An Introduction to Enumeration Schemes

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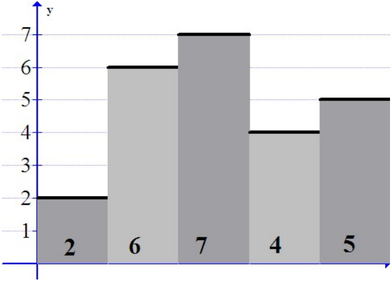
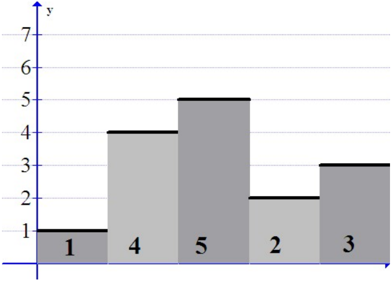
Outline

Reduction

Given a string of numbers $q = q_1 \cdots q_m$,
the reduction of q is the string obtained by replacing the i^{th} smallest number of q with i.

For example, the reduction of 26745 is 14523.





Pattern Avoidance/Containment

Given permutations $\pi = \pi_1 \cdots \pi_n$ and $q = q_1 \cdots q_m$,

- π contains q as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $\pi_{i_1} \cdots \pi_{i_m}$ reduces to q;
- otherwise π avoids q.

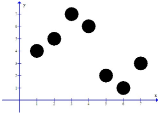
For example,

- 4576213 contains 312 (4576213).
- 4576213 avoids 1234.

Permutations as Functions

We can also think of a permutation as a function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.

$$\pi = 4576213$$



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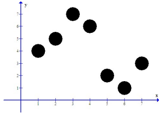
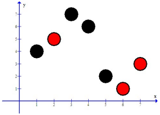
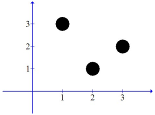


Then, permutation π contains permutation q if the graph of π contains the graph of q.





4576213 contains 312.

Two Questions

Easy: Given $\pi \in S_n$, what patterns does π contain?

Hard: Given $q \in S_m$,

- Let $S_n(q) = \{\pi \in S_n \mid \pi \text{ avoids } q\}$.
- Find an expression for $|S_n(q)|$.

Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

What is $|S_n(12)|$?

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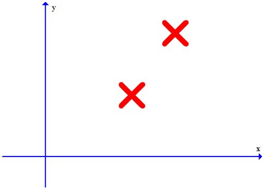

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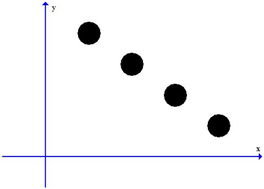

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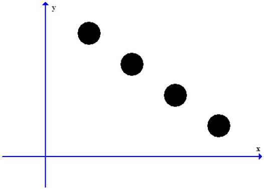

$|S_n(12)| = 1$ (for $n \geq 0$).

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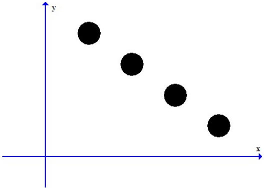

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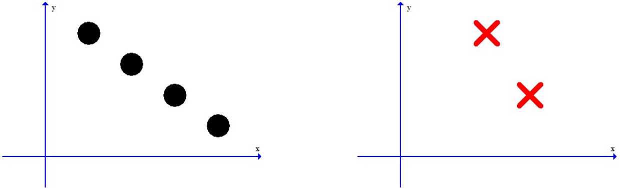

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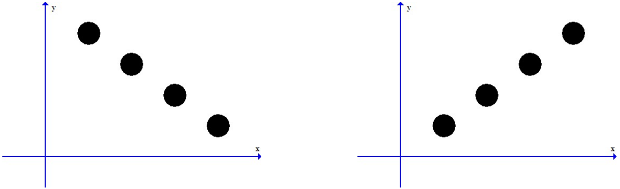

