Counting Pattern-Avoiding Permutations

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Trinity University Mathematics Majors’ Seminar
November 19, 2009
A permutation of length $n$ is a string of numbers using each of 1, \ldots, $n$ exactly once.
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Write $S_n$ for the set of permutations of length $n$. For example:

- $S_1 = \{1\}$
- $S_2 = \{12, 21\}$
- $S_3 = \{123, 132, 213, 231, 312, 321\}$
Given a string of numbers $q = q_1 \cdots q_m$, the **reduction** of $q$ is the string obtained by replacing the $i^{th}$ smallest number of $q$ with $i$.

For example, the reduction of 26745 is 14523.
Pattern Avoidance/Containment

Given permutations $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$,

- $p$ contains $q$ as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $p_{i_1} \cdots p_{i_m}$ reduces to $q$;
- otherwise $p$ avoids $q$.

For example,

- $4576213$ contains $312$ ($4576213$).
- $4576213$ avoids $1234$. 
We can also think of a permutation as a function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. 

$$p = 4576213$$
Permutations as Functions

We can also think of a permutation as a function from \(\{1, \ldots, n\}\) to \(\{1, \ldots, n\}\).

\[ p = 4576213 \]

Then, permutation \(p\) contains permutation \(q\) if the graph of \(p\) contains the graph of \(q\).

\[ 4576213 \text{ contains } 312. \]
Easier Question: Fix $p$. What patterns are contained in $p$?
For example, $p = 1423$
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$1423$ contains the pattern $1$. 
Easier Question: Fix \( p \). What patterns are contained in \( p \)?
For example, \( p = 1423 \)

1423 contains the pattern 1.
**Pattern Avoidance in Permutations**

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1423 contains the patterns $1$, $12$. 
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1423 contains the patterns 1, 12, 21, 132, 123, 312.
**Easier Question:** Fix $p$. What patterns are contained in $p$?

For example, $p = 1423$

1423 contains the patterns 1, 12, 21, 132, 123, 312, and 1423, and avoids all other patterns.
Harder Question: Fix a pattern $q$. Enumerate $S_n(q) := \{ p \in S_n \mid p \text{ avoids } q \}$.
Avoiding a Pattern of Length 1

There is one pattern of length 1: 1.

What is $|S_n(1)|$?
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What is $|S_n(1)|$?

$$|S_n(1)| = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}.$$
Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

What is \(|S_n(12)|\)?
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$|S_n(12)| = |S_n(21)| = 1$ (for $n \geq 0$).
Useful Observation (Wilf Equivalence)

For any pattern \( q \), we have:

\[ |S_n(q)| = |S_n(q^r)| = |S_n(q^c)| = |S_n(q^{-1})| \]
Avoiding a Pattern of Length 3

There are six patterns of length 3: 123, 132, 213, 231, 312, 321.

Using the useful observation, we have

\[ |S_n(123)| = |S_n(321)| \quad \text{and} \quad |S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|. \]
Avoiding a Pattern of Length 3

There are six patterns of length 3: 123, 132, 213, 231, 312, 321.

Using the useful observation, we have
\[ |S_n(123)| = |S_n(321)| \text{ and } |S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|. \]
\[ |S_n(123)| = |S_n(132)| \text{ (Simion and Schmidt, 1985).} \]
Avoiding the pattern 132

What is $|S_n(132)|$?
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\[
|S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)||S_{n-i}(132)|
\]

For $n > 0$, $|S_n(132)| = \frac{2^n n!}{n+1}$, which is the $n$th Catalan number.
Avoiding the pattern 132

What is $|S_n(132)|$?

$$|S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \quad (\text{for } n > 0)$$
What is a pattern-avoiding permutation?

What is $|S_n(132)|$?

\[ |S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \quad \text{(for } n > 0) \]

\[ |S_n(132)| = \frac{(2n)}{n+1} = n\text{th Catalan number} \]
Avoiding a Pattern of Length 4

There are 24 patterns of length 4.

Using the useful observation and similar bijections, we can narrow our work to 3 cases: $S_n(1342)$, $S_n(1234)$, and $S_n(1324)$. 
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Using the useful observation and similar bijections, we can narrow our work to 3 cases: $S_n(1342)$, $S_n(1234)$, and $S_n(1324)$.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n(1342)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>23</td>
<td>103</td>
<td>512</td>
<td>2740</td>
<td>15485</td>
</tr>
<tr>
<td>$S_n(1234)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>23</td>
<td>103</td>
<td>513</td>
<td>2761</td>
<td>15767</td>
</tr>
<tr>
<td>$S_n(1324)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>23</td>
<td>103</td>
<td>513</td>
<td>2762</td>
<td>15793</td>
</tr>
</tbody>
</table>
A **stack** is a *last in, first out* data structure.

132 → 32 → 32 → 2

1 → 1 → 1 → 123

Question: What permutations can be sorted by passing them through the stack exactly one time?
A **stack** is a *last in, first out* data structure.

