Counting Pattern-Avoiding Permutations

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Wabash College Math & CS Colloquium
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Outline

Permutations

A permutation of length n is a string of numbers using each of $1, \ldots, n$ exactly once.

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A permutation of length n is a string of numbers using each of $1, \ldots, n$ exactly once.

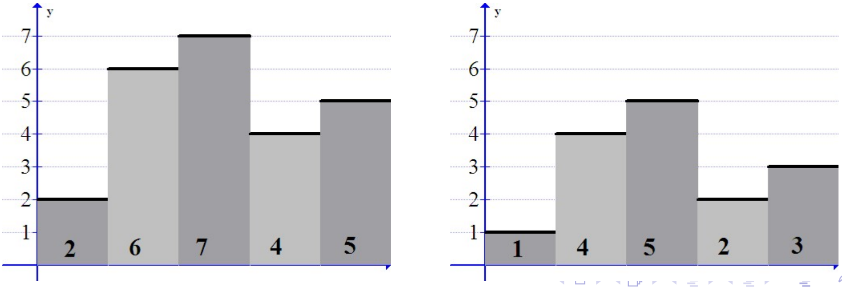
Write S_n for the set of permutations of length n.
For example:

- $S_1 = \{1\}$
- $S_2 = \{12, 21\}$
- $S_3 = \{123, 132, 213, 231, 312, 321\}$

Reduction

Given a string of numbers $q = q_1 \cdots q_n$,
the reduction of q is the string obtained by replacing the i^{th} smallest number of q with i.

For example, the reduction of 26745 is 14523.



Pattern Avoidance/Containment

Given permutations $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$,

- p contains q as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $p_{i_1} \cdots p_{i_m}$ reduces to q;
- otherwise p avoids q.

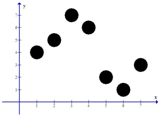
For example,

- 4576213 contains 312 (4576213).
- 4576213 avoids 1234.

Permutations as Functions

We can also think of a permutation as a function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$$p = 4576213$$



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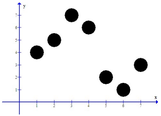
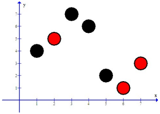
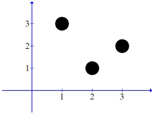


Then, permutation p contains permutation q if the graph of p contains the graph of q.

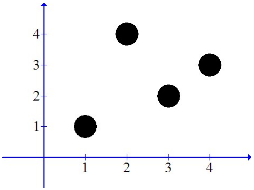




4576213 contains 312.

Pattern Avoidance in Permutations

Easier Question: Fix p. What patterns are contained in p?
For example, $p = 1423$



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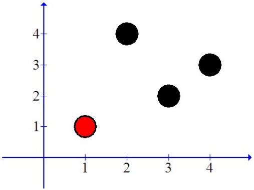

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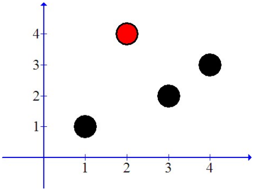

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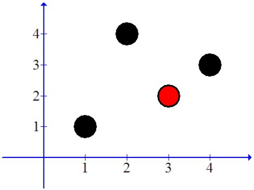

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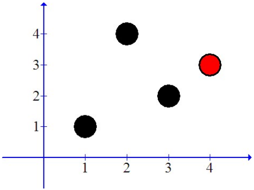

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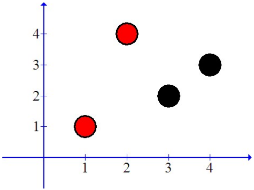

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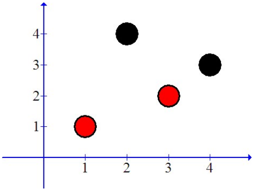

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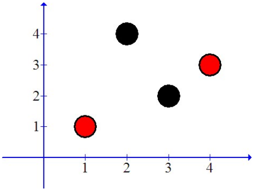

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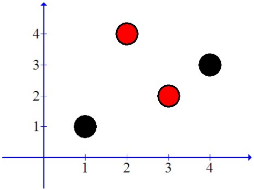

1423 contains the patterns 1, 12, 21.

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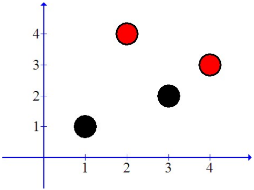

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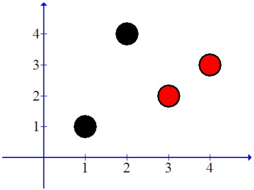

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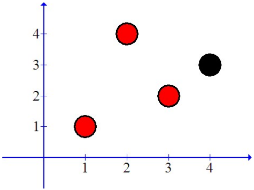

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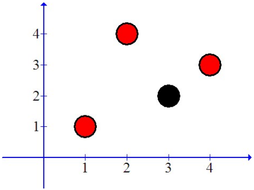

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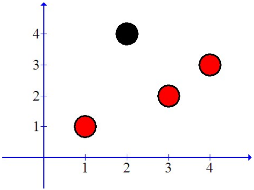

1423 contains the patterns 1, 12, 21, 132, 123.