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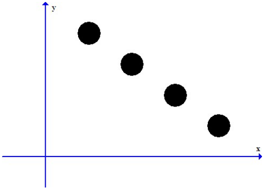
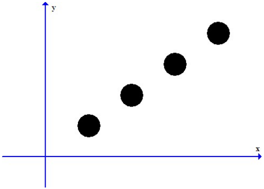
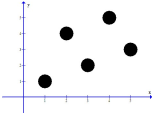
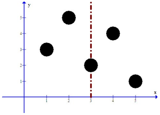
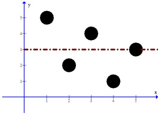
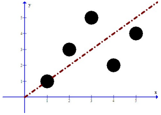




$|S_n(12)| = |S_n(21)| = 1$ (for $n \geq 0$).

Useful Observation (Wilf Equivalence)

q (any pattern)	q^r (reverse)	q^c (complement)	q^{-1} (inverse)
			

For any pattern q, we have:

$$|S_n(q)| = |S_n(q^r)| = |S_n(q^c)| = \left|S_n(q^{-1})\right|$$

Avoiding a Pattern of Length 3

There are six patterns of length 3:
123, 132, 213, 231, 312, 321.

Using Wilf equivalence, we have
$|S_n(123)| = |S_n(321)|$ and
$|S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|$.

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$|S_n(123)| = |S_n(132)|$ (Simion and Schmidt, 1985).

Avoiding the pattern 132

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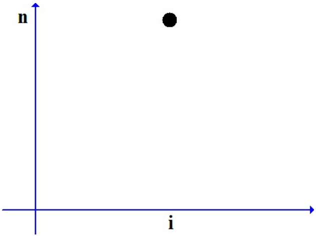

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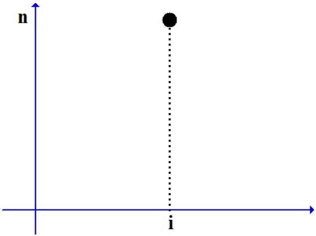

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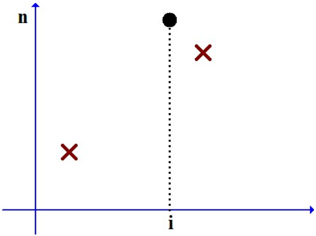

Avoiding the pattern 132

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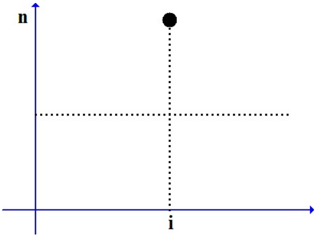

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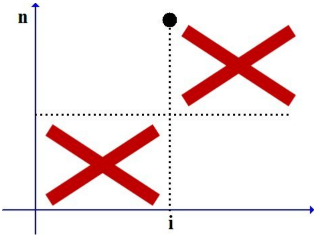

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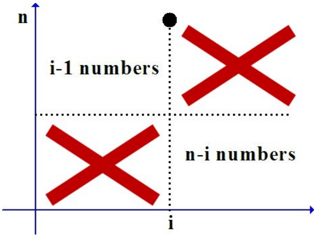

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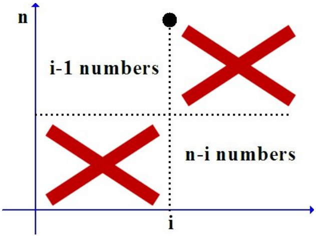

$$|S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \text{ (for } n > 0)$$

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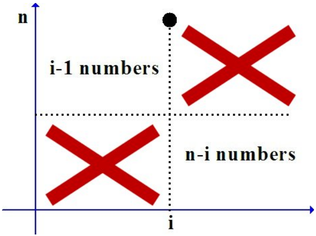

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$$|S_n(132)| = \frac{\binom{2n}{n}}{n+1} = n\text{th Catalan number}$$

Avoiding a Pattern of Length 4

There are 24 patterns of length 4.

Using Wilf equivalence and similar bijections, we can narrow our work to 3 cases:
$S_n(1342), S_n(1234),$ and $S_n(1324)$.

