An Introduction to Enumeration Schemes

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Outline

1 Pattern-Avoiding Permutations
   - Definitions
   - Counting Results
   - Motivation

2 Enumeration Schemes
   - Divide
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   - Putting It All Together...

3 Summary
Given a string of numbers \( q = q_1 \cdots q_m \), the reduction of \( q \) is the string obtained by replacing the \( i^{th} \) smallest number of \( q \) with \( i \).

For example, the reduction of 26745 is 14523.
Given permutations $\pi = \pi_1 \cdots \pi_n$ and $q = q_1 \cdots q_m$,

- $\pi$ contains $q$ as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $\pi_{i_1} \cdots \pi_{i_m}$ reduces to $q$;
- otherwise $\pi$ avoids $q$.

For example,

- 4576213 contains 312 (4576213).
- 4576213 avoids 1234.
We can also think of a permutation as a function from \( \{1, \ldots, n\} \) to \( \{1, \ldots, n\} \).

\[ \pi = 4576213 \]
Permutations as Functions

We can also think of a permutation as a function from \(\{1, \ldots, n\}\) to \(\{1, \ldots, n\}\).

\[
\pi = 4576213
\]

Then, permutation \(\pi\) contains permutation \(q\) if the graph of \(\pi\) contains the graph of \(q\).

4576213 contains 312.
Two Questions

Easy: Given $\pi \in S_n$, what patterns does $\pi$ contain?

Hard: Given $q \in S_m$,
- Let $S_n(q) = \{ \pi \in S_n \mid \pi \text{ avoids } q \}$.
- Find an expression for $|S_n(q)|$.  

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An Introduction to Enumeration Schemes
Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

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Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

What is $|S_n(12)|$? What is $|S_n(21)|$?

$|S_n(12)| = |S_n(21)| = 1$ (for $n \geq 0$).
For any pattern \( q \), we have:

\[
|S_n(q)| = |S_n(q^r)| = |S_n(q^c)| = |S_n(q^{-1})|
\]
There are six patterns of length 3: 123, 132, 213, 231, 312, 321.

Using Wilf equivalence, we have
\[ |S_n(123)| = |S_n(321)| \quad \text{and} \]
\[ |S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|. \]
Avoiding a Pattern of Length 3

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\[ |S_n(123)| = |S_n(132)| \] (Simion and Schmidt, 1985).
Avoiding the pattern 132

What is $|S_n(132)|$?
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$$|S_n(132)| = \binom{2n}{n} = \frac{n}{n+1}$$
Avoiding the pattern 132

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$$|S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \text{ (for } n > 0)$$
Avoiding the pattern 132

What is $|S_n(132)|$?

\[ |S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \text{ (for } n > 0) \]

\[ |S_n(132)| = \frac{(2n)}{n + 1} = n \text{th Catalan number} \]
Avoiding a Pattern of Length 4

There are 24 patterns of length 4.

Using Wilf equivalence and similar bijections, we can narrow our work to 3 cases:
\(S_n(1342), S_n(1234), \text{ and } S_n(1324)\).
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<td>23</td>
<td>103</td>
<td>513</td>
<td>2762</td>
<td>15793</td>
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Pattern-Avoidance Sightings

Pattern-avoiding permutations appear in the context of...

- sorting algorithms
- Schubert varieties
- experimental mathematics
Most techniques studying $|S_n(q)|$ find formulas for a specific $q$.

1998: Zeilberger’s *prefix enumeration schemes*, i.e. a system of recurrences to compute $|S_n(q)|$.