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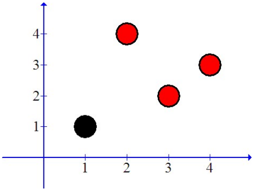

1423 contains the patterns 1, 12, 21, 132, 123, 312.

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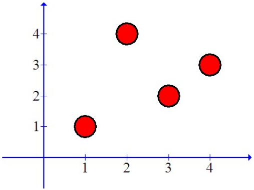

1423 contains the patterns 1, 12, 21, 132, 123, 312, and 1423, and avoids all other patterns.

Pattern Avoidance in Permutations

Harder Question: Fix a pattern q.
Enumerate $S_n(q) := \{p \in S_n \mid p \text{ avoids } q\}$.

Avoiding a Pattern of Length 1

There is one pattern of length 1: 1.

What is $|S_n(1)|$?

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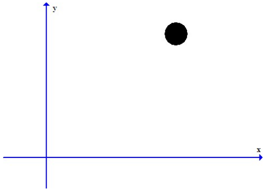

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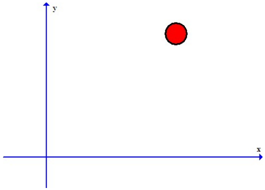

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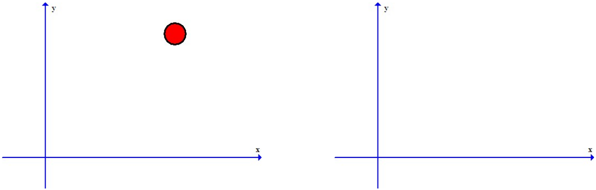

$$|S_n(1)| = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1. \end{cases}$$

Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

What is $|S_n(12)|$?

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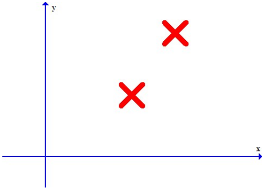

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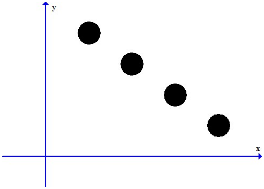

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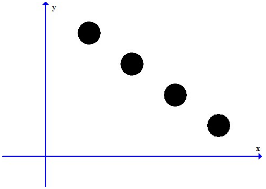

$|S_n(12)| = 1$ (for $n \geq 0$).

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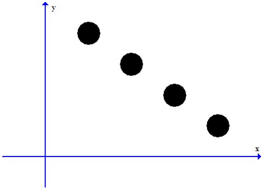

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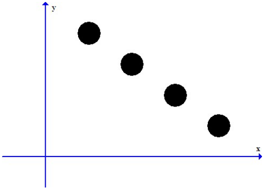

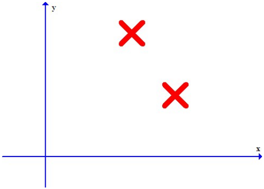

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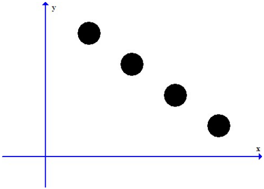

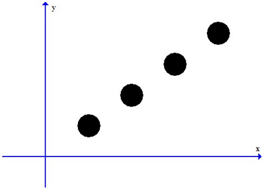

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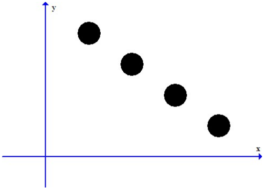
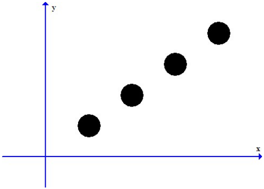
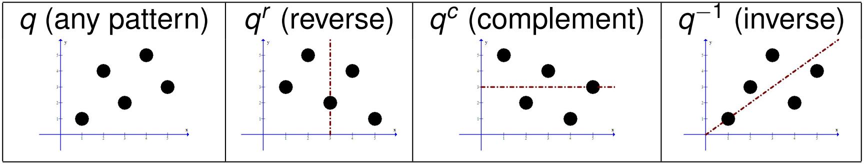




$|S_n(12)| = |S_n(21)| = 1$ (for $n \geq 0$).

Useful Observation (Wilf Equivalence)



For any pattern q, we have:

$$|S_n(q)| = |S_n(q^r)| = |S_n(q^c)| = \left|S_n(q^{-1})\right|$$

Avoiding a Pattern of Length 3

There are six patterns of length 3:
123, 132, 213, 231, 312, 321.

