What's in your wallet?!

Prof. Pudwell

MathPath plenary
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The Coin Keeper

What percentage of coins in the jar are pennies?

Assumptions...

1. The fractional parts of prices are distributed uniformly between 0 and 99 cents.

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1. The fractional parts of prices are distributed uniformly between 0 and 99 cents.
2. Cashiers give change in a predictable way.

Making change

Make change for....

- 4 cents:

Making change

Make change for....

- 4 cents: 

Making change

Make change for....

- 4 cents: 
- 6 cents:

Making change

Make change for....

- 4 cents: 
- 6 cents: 

  or 

Making change

Make change for....

- 4 cents: 
- 6 cents: 

  or 
- 41 cents:

Making change

Make change for....

- 4 cents: 
- 6 cents: 

  or 
- 41 cents: 

  or 

  or 

  or 

  or.... (27 other ways)

That's greedy!

How to give c cents in change:

1. Give q quarters where $25q \leq c < 25(q+1)$.
2. Give d dimes where $10d \leq c - 25q < 10(d+1)$.
3. Give n nickels where $5n \leq c - 25q - 10d < 5(n+1)$.
4. Give p pennies where $p = c - 25q - 10d - 5n$.

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Example: 47 cents:

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Example: 47 cents: 

Is this really the most efficient way to make change?

In another world...

Make change for 6 cents using 1-cent, 3-cent, and 4-cent coins,....

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Greedy: $4 + 1 + 1 = 6$

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Greedy: $4 + 1 + 1 = 6$
vs.
Most efficient: $3 + 3 = 6$

In another world...

Make change for 6 cents using 1-cent, 3-cent, and 4-cent coins,....

Greedy: $4 + 1 + 1 = 6$
vs.
Most efficient: $3 + 3 = 6$

But sometimes greedy *is* best!
David Pearson, A polynomial-time algorithm for the change-making problem, *Operations Research Letters* **33** (2005), 231–234.

The Coin Keeper

Change from...

- \$1.00 is nothing
- \$0.99 is 1 penny
- ⋮
- \$0.76 is 2 dimes, 4 pennies
- ⋮

The Coin Keeper

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- $0.99 is 1 penny
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- $0.76 is 2 dimes, 4 pennies
- ⋮

Change from all 100 transactions is

- 150 quarters (31.9%)
- 80 dimes (17%)
- 40 nickels (8.5%)
- 200 pennies (42.6%)

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except when he spends $5.20 cash on a latte.
What does his wallet look like?

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Start: 0 cents

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Start: 0 cents
Then: 100-20 = 80 cents

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What does his wallet look like?

Start: 0 cents
Then: 100-20 $=$ 80 cents
Then: 80-20 $=$ 60 cents

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Then: 100-20 = 80 cents
Then: 80-20 = 60 cents
Then: 60-20 = 40 cents

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Then: 40-20 = 20 cents

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Eric usually uses his debit card...
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What does his wallet look like?

Start: 0 cents
Then: 100-20 = 80 cents
Then: 80-20 = 60 cents
Then: 60-20 = 40 cents
Then: 40-20 = 20 cents
Then: 20-20= 0 cents
...and repeat!

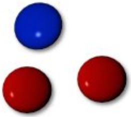
1. The Coin Keeper
2. The Simple Spender
3. The Gamblers
4. The Small Spender
5. The Whole Shebang
6. ...and More Fun

On the planet *Markovia*, coins aren't for spending money.
They change colors and they're for playing the lottery.

- Original coin has a 50% chance of being red, 50% chance of being blue.
- For every round of the lottery,
  - 1/3 of red coins turn blue.
  - 3/4 of blue coins turn red.
  - After 10 rounds, all the players with blue coins share the prize.

Shorthand:

	red	*blue*
start	$\frac{1}{2}$	$\frac{1}{2}$
red	$\frac{2}{3}$	$\frac{1}{3}$
blue	$\frac{3}{4}$	$\frac{1}{4}$



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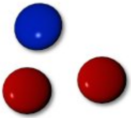

Question: If I have a blue coin now, what's the probability that it will be red in the next round, and blue in the round after that?