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	1	2	3	4	5	6	7	8	
$\lvert S_n(1342)\rvert$	1	2	6	23	103	512	2740	15485	$\sim 8^n$
$\lvert S_n(1234)\rvert$	1	2	6	23	103	513	2761	15767	$\sim 9^n$
$\lvert S_n(1324)\rvert$	1	2	6	23	103	513	2762	15793	$\sim 9.3^n$

Pattern-Avoidance Sightings

Pattern-avoiding permutations appear in the context of...

- sorting algorithms
- Schubert varieties
- experimental mathematics

Algorithms for Pattern-Avoiding Permutations

- Most techniques studying $|S_n(q)|$ find formulas for a *specific* q.
- 1998: Zeilberger's *prefix enumeration schemes*, i.e. a system of recurrences to compute $|S_n(q)|$.
- 2005: Vatter's modified schemes automate the computation of $|S_n(q)|$ for even more patterns q.

Refinement Notation

Goal: Divide $S_n(q)$ into disjoint subsets.

$$S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \;\middle|\; \begin{array}{l} \pi \text{ avoids } q \\ \pi \text{ has prefix } p_1 \cdots p_l \end{array} \right\}$$

For example, $S_3(123) = \{132, 213, 231, 312, 321\}$, so
$S_3(123; 12) = \{132, 231\}$, and
$S_3(123, 21) = \{213, 312, 321\}$.

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$$S_n\left(q; \begin{matrix} p_1 \cdots p_l \\ i_1 \cdots i_l \end{matrix}\right) := \left\{ \pi \in S_n \;\middle|\; \begin{matrix} \pi \text{ avoids } q \\ \pi \text{ has prefix } p_1 \cdots p_l \\ \pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n \end{matrix} \right\}$$

We have seen $S_3(123; 12) = \{132, 231\}$, so

$S_3\left(123, \begin{matrix} 12 \\ 13 \end{matrix}\right) = \{132\}$,

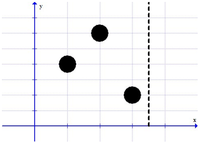
$S_3\left(123, \begin{matrix} 12 \\ 23 \end{matrix}\right) = \{231\}$, and

$S_3\left(123, \begin{matrix} 12 \\ 12 \end{matrix}\right) = \{\}$.

Refinement

Given a prefix p of length l, the refinements of p ($Ref(p)$) are the permutations of length $l+1$ whose first l letters reduce to p.

For example, $Ref(231) = \{3421, 3412, 2413, 2314\}$.



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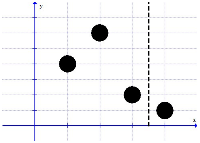

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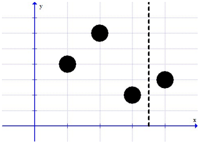

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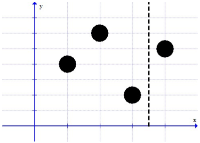

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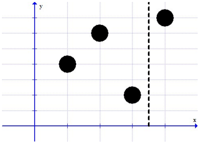

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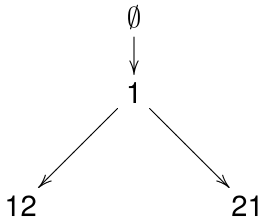
$$S_n(q;p) = \bigcup_{r \in Ref(p)} S_n(q;r)$$

Refinement Example

For any pattern q,
$S_n(q) = S_n(q;1) = S_n(q;12) \cup S_n(q;21) = \ldots.$

or graphically:



Reversibly Deletable Positions

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Given a pattern q and a prefix p, p_r is reversibly deletable if

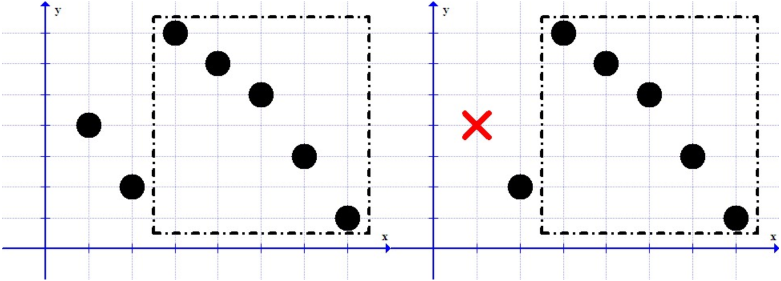
- Deleting p_r from $\pi \in S_n(q; p_1 \cdots p_l)$ produces a q-avoiding permutation of length $n-1$, and
- Inserting p_r into $\pi \in S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)$ produces a q-avoiding permutation of length n.