Using the useful observation, we have
$|S_n(123)| = |S_n(321)|$ and
$|S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|$.

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$|S_n(123)| = |S_n(132)|$ (Simion and Schmidt, 1985).

Avoiding the pattern 132

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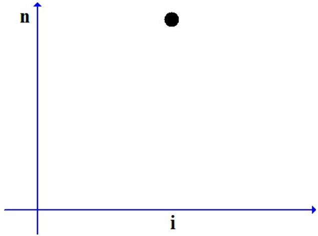

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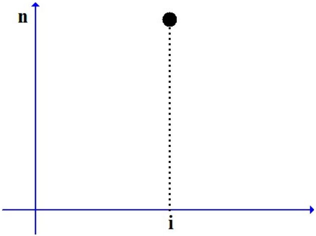

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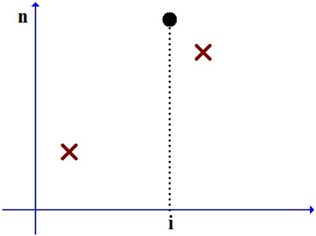

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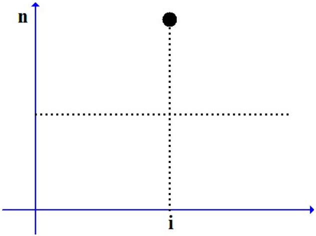

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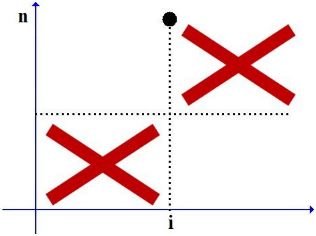

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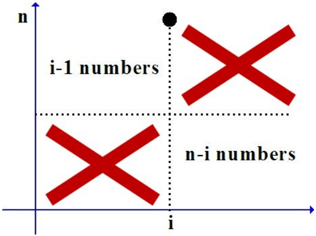

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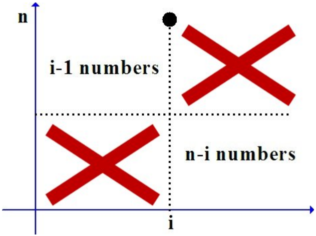

$$|S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \text{ (for } n > 0)$$

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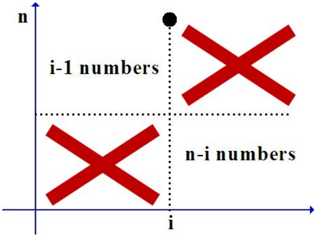

$$|S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \text{ (for } n > 0)$$

$$|S_n(132)| = \frac{\binom{2n}{n}}{n+1} = n\text{th Catalan number}$$

Avoiding a Pattern of Length 4

There are 24 patterns of length 4.

Using the useful observation and similar bijections, we can narrow our work to 3 cases:
$S_n(1342), S_n(1234),$ and $S_n(1324)$.

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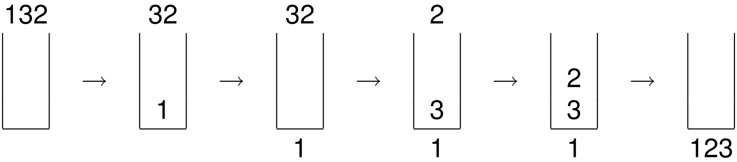
	1	2	3	4	5	6	7	8	
$\|S_n(1342)\|$	1	2	6	23	103	512	2740	15485	$\sim 8^n$
$\|S_n(1234)\|$	1	2	6	23	103	513	2761	15767	$\sim 9^n$
$\|S_n(1324)\|$	1	2	6	23	103	513	2762	15793	$\sim 9.3^n$

Stack Sorting

A stack is a *last in, first out* data structure.



Question: What permutations can be sorted by passing them through the stack exactly one time?

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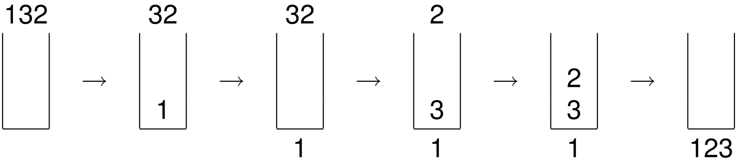
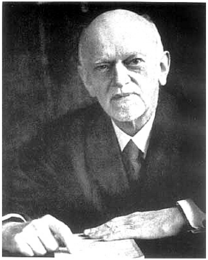


Question: What permutations can be sorted by passing them through the stack exactly one time?

Answer: (Knuth, 1973) Exactly the permutations that avoid 231.