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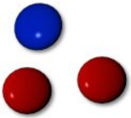

Question: If I have a blue coin now, what's the probability that it will be red in the next round, and blue in the round after that?

Answer: $\frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4} = 0.25$

Shorthand:

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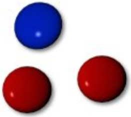
red	$\frac{2}{3}$	$\frac{1}{3}$
blue	$\frac{3}{4}$	$\frac{1}{4}$



Question: What is the probability of starting with a blue coin and having it stay blue for all 10 rounds?

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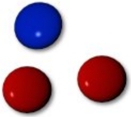

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Answer: $\frac{1}{2} \cdot \left(\frac{1}{4}\right)^{10} = \frac{1}{2097152} \approx .0000004768371582$

Shorthand:

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red	$\frac{2}{3}$	$\frac{1}{3}$
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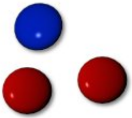

Question: If I play the lottery, what's the probability that I'll have a blue coin at the end of 10 rounds?

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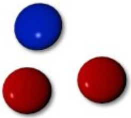

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Answer: Markov chains!

Side note: Matrix Multiplication

We have:

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start	$\frac{1}{2}$	$\frac{1}{2}$
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Represent this with two matrices:
Initial state matrix: $v^{(0)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
Transition probability matrix: $P = \begin{pmatrix} 2/3 & 1/3 \\ 3/4 & 1/4 \end{pmatrix}$

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Here's how to multiply a 1×2 matrix times a 2×2 matrix:

$$\begin{pmatrix} A & B \end{pmatrix} \times \begin{pmatrix} C & D \\ E & F \end{pmatrix} = \begin{pmatrix} ? & ? \end{pmatrix}$$

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$$\begin{aligned} v^{(1)} &= \begin{pmatrix} \left(\frac{1}{2} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{3}{4}\right) & \left(\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4}\right) \end{pmatrix} \\ &= \begin{pmatrix} \frac{17}{24} & \frac{7}{24} \end{pmatrix} \approx \begin{pmatrix} 0.7083 & 0.2917 \end{pmatrix} \end{aligned}$$

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$$v^{(2)} = v^{(0)}P^2 = v^{(1)}P \approx \begin{pmatrix} 0.6910 & 0.3090 \end{pmatrix}$$

and

$$v^{(10)} = v^{(0)}P^{10} \approx \begin{pmatrix} 0.6923 & 0.3077 \end{pmatrix}$$

After 10 rounds, you have a 30.77% chance of winning the Markovian lottery!

Markov chain behaviors:

1. *absorbing* – there are states where you can get stuck for forever.
2. *cyclic* – there exist some states where you cycle between them for forever.
3. *regular* – for some positive integer n, P^n has no zero entries.

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The Markovian lottery is *regular*.

Question: What if we played the Markovian lottery for infinitely many rounds?

For regular Markov chains

- Have transition probability matrix P.
- Want long term probability matrix L of ending up in each state.

Big idea: $LP = L$ (and the entries in L sum to 1.)

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Solve:

- $2/3p_r + 3/4p_b = p_r$
- $1/3p_r + 1/4p_b = p_b$
- $p_r + p_b = 1$

$$p_r = \frac{9}{13} \approx 0.6923, p_b = \frac{4}{13} \approx 0.3077$$

Markov chains for coins

In the land of *simplicity* there are 25-cent and 50-cent coins.
All prices end in 0, 25, 50, or 75 cents.