2005: Vatter’s modified schemes automate the computation of $|S_n(q)|$ for even more patterns $q$. 
Refinement Notation

Goal: Divide $S_n(q)$ into disjoint subsets.

$$S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \mid \pi \text{ avoids } q \right. \quad \left. \pi \text{ has prefix } p_1 \cdots p_l \right\}$$

For example, $S_3(123) = \{132, 213, 231, 312, 321\}$, so
$S_3(123; 12) = \{132, 231\}$, and
$S_3(123, 21) = \{213, 312, 321\}$.
Refinement Notation

Goal: Divide $S_n(q)$ into subsets.

$$S_n\left(q; p_1 \cdots p_l \mid i_1 \cdots i_l\right) := \left\{ \pi \in S_n \mid \begin{array}{l}
\pi \text{ avoids } q \\
\pi \text{ has prefix } p_1 \cdots p_l \\
\pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n
\end{array} \right\}$$

We have seen $S_3(123; 12) = \{132, 231\}$, so

$S_3(123, \begin{pmatrix} 12 \\ 13 \end{pmatrix}) = \{132\}$,

$S_3(123, \begin{pmatrix} 12 \\ 23 \end{pmatrix}) = \{231\}$, and

$S_3(123, \begin{pmatrix} 12 \\ 12 \end{pmatrix}) = \{\}$. 
Refinement

Given a prefix $p$ of length $l$, the refinements of $p$ (Ref($p$)) are the permutations of length $l + 1$ whose first $l$ letters reduce to $p$.

For example, $\text{Ref}(231) = \{3421, 3412, 2413, 2314\}$.

We have

$$S_n(q; p) = \bigcup_{r \in \text{Ref}(p)} S_n(q; r)$$
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We have

$$S_n(q; p) = \bigcup_{r \in \text{Ref}(p)} S_n(q; r)$$
For any pattern $q$,

$$S_n(q) = S_n(q; 1) = S_n(q; 12) \cup S_n(q; 21) = \ldots.$$ 

or graphically:
Reversibly Deletable Positions

Given a pattern $q$ and a prefix $p$, $p_r$ is reversibly deletable if

- Deleting $p_r$ from $\pi \in S_n(q; p_1 \cdots p_l)$ produces a $q$-avoiding permutation of length $n - 1$, and
- Inserting $p_r$ into $\pi \in S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)$ produces a $q$-avoiding permutation of length $n$.

If $p_r$ is reversibly deletable then,

$$|S_n(q; p_1 \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|.$$
For 123-avoiding permutations that begin with $p = 21$, $p_1$ is reversibly deletable.
For 123-avoiding permutations that begin with \( p = 21 \), \( p_1 \) is reversibly deletable.

\[
\left| S_n \left( 123; \frac{21}{ij} \right) \right| = \left| S_{n-1} \left( 123; \frac{1}{j} \right) \right|
\]
For 123-avoiding permutations that begin with $p = 21$, $p_2$ is not reversibly deletable.
For 123-avoiding permutations that begin with $p = 21$, $p_2$ is not reversibly deletable.

While deleting $p_2$ gives a smaller 123-avoiding permutation, inserting $p_2$ into a member of $S_{n-1}(123)$ doesn’t always give a 123-avoiding permutation.
Algorithm to find Reversibly Deletable Elements

Brute force:
- List all scenarios in which $p_r$ can participate in a forbidden $q$-pattern.
- Delete $p_r$ from each scenario. If every resulting permutation contains $q$, then $p_r$ is reversibly deletable.

In practice:

Theorem (Vatter, 2005)

If $|S_n(q; p_1 \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$ for all $n \leq |p| + |q| - 1$ then $p_r$ is reversibly deletable.
Graphically, for $q = 123$, we have:
Graphically, for $q = 123$, we have:
Gap vectors give a condition for which choices of $i_1, \ldots, i_l$ yield

$$\left| S_n \left( q; \frac{p_1}{i_1} \ldots \frac{p_r}{i_r} \ldots \frac{p_l}{i_l} \right) \right| = 0.$$
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$$\left| S_n \left( q; \frac{p_1 \cdots p_r \cdots p_l}{i_1 \cdots i_r \cdots i_l} \right) \right| = 0.$$ 

Since there are no members of $S_n(123; 12)$ where $v_3 = 1$, we say $(0, 0, 1)$ is a gap vector for $p = 12$.
Gap Vectors

Knowing that \((0, 0, 1)\) is a gap vector for \(q = 123\) and \(p = 12\) can help us determine more reversibly deletable positions.

\[ S_n(123; 12) = 0 \text{ if } j < n \]

\[ S_n(123; 12) = S_{n-1}(123; 1) \]

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\[
\left| S_n \left( 123; \frac{12}{in} \right) \right| = \left| S_{n-1} \left( 123; \frac{1}{i} \right) \right|
\]
Algorithm to find Gap Vectors

Brute force:
- List all permutations \( \pi \) that begin with prefix \( p \) and obey vector \( v \).
- If every element of this set contains \( q \), then \( v \) is a gap vector.