If p_r is reversibly deletable then,

$$|S_n(q; p_1 \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)| .$$

Reversibly Deletable Positions

For 123-avoiding permutations that begin with $p = 21$, p_1 is reversibly deletable



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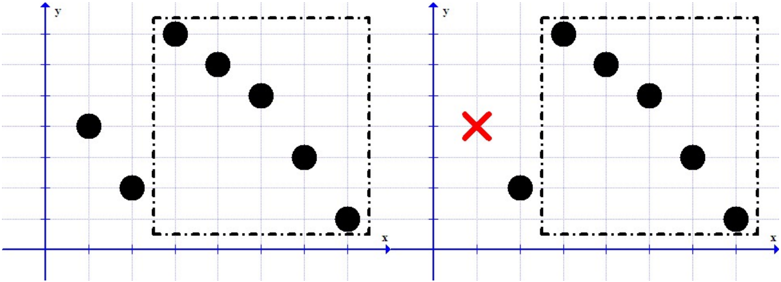
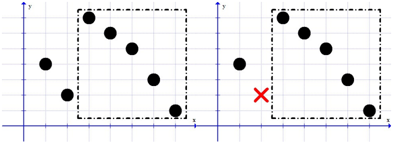


$$\left|S_n\left(123; \begin{matrix}21\\ ij\end{matrix}\right)\right| = \left|S_{n-1}\left(123; \begin{matrix}1\\ j\end{matrix}\right)\right|$$

Reversibly Deletable Positions

For 123-avoiding permutations that begin with $p = 21$, p_2 is not reversibly deletable.



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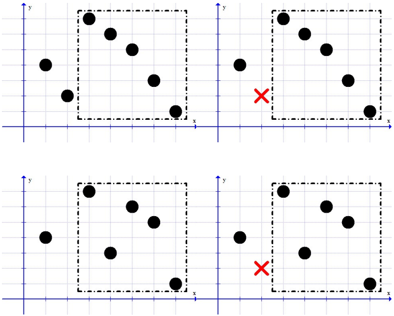


While deleting p_2 gives a smaller 123-avoiding permutation, inserting p_2 into a member of $S_{n-1}(123)$ doesn't always give a 123-avoiding permutation.

Algorithm to find Reversibly Deletable Elements

Brute force:

- List all scenarios in which p_r can participate in a forbidden q-pattern.
- Delete p_r from each scenario. If every resulting permutation contains q, then p_r is reversibly deletable.

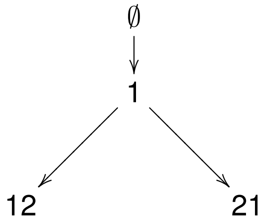
In practice:

Theorem (Vatter, 2005)

If $|S_n(q; p_1 \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$ for all $n \leq |p| + |q| - 1$ then p_r is reversibly deletable.

Reversibly Deletable Example

Graphically, for $q = 123$, we have:



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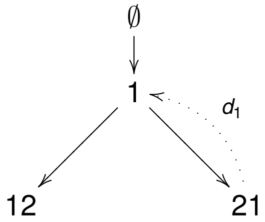
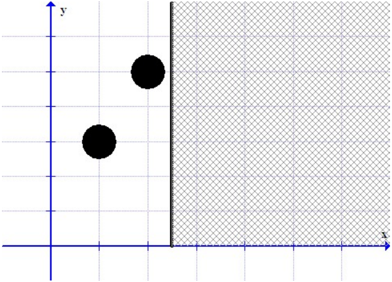