132 → 32 → 32 → 2 → 1 → 1 → 3 → 2 → 3 → 1 → 123

Question: What permutations can be sorted by passing them through the stack exactly one time?

Answer: (Knuth, 1973) Exactly the permutations that avoid 231.
1900: Hilbert’s 15th problem is to find a "rigorous foundation of Schubert’s enumerative calculus".

1900s: In algebraic geometry, Schubert varieties are the most commonly studied type of singular variety.

Question: Which Schubert varieties are smooth?
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1900s: In algebraic geometry, Schubert varieties are the most commonly studied type of singular variety.

Question: Which Schubert varieties are smooth?

Answer: (Lakshmibai and Sandhya, 1990): Exactly the varieties whose indexing permutation avoids 4231 and 3412.
The techniques to count \( |S_n(q)| \) usually depend on \( q \).

Goal: Find an algorithm to count \( |S_n(q)| \) that works well regardless of what \( q \) is.
We may consider permutations which simultaneously avoid more than one pattern:

\[ S_n(Q) := \{ p \in S_n \mid p \text{ avoids } q \text{ for every } q \in Q \}. \]

Some particularly nice results include:

- \( S_n(\{123, 132\}) = 2^{n-1} \)
- \( S_n(\{132, 213, 321\}) = n \)
- \( S_n(\{123, 132, 213\}) = F_n \) (Fibonacci numbers)
Pattern-Avoiding Words

Instead of pattern-avoiding permutations, we may consider strings with repeated letters:

\[ A[a_1, \ldots, a_k](Q) := \left\{ w \in [k]^{\sum a_i} \mid w \text{ has } a_i \text{'s, } \forall q \in Q \right\} \]

Notice \( A[1, \ldots, 1](Q) = S_n(Q) \).

\[ A[2, \ldots, 2](\{132, 231, 2134\}) = 2n^2 + 6n - 2 \]
Consider patterns where there may or may not be a dash between each pair of numbers.
E.g. 3 − 12

A dash indicates those two numbers can be arbitrarily far apart, no dash indicates they must be adjacent.
E.g. 241653 contains 12 – 3 (241653), but not 1 – 23

Further, we can specify exact distances between numbers of a pattern.
A barred permutation pattern is a permutation where each number may or may not have a bar over it. E.g. \( q = \overline{31542} \) is a barred pattern.
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A barred pattern $q$ encodes two permutation patterns,

1. The smaller pattern $q_s$ formed by the unbarred numbers of $q$. (in this case, 542 forms a 321 pattern.)
2. The larger pattern $q_\ell$ formed by all numbers of $q$. (in this case, 31542.)
We say that permutation $p$ avoids the barred pattern $q$ iff every copy of $q_s$ in $p$ is part of a copy of $q_\ell$ in $p$.

Example: $q = \overline{31542}$
We say that permutation \( p \) avoids the barred pattern \( q \) iff every copy of \( q_s \) in \( p \) is part of a copy of \( q_\ell \) in \( p \).

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Bar Notation

We say that permutation $p$ avoids the barred pattern $q$ iff every copy of $q_s$ in $p$ is part of a copy of $q_\ell$ in $p$.

Example: $q = \overline{31542}$
Barred Pattern Avoidance

Two friendly examples:
- $S_n(\overline{132}) = (n - 1)!$
- $S_n(\overline{1423}) = B_n$ (Bell numbers)
Barred Pattern Avoidance

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- \( S_n(\overline{132}) = (n - 1)! \)
- \( S_n(\overline{1423}) = B_n \) (Bell numbers)

Two Nice Theorems:
- West, 1990: A permutation is 2-stack sortable if and only if it avoids 2341 and 3\overline{5241}.
- Woo and Yong, 2006: A Schubert variety \( X_w \) is locally factorial if and only if \( w \) avoids the patterns 1324 and \( 21\overline{354} \).
Open Problems

- **Enumeration**
  - What is $S_n(1324)$?, $S_n(q)$, where $|q| \geq 5$?
  - What can you say about permutations avoiding generalized or barred patterns of length $\geq 5$?
What is a pattern-avoiding permutation?

**Enumeration**

**Motivational Interlude**

**Variations of Pattern Avoidance**

**Open Problems**

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**Open Problems 2**

- **Algebra**
  - There are Wilf equivalences other than the symmetries of the square. What other equivalences can you find?
  - What Wilf equivalences carry over to subsets or subgroups of $S_n$?

- **Asymptotics**
  - Given any pattern $q$, there exists a constant $c_q$ such that
    \[
    \lim_{n \to \infty} |S_n(q)| \to c_q^n.
    \]
  - What values of $c_q$ are possible?

Lara Pudwell

Counting Pattern-Avoiding Permutations
Open Problems 3

Applications

- (Stack Sorting) What is a characterization for 3-stack-sortable permutations?
- (Schubert calculus, etc.) $S_n$ is an example of a Coxeter group (a group generated by reflections). What can you say about pattern avoidance in other Coxeter groups?
- (Experimental Mathematics) Can you find a single method that efficiently counts $|S_n(Q)|$ for many different examples of $Q$?
Thank You!