In practice:

**Theorem (Vatter, 2005)**
1. If \( v \) is a gap vector for \((q; p)\), and \( u \geq v \) componentwise, then \( u \) is a gap vector for \((q; p)\).
2. Minimal gap vectors for \((q; p)\) have \( \|v\| \leq \|q\| - 1 \).
Gap Vectors

Graphically, for $q = 123$, we have:

```
    0
   ↓
   1
  / \  \\
 12  21
```

Where $d_1$ represents the gap vector.
Graphically, for $q = 123$, we have:

\[
\begin{align*}
\emptyset & \quad \downarrow \\
1 & \quad \downarrow \\
12 & \quad \rightarrow d_1 \\
\geq (0, 0, 1) & \quad \rightarrow 21
\end{align*}
\]
Graphically, for $q = 123$, we have:
An enumeration scheme is a set of triples \([p_i, G_i, R_i]\) such that for each triple

- \(p_i\) is a reduced prefix of length \(n\)
- \(G_i\) is a set of vectors of length \(n + 1\)
- \(R_i\) a subset of \(\{1, \ldots, n\}\)

and

- either \(R_i\) is non-empty or all refinements of \(p_i\) are also in the scheme.
An enumeration scheme is a set of triples \([p_i, G_i, R_i]\) such that for each triple

- \(p_i\) is a reduced prefix of length \(n\) (prefix)
- \(G_i\) is a set of vectors of length \(n + 1\) (gap vectors)
- \(R_i\) a subset of \(\{1, \ldots, n\}\) (reversibly deletable positions)
  and
- either \(R_i\) is non-empty or all refinements of \(p_i\) are also in the scheme.
For the pattern $q = 123$, we have constructed the following scheme:

$S = \{[\emptyset, \emptyset, \emptyset]\}$
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S = \{ [\emptyset, \emptyset, \emptyset], [1, \emptyset, \emptyset], [12, \{(0, 0, 1)\}, \{2\}], [21, \emptyset, \{1\}] \}
\]
The scheme can
\[ S = \{[\emptyset, \emptyset, \emptyset], [1, \emptyset, \emptyset], [12, \{(0, 0, 1)\}, \{2\}], [21, \emptyset, \{1\}]\} \]
can be seen as a recurrence to count the elements of \( S_n(123) \).

\[ |S_n(123)| = \sum_{i=1}^{n} \left| S_n\left(123, \binom{1}{i}\right)\right| \]
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\[ = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \left| S_n \left(123, \binom{12}{ij} \right) \right| + \sum_{i=1}^{n} \sum_{h=1}^{i-1} \left| S_n \left(123, \binom{21}{ih} \right) \right| \]
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\]

\[
= \sum_{i=1}^{n} \left| S_n \left(123; \frac{12}{in}\right) \right| + \sum_{i=1}^{n} \sum_{h=1}^{i-1} \left| S_{n-1} \left(123; \frac{1}{h}\right) \right|
\]
Enumeration Scheme Example

The scheme can
\[ S = \{[\emptyset, \emptyset, \emptyset], [1, \emptyset, \emptyset], [12, \{(0, 0, 1)\}, \{2\}], [21, \emptyset, \{1\}]\} \]
can be seen as a recurrence to count the elements of \( S_n(123) \).