Gap Vectors

Gap vectors give a condition for which choices of $i_1, \ldots, i_l$ yield

$$\left| S_n \left(q; \begin{matrix} p_1 \cdots p_r \cdots p_l \\ i_1 \cdots i_r \cdots i_l \end{matrix} \right) \right| = 0.$$



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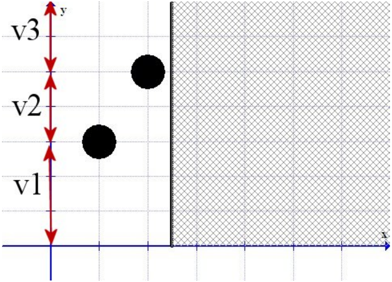

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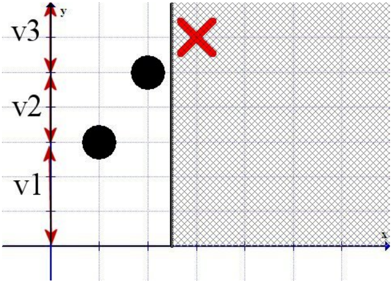
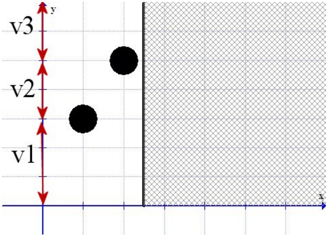
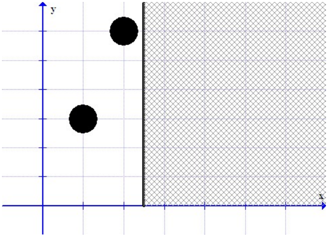


Since there are no members of $S_n(123; 12)$ where $v_3 = 1$, we say $(0, 0, 1)$ is a gap vector for $p = 12$

Gap Vectors

Knowing that $(0, 0, 1)$ is a gap vector for $q = 123$ and $p = 12$ can help us determine more reversibly deletable positions.





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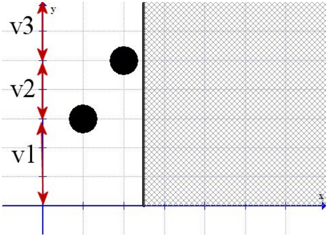

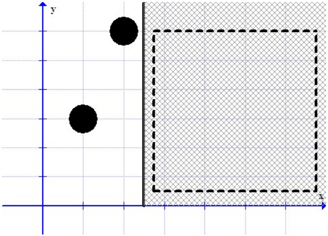

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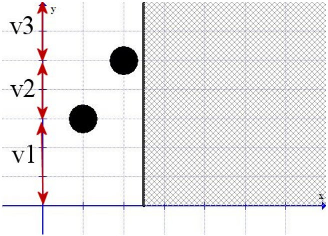
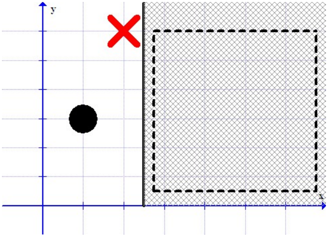




$$\left| S_n \left(123; \begin{matrix} 12 \\ ij \end{matrix} \right) \right| = 0 \text{ if } j < n$$
$$\left| S_n \left(123; \begin{matrix} 12 \\ in \end{matrix} \right) \right| = \left| S_{n-1} \left(123; \begin{matrix} 1 \\ i \end{matrix} \right) \right|$$

Algorithm to find Gap Vectors

Brute force:

- List all permutations π that begin with prefix p and obey vector v.
- If every element of this set contains q, then v is a gap vector.

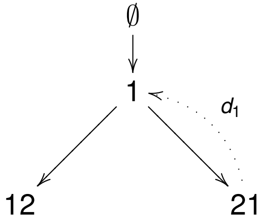
In practice:

Theorem (Vatter, 2005)

1. If v is a gap vector for $(q; p)$, and $u \geq v$ componentwise, then u is a gap vector for $(q; p)$.
2. Minimal gap vectors for $(q; p)$ have $||v|| \leq |q| - 1$.