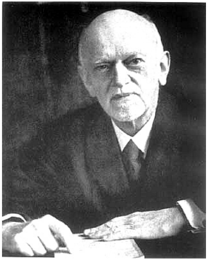
Schubert Varieties



- 1900: Hilbert's 15th problem is to find a "rigorous foundation of Schubert's enumerative calculus".
- 1900s: In algebraic geometry, Schubert varieties are the most commonly studied type of singular variety.

Question: Which Schubert varieties are smooth?

Schubert Varieties



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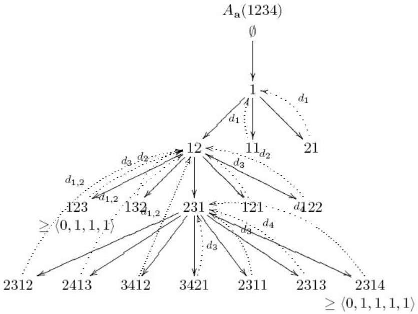
Question: Which Schubert varieties are smooth?

Answer: (Lakshmibai and Sandhya, 1990): Exactly the varieties whose indexing permutation avoids 4231 and 3412.

Experimental Mathematics

- The techniques to count $|S_n(q)|$ usually depend on q.
- Goal: Find an algorithm to count $|S_n(q)|$ that works well regardless of what q is.



Avoiding Sets of Patterns

We may consider permutations which simultaneously avoid more than one pattern:

$$S_n(Q) := \{p \in S_n \mid p \text{ avoids } q \text{ for every } q \in Q\}.$$

Some particularly nice results include:

- $S_n(\{123, 132\}) = 2^{n-1}$
- $S_n(\{132, 213, 321\}) = n$
- $S_n(\{123, 132, 213\}) = F_n$ (Fibonacci numbers)

Generalized and Distanced Patterns

Consider patterns where there may or may not be a dash between each pair of numbers.
E.g. $3-12$

A dash indicates those two numbers can be arbitrarily far apart, no dash indicates they must be adjacent.
E.g. 241653 contains $12-3$ (241653), but not $1-23$

Further, we can specify *exact* distances between numbers of a pattern.

Bar Notation

A *barred permutation pattern* is a permutation where each number may or may not have a bar over it.
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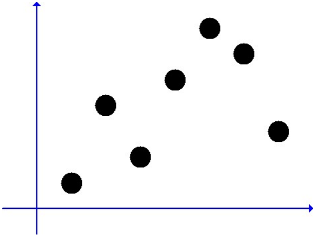
A barred pattern q encodes two permutation patterns,

1. The smaller pattern q_s formed by the unbarred numbers of q.
   (in this case, 542 forms a 321 pattern.)
2. The larger pattern q_ℓ formed by all numbers of q.
   (in this case, 31542.)

Bar Notation

We say that permutation p avoids the barred pattern q iff every copy of q_s in p is part of a copy of q_ℓ in p.

Example: $q = \overline{31}542$



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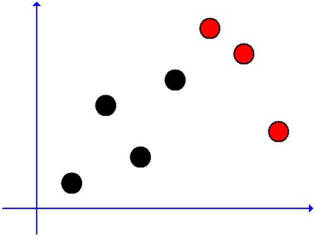

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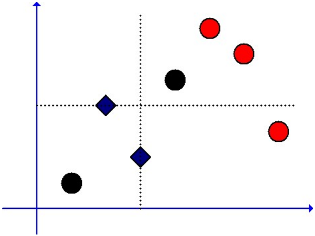

Barred Pattern Avoidance

Two friendly examples:

- $S_n(\bar{1}32) = (n-1)!$
- $S_n(1\bar{4}23) = B_n$ (Bell numbers)

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Two Nice Theorems:

- West, 1990: A permutation is 2-stack sortable if and only if it avoids 2341 and $3\bar{5}241$.
- Woo and Yong, 2006: A Schubert variety X_w is locally factorial if and only if w avoids the patterns 1324 and $21\bar{3}54$.

Open Problems

- Enumeration
  - What is $S_n(1324)$?, $S_n(q)$, where $|q| \geq 5$?
  - What can you say about permutations avoiding generalized or barred patterns of length ≥ 5?

Open Problems 2

- Algebra
  - There are Wilf equivalences other than the symmetries of the square. What other equivalences can you find?
  - What Wilf equivalences carry over to subsets or subgroups of S_n?
- Asymptotics
  - Given any pattern q, there exists a constant c_q such that

$$\lim_{n\to\infty} |S_n(q)| \to c_q^n.$$

    What values of c_q are possible?

Open Problems 3

- Applications
  - (Stack Sorting) What is a characterization for 3-stack-sortable permutations?
  - (Schubert calculus, etc.) S_n is an example of a Coxeter group (a group generated by reflections). What can you say about pattern avoidance in other Coxeter groups?
  - (Experimental Mathematics) Can you find a single method that efficiently counts $|S_n(Q)|$ for many different examples of Q?

Thank You!