Possible wallet states?

charged / start	0	25	50	75
empty				

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{25}				
{50}				
{25,50}				

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,50}	{25,50}	{50}	{25}	empty

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{50}	{50}	{25}	empty	{25,50}
{25,50}	{25,50}	{50}	{25}	empty

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,25}				
{25,50}	{25,50}	{50}	{25}	empty

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,25}	{25,25}	{25}	empty	{25,25,25}
{25,50}	{25,50}	{50}	{25}	empty

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,25}	{25,25}	{25}	empty	{25,25,25}
{25,50}	{25,50}	{50}	{25}	empty

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,25}	{25,25}	{25}	empty	{25,25,25}
{25,50}	{25,50}	{50}	{25}	empty

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,25}	{25,25}	{25}	empty	{25,25,25}
{25,50}	{25,50}	{50}	{25}	empty
{25,25,25}	{25,25,25}	{25,25}	{25}	empty

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{25}	{25}	empty	{25,50}	{25,25}
{50}	{50}	{25}	empty	{25,50}
{25,25}	{25,25}	{25}	empty	{25,25,25}
{25,50}	{25,50}	{50}	{25}	empty
{25,25,25}	{25,25,25}	{25,25}	{25}	empty

Simplicity wallet states

(empty), (25), (50), (25, 25), (50, 25), (25, 25, 25)

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 1/4 & 1/4 & 1/4 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 0 & 1/4 \end{pmatrix}$$

Simplicity in the long run

We want: $L = \begin{bmatrix} p_{(empty)} & p_{(25)} & p_{(50)} & p_{(25,25)} & p_{(50,25)} & p_{(25,25,25)} \end{bmatrix}$

Setup: $LP = L$

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Setup: $LP = L$

$$
\begin{array}{llllllll}
\frac{p(empty)}{4} & +\frac{p(25)}{4} & +\frac{p(50)}{4} & +\frac{p(25,25)}{4} & +\frac{p(50,25)}{4} & +\frac{p(25,25,25)}{4} & = p(empty) \\
\frac{p(empty)}{4} & +\frac{p(25)}{4} & +\frac{p(50)}{4} & +\frac{p(25,25)}{4} & +\frac{p(50,25)}{4} & +\frac{p(25,25,25)}{4} & = p(25) \\
\frac{p(empty)}{4} & & +\frac{p(50)}{4} & & +\frac{p(50,25)}{4} & & = p(50) \\
 & \frac{p(25)}{4} & & +\frac{p(25,25)}{4} & & +\frac{p(25,25,25)}{4} & = p(25,25) \\
\frac{p(empty)}{4} & +\frac{p(25)}{4} & +\frac{p(50)}{4} & & +\frac{p(50,25)}{4} & & = p(50,25) \\
 & & & \frac{p(25,25)}{4} & & +\frac{p(25,25,25)}{4} & = p(25,25,25) \\
p(empty) & + p(25) & + p(50) & + p(25,25) & + p(50,25) & + p(25,25,25) & = 1
\end{array}
$$

Simplicity in the long run

We want: $L = \left[\; p(\mathit{empty}) \quad p(25) \quad p(50) \quad p(25{,}25) \quad p(50{,}25) \quad p(25{,}25{,}25) \;\right]$

Setup: $LP = L$

$$
\begin{array}{llllll}
\frac{p(empty)}{4} + \frac{p(25)}{4} & + \frac{p(50)}{4} + \frac{p(25,25)}{4} & + \frac{p(50,25)}{4} + \frac{p(25,25,25)}{4} & = p(empty) \\
\frac{p(empty)}{4} + \frac{p(25)}{4} & + \frac{p(50)}{4} + \frac{p(25,25)}{4} & + \frac{p(50,25)}{4} + \frac{p(25,25,25)}{4} & = p(25) \\
\frac{p(empty)}{4} & + \frac{p(50)}{4} & + \frac{p(50,25)}{4} & = p(50) \\
\frac{p(25)}{4} & + \frac{p(25,25)}{4} & + \frac{p(25,25,25)}{4} & = p(25,25) \\
\frac{p(empty)}{4} + \frac{p(25)}{4} & + \frac{p(50)}{4} & + \frac{p(50,25)}{4} & = p(50,25) \\
 & \frac{p(25,25)}{4} & + \frac{p(25,25,25)}{4} & = p(25,25,25) \\
p(empty) + p(25) & + p(50) + p(25,25) & + p(50,25) + p(25,25,25) & = 1
\end{array}
$$

Solve to get: $\left[\; \frac{1}{4} \quad \frac{1}{4} \quad \frac{5}{32} \quad \frac{3}{32} \quad \frac{7}{32} \quad \frac{1}{32} \;\right]$

or $\left[\; 0.25 \quad 0.25 \quad 0.15625 \quad 0.09375 \quad 0.21875 \quad 0.03125 \;\right]$

More assumptions

1. The fractional parts of prices are distributed uniformly between 0 and 99 cents.
2. Cashiers return change using the greedy algorithm.