Gap Vectors

Graphically, for $q = 123$, we have:



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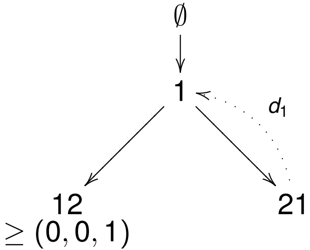

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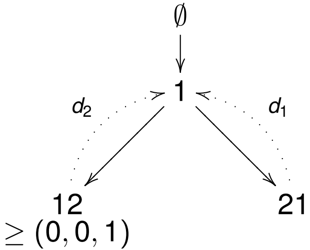

Enumeration Scheme Definition

An enumeration scheme is a set of triples $[p_i, G_i, R_i]$ such that for each triple

- p_i is a reduced prefix of length n
- G_i is a set of vectors of length $n+1$
- R_i a subset of $\{1, \ldots, n\}$
  and
- either R_i is non-empty or all refinements of p_i are also in the scheme.

Enumeration Scheme Definition

An enumeration scheme is a set of triples $[p_i, G_i, R_i]$ such that for each triple

- p_i is a reduced prefix of length n (prefix)
- G_i is a set of vectors of length $n+1$ (gap vectors)
- R_i a subset of $\{1, \ldots, n\}$ (reversibly deletable positions) and
- either R_i is non-empty or all refinements of p_i are also in the scheme.

Enumeration Scheme Example

For the pattern $q = 123$, we have constructed the following scheme:
$S = \{[\emptyset, \emptyset, \emptyset]\}$

$\emptyset$

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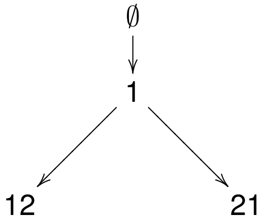
$S = \{[\emptyset, \emptyset, \emptyset], [1, \emptyset, \emptyset], [12, G_{12}, R_{12}], [21, G_{21}, R_{21}]\}$



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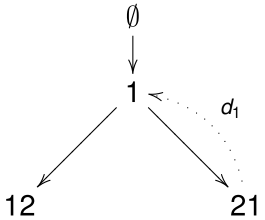

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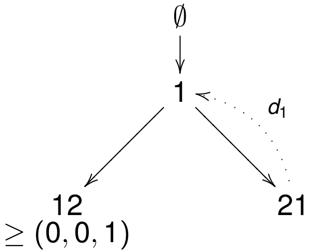

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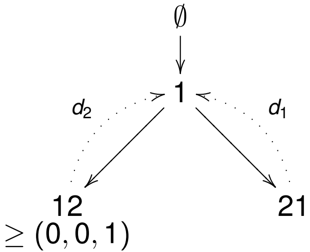

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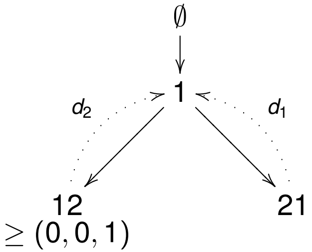

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$S = \{[\emptyset, \emptyset, \emptyset], [1, \emptyset, \emptyset], [12, \{(0, 0, 1)\}, \{2\}], [21, \emptyset, \{1\}]\}$
can be seen as a recurrence to count the elements of $S_n(123)$.

$|S_n(123)| = \sum_{i=1}^{n} \left| S_n \left(123, \begin{matrix} 1 \\ i \end{matrix} \right) \right|$

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Enumeration Scheme Example

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$$= \sum_{i=1}^{n} \left(\left| S_{n-1}\left(123; \begin{matrix}1\\ i\end{matrix}\right) \right| + \sum_{h=1}^{i-1} \left| S_{n-1}\left(123; \begin{matrix}1\\ h\end{matrix}\right) \right| \right)$$

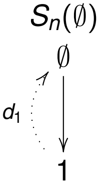
$$= \sum_{i=1}^{n}\sum_{h=1}^{i} \left| S_{n-1}\left(123; \begin{matrix}1\\ h\end{matrix}\right) \right|$$

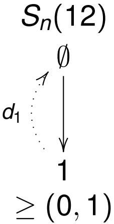
Enumeration Schemes

- Refinements
- Reversibly deletable elements
- Gap vectors

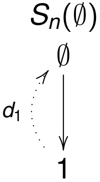
can all be found completely automatically, so we have an algorithm to compute enumeration schemes for pattern-avoiding permutations.