\[
|S_n(123)| = \sum_{i=1}^{n} S_n \left( 123, \begin{array}{c} 1 \\ 1 \end{array} \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=i+1}^{n} S_n \left( 123; \begin{array}{c} 12 \\ ij \end{array} \right) + \sum_{i=1}^{n} \sum_{h=1}^{i-1} S_n \left( 123; \begin{array}{c} 21 \\ ih \end{array} \right)
\]

\[
= \sum_{i=1}^{n} S_n \left( 123; \begin{array}{c} 12 \\ in \end{array} \right) + \sum_{i=1}^{n} \sum_{h=1}^{i-1} S_{n-1} \left( 123; \begin{array}{c} 1 \\ h \end{array} \right)
\]

\[
= \sum_{i=1}^{n} \left( S_{n-1} \left( 123; \begin{array}{c} 1 \\ i \end{array} \right) + \sum_{h=1}^{i-1} S_{n-1} \left( 123; \begin{array}{c} 1 \\ h \end{array} \right) \right)
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can be seen as a recurrence to count the elements of \( S_n(123) \).

\[
|S_n(123)| = \sum_{i=1}^{n} |S_n\left(123, \frac{1}{i}\right)|
\]

\[
= \sum_{i=1}^{n} \sum_{j=i+1}^{n} |S_n\left(123; \frac{12}{ij}\right)| + \sum_{i=1}^{n} \sum_{h=1}^{i-1} |S_n\left(123; \frac{21}{ih}\right)|
\]

\[
= \sum_{i=1}^{n} |S_n\left(123; \frac{12}{in}\right)| + \sum_{i=1}^{n} \sum_{h=1}^{i-1} |S_{n-1}\left(123; \frac{1}{h}\right)|
\]

\[
= \sum_{i=1}^{n} \left(|S_{n-1}\left(123; \frac{1}{i}\right)| + \sum_{h=1}^{i-1} |S_{n-1}\left(123; \frac{1}{h}\right)|\right)
\]

\[
= \sum_{i=1}^{n} \sum_{h=1}^{i} |S_{n-1}\left(123; \frac{1}{h}\right)|
\]
Refinements
Reversibly deletable elements
Gap vectors
can all be found completely automatically, so we have an algorithm to compute enumeration schemes for pattern-avoiding permutations.
$S_n(\emptyset)$ and $S_n(12)$

$S_n(\emptyset)$

$\emptyset$

$\downarrow$

$d_1$

$1$

$\geq (0, 1)$
\[ S_n(\emptyset) \text{ and } S_n(12) \]

\[
S_n(\emptyset) \quad \begin{array}{c} \emptyset \\ \downarrow \\ 1 \\ d_1 \\ \uparrow \\ 1 \quad S_n(12) \\ \begin{array}{c} \emptyset \\ \downarrow \\ 1 \\ \geq (0, 1) \end{array} 
\]

\[
|S_n(\emptyset)| = \sum_{i=1}^{n} \left| S_n \left( \emptyset; \begin{array}{c} 1 \\ i \end{array} \right) \right| = \sum_{i=1}^{n} \left| S_{n-1} (\emptyset) \right| = n \left| S_{n-1}(\emptyset) \right|
\]
\[ |S_n(\emptyset)| = \sum_{i=1}^{n} |S_n(12; \frac{1}{i})| = |S_n(12; \frac{1}{n})| = |S_{n-1}(12)| \]
$S_n(123)$ and $S_n(132)$

$S_n(123)$

$\emptyset$ \quad 1 \quad 12 \quad \geq (0, 0, 1)

$S_n(132)$

$\emptyset$ \quad 1 \quad 12 \quad \geq (0, 1, 0)$
Pattern-Avoiding Permutations
Enumeration Schemes
Summary

Divide
Conquer
Putting It All Together...

$S_n(1234)$

$S_n(1234)$

$\emptyset$

$1$

$d_1$

$12$

$21$

$d_1,2$

$123$

$d_3$

$\geq (0,0,0,1)$

$132$

$d_2$

$231$

$d_1,2$

$\geq (0,0,1,0)$

$2314$

$d_4$

$3421$

$d_3$

$3412$

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An Introduction to Enumeration Schemes
There are few techniques to count many classes of pattern-avoiding permutations. Zeilberger’s and Vatter’s schemes give a good success rate for counting the elements of $S_n(q)$.

Enumeration schemes have also been successfully used to count:
- pattern-avoiding words (strings with repeated letters)
- permutations avoiding barred patterns
Thank You!