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5. If there are multiple ways to overpay as little as possible, the spender favors spending a bigger coin over a smaller coin.

What's (the most) in your wallet?

- If you have at most 99 cents before a transaction, you'll have at most 99 cents after.
  - Case 1: (price $\leq$ wallet):
    You pay, and have less money in your wallet.

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  - Case 1: (price $\leq$ wallet):
    You pay, and have less money in your wallet.
  - Case 2: (price $>$ wallet):
    You get $(100 - p)$ in change, and end up with
    $(100 - p) + w = 100 - (p - w) < 100$.

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To get from any wallet to the empty wallet, imagine you have exact change.

To get from empty wallet to $\{p$ pennies , n nickels , d dimes , q quarters$\}$, imagine:

- q 75 cent charges
- d 90 cent charges
- n 95 cent charges
- p 99 cent charges

Counting states

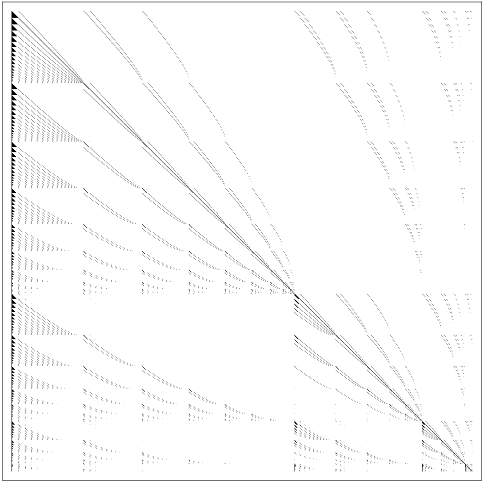
Known: Must have at most 99 cents. In other words, at most...

- 99 pennies
- 19 nickels
- 9 dimes
- 3 quarters

$$100 \times 20 \times 10 \times 4 = 80,000 \text{ possible states.}$$

... but that's overkill.
There are **6720** combinations of coins with at most 99 cents.

That's one big matrix...



Goal: Find L where $LP = L$.

25 CPU hours later...

Wallet state	p_{state}	Wallet state	p_{state}
0 pennies	.01000	{25, 1, 1, 1}	.00453
1 penny	.01000	{5, 1, 1, 1}	.00448
2 pennies	.01000	{10, 5, 1, 1, 1, 1}	.00439
3 pennies	.01000	{25, 1, 1}	.00429
4 pennies	.01000	{10, 1, 1, 1}	.00420
5 pennies	.00813	{10, 1, 1, 1, 1, 1}	.00414
6 pennies	.00732	{25, 1}	.00405
7 pennies	.00644	{10, 1, 1, 1, 1, 1, 1}	.00391
8 pennies	.00551	{25}	.00379
{5, 1, 1, 1, 1}	.00543	{10, 5, 1, 1, 1, 1, 1, 1}	.00377
{25, 1, 1, 1, 1}	.00475	{25, 1, 1, 1, 1, 1}	.00376
{10, 1, 1, 1, 1}	.00467	{10, 5, 1, 1, 1, 1, 1}	.00375
9 pennies	.00456	{5, 1, 1, 1, 1, 1}	.00374

In case you were wondering...

- Expected number of coins in your wallet: 10.04
  - Expected number of quarters: 1.06 (10.6%)
  - Expected number of dimes: 1.15 (11.4%)
  - Expected number of nickels: 0.91 (9.1%)
  - Expected number of pennies: 6.92 (68.9%)

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  - Expected number of pennies: 6.92 (68.9%)
- Probability of empty wallet: 0.01
- Probability of having at least one nickel: 0.58085
- Probability of having at least one penny: 0.95975
- Probability of having only pennies (and a non-empty wallet): 0.08430
- Probability of being able to pay any price with exact change: 0.00831

And that's not all!