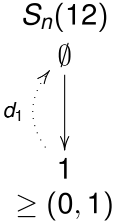
$S_n(\emptyset)$ and $S_n(12)$





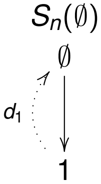
$S_n(\emptyset)$ and $S_n(12)$

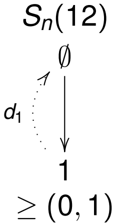




$$|S_n(\emptyset)| = \sum_{i=1}^{n} \left| S_n \left(\emptyset; \begin{matrix} 1 \\ i \end{matrix} \right) \right| = \sum_{i=1}^{n} |S_{n-1}(\emptyset)| = n\,|S_{n-1}(\emptyset)|$$

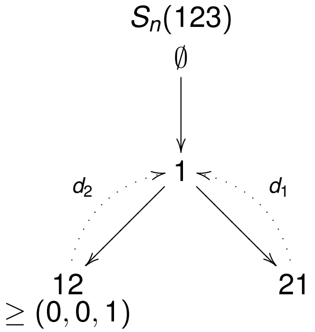
$S_n(\emptyset)$ and $S_n(12)$

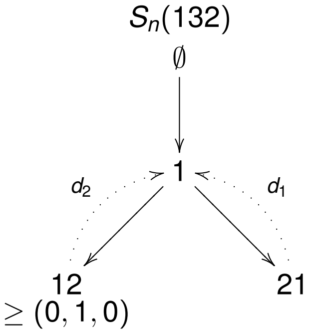




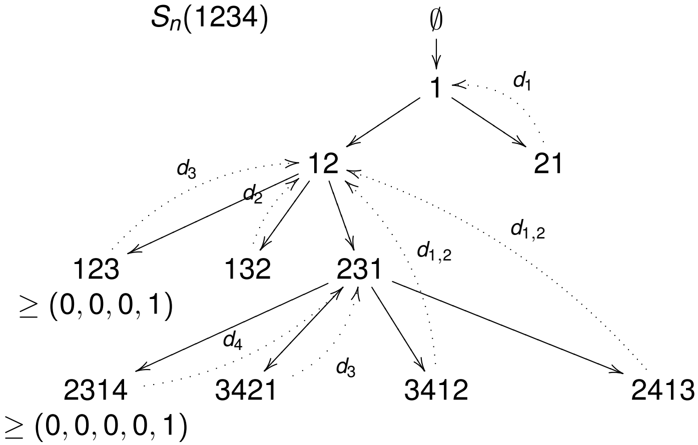
$$|S_n(12)| = \sum_{i=1}^{n} \left| S_n \begin{pmatrix} 12; & 1 \\ & i \end{pmatrix} \right| = \left| S_n \begin{pmatrix} 12; & 1 \\ & n \end{pmatrix} \right| = |S_{n-1}(12)|$$

$S_n(123)$ and $S_n(132)$





$S_n(1234)$


$S_n(1234)$
$\emptyset$
1
d_1
12
21
d_3
d_2
$d_{1,2}$
$d_{1,2}$
123
$\geq (0,0,0,1)$
132
231
d_4
d_3
2314
$\geq (0,0,0,0,1)$
3421
3412
2413

Summary

- There are few techniques to count many classes of pattern-avoiding permutations.
- Zeilberger's and Vatter's schemes give a good success rate for counting the elements of $S_n(q)$.

- Enumeration schemes have also been successfully used to count:
  - pattern-avoiding words (strings with repeated letters)
  - permutations avoiding barred patterns

Thank You!