while being above the upper one assures disease prevalence; when the infection rate is between the two thresholds the number of infected grows exponentially but the proportion of them in the population decreases to zero. The somewhat unexpected result we found is that, even though the disease does not increase mortality, it may lead to population extinction in the case of a sufficiently large isolation rate of infected, if at the same time the infection rate is large enough that it causes a depletion of the susceptibles.

For the logistic model, we find exactly the same result as for the exponential model concerning the population going extinct for lack of sufficient reproduction. However, in this case we also find that when the Malthusian rate $r = \beta - \mu$ is less than the isolation rate for the healthy group, the trivial steady state is locally asymptotically stable, and it is unstable when r exceeds that isolation rate. Moreover, in such case there exists a unique disease-free equilibrium that is locally asymptotically stable if the *per capita* infection rate lies in a certain interval that guarantees enough contagion for the disease to persist in the absence of isolation of infected individuals, but not large enough for it to persist in the presence of isolation. When the latter condition is violated a unique endemic equilibrium exists, for which we were not able to establish stability in general. However, in the case that the isolation rates from the groups of healthy and infected individuals are equal, we were able to show it is, in fact, globally stable.

We see an important difference between the long-term persistence of the disease in the exponential and logistic cases. Since the demographic model of preference for long-term projections is the logistic one, we see from our results that if the contact rate between healthy and infected individuals is sufficiently large, the disease cannot be eliminated just by isolating individuals without curtailing the normal development of the population and leading it to extinction. This conclusion, of course, may have an impact in the recommended action to contain such a disease.

A more accurate description of the problem analyzed here can be achieved by using demographic two-sex models that include couple formation between one infected and one healthy person as a source for the group of infected couples who in turn, by separation or death of one partner, provide a source for the newly single infected people (since infection of the partner is assumed instantaneous). The resulting dynamical systems are much more complex since they involve more subclasses and interactions, as well as nonlinear (and possibly degenerate) marriage functions. This research is presently underway and will be reported later. Other ways to extend the models we presented to provide more realistic ones include modeling treatment, recovery, and other transitions from one subclass to another (e.g. from healthy isolated to healthy sexually active). We expect to begin that research in the near future.

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