Some (common?) variations

- The pennyless purchaser
- The quarter hoarder
- The pennies-first spender
- The Shallit currency

And that's not all!

Some (common?) variations

- The pennyless purchaser (5, 10, and 25-cent pieces)
- The quarter hoarder (1, 5, and 10-cent pieces)
- The pennies-first spender (1, 5, 10, and 25-cent pieces)
- The Shallit currency (1, 5, 18, and 25-cent pieces)

And that's not all!

Some (common?) variations

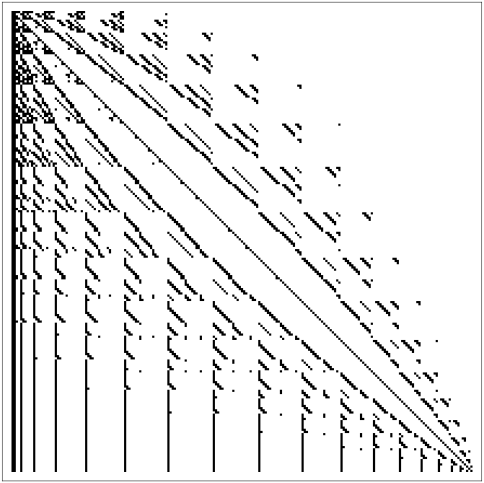
- The pennyless purchaser (5, 10, and 25-cent pieces) Go
- The quarter hoarder (1, 5, and 10-cent pieces) Go
- The pennies-first spender (1, 5, 10, and 25-cent pieces) Go
- The Shallit currency (1, 5, 18, and 25-cent pieces) Go

Jeffrey Shallit, What this country needs is an 18¢ piece, *The Mathematical Intelligencer* **25** (2003) 20–23.

Go

Pennyless purchaser

213 states



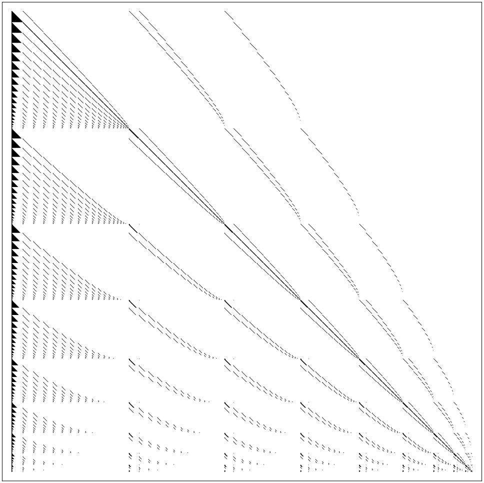
Pennyless purchaser results

Wallet state	p_{pp}	Wallet state	p_{pp}
{}	.05000	14 nickels	1.29×10^{-11}
{5}	.05000	2 dimes and 15 nickels	3.37×10^{-12}
{10, 5}	.03916	1 dime and 15 nickels	2.28×10^{-12}
{25, 10, 5}	.03093	15 nickels	9.90×10^{-13}
{25, 5}	.02847	1 dime and 16 nickels	1.76×10^{-13}
{10, 5, 5}	.02731	16 nickels	6.23×10^{-14}
{25, 25, 10, 5}	.02625	1 dime and 17 nickels	1.27×10^{-14}
{5, 5}	.02536	17 nickels	3.96×10^{-15}
{10}	.02463	18 nickels	2.09×10^{-16}
{25, 10, 5, 5}	.02417	19 nickels	1.10×10^{-17}

Go

Quarter hoarder

4125 states



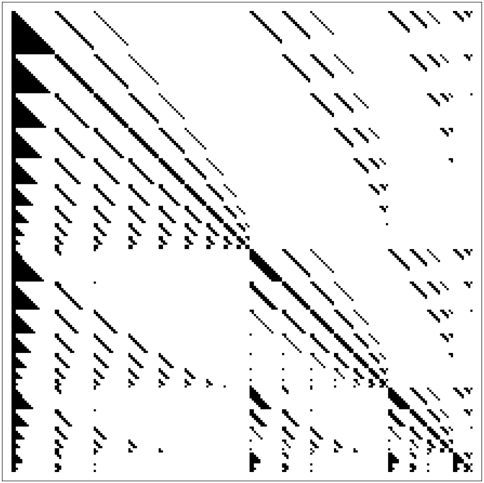
Quarter hoarder results

Wallet state	p_{qh}	**Wallet state**	p_{qh}
{1, 1, 1, 1}	.01164	10 pennies	.00713
{1, 1, 1}	.01129	{10,5,1,1,1,1,1,1,1,1}	.00651
{1, 1}	.01095	{10, 5, 1, 1, 1, 1, 1, 1, 1}	.00642
5 pennies	.01084	11 pennies	.00638
{1}	.01062	{10, 5, 1, 1, 1, 1, 1, 1, 1, 1, 1}	.00637
6 pennies	.01039	{10, 5, 1, 1, 1, 1, 1, 1}	.00614
{}	.01030	{10, 5, 1, 1, 1, 1, 1}	.00569
7 pennies	.00984	12 pennies	.00564
8 pennies	.00919	{10, 5, 1, 1, 1, 1}	.00549
9 pennies	.00844	{10, 1, 1, 1, 1, 1, 1}	.00523

Go

Pennies-first spender

1065 states



Pennies-first results

Expected pennies-first coins in your wallet: 5.74

- Expected quarters: 1.12
- Expected dimes: 1.27
- Expected nickels: 1.35
- Expected pennies: 2.00

Expected number of coins in your wallet: 10.04

- Expected quarters: 1.06
- Expected dimes: 1.15
- Expected nickels: 0.91
- Expected pennies: 6.92

Go

The Shallit currency

Idea: replacing a dime with an 18-cent coin minimizes coins used per transaction

Two catches:

- Greedy algorithm isn't always best!

  Example: 28 cents
  Greedy: 25+1+1+1
  Efficient: 18+5+5

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  Example: 77 cents
  $25 + 25 + 25 + 1 + 1 = 77$
  $18 + 18 + 18 + 18 + 5 = 77$

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  $25 + 25 + 25 + 1 + 1 = 77$
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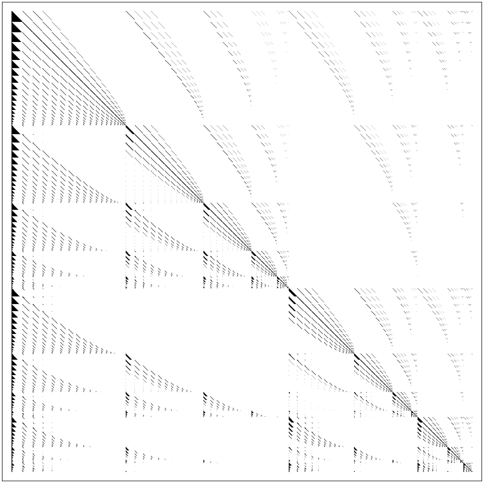
Assumptions:

- Spenders: still break ties by using bigger coins.
- Cashiers: break ties by using each "best" change equally often.

The Shallit currency

4238 states



Shallit currency results

Expected Shallit coins in your wallet: 8.63

- Expected quarters: 0.66
- Expected 18-cents: 0.98
- Expected nickels: 2.10
- Expected pennies: 4.89

Expected number of coins in your wallet: 10.04

- Expected quarters: 1.06
- Expected dimes: 1.15
- Expected nickels: 0.91
- Expected pennies: 6.92

Go

Cashing in...

I sometimes think that the best way to change the public attitude to math would be to stick a red label on everything that uses mathematics. "Math inside." There would be a label on every computer, of course, and I suppose if we were to take the idea literally, we ought to slap one on every math teacher. But we should also place a red math sticker on every airline ticket, every telephone, every car, every airplane, every traffic light, every vegetable...

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More details at...

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Thanks for listening!