Counting Pattern-Avoiding Permutations

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Trinity REU
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Given a string of numbers $q = q_1 \cdots q_n$, the \textit{reduction} of $q$ is the string obtained by replacing the $i^{th}$ smallest number(s) of $q$ with $i$. 
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For example, the reduction of \( 2674425 \) is \( 1\bullet\bullet\bullet1\bullet \).
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For example, the reduction of $2674425$ is $1\bullet\bullet\underline{2}213$. 
Given a string of numbers $q = q_1 \cdots q_n$, the reduction of $q$ is the string obtained by replacing the $i^{th}$ smallest number(s) of $q$ with $i$.

For example, the reduction of $2674425$ is $14\bullet2213$. 
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For example, the reduction of $2674425$ is $1452213$. 
A permutation of length $n$, is a string of numbers using each of 1, \ldots, $n$ exactly once.
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- \( S_2 = \{12, 21\} \)
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- \( S_1 = \{1\} \)
- \( S_2 = \{12, 21\} \)
- \( S_3 = \{123, 132, 213, 231, 312, 321\} \)
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Write $S_n$ for the set of permutations of length $n$. For example:

- $S_1 = \{1\}$
- $S_2 = \{12, 21\}$
- $S_3 = \{123, 132, 213, 231, 312, 321\}$
- $S_4 = \{1234, 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321\}$
Given permutations $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$, 

- $p$ contains $q$ as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $p_{i_1} \cdots p_{i_m}$ reduces to $q$; 
- otherwise $p$ avoids $q$. 

For example, $4576213$ contains $312$ ($4576213$) and $4576213$ avoids $1234$. 

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For example,

- $4576213$ contains $312$ ($4576213$).
- $4576213$ avoids $1234$. 
Pattern Avoidance in Permutations

**Easier Question:** Fix $p$. What patterns are contained in $p$?

For example,

- $p = 1423$ contains $1$.
- $p = 1423$ contains $12, 21$.
- $p = 1423$ contains $123, 132, 312$; avoids $213, 231, 321$.
- $p = 1423$ contains $1423$; avoids all 23 other patterns of length 4.
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For example, $p = 1423$
- length 1
  - 1423 (1)
  - 1423 (1), 1423 (1), 1423 (1), 1423 (1)
  - contains 1.
Easier Question: Fix $p$. What patterns are contained in $p$?
For example, $p = 1423$

- length 1
  - $1423$ (1), $1423$ (1), $1423$ (1), $1423$ (1)
  - contains 1.

- length 2
  - $1423$, $1423$, $1423$, $1423$, $1423$, $1423$, $1423$
Easier Question: Fix $p$. What patterns are contained in $p$?

For example, $p = 1423$

- **length 1**
  - $1423 (1), 1423 (1), 1423 (1), 1423 (1)$
  - contains 1.

- **length 2**
  - $1423 (12), 1423 (12), 1423 (12), 1423 (21), 1423 (21), 1423 (12)$
  - contains 12, 21.
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For example, $p = 1423$

- **length 1**
  - $1423 (1), 1423 (1), 1423 (1), 1423 (1)$
  - contains 1.

- **length 2**
  - $1423 (12), 1423 (12), 1423 (12), 1423 (21), 1423 (21), 1423 (12)$
  - contains 12, 21.

- **length 3**
  - $1423 , 1423 , 1423 , 1423$
Easier Question: Fix $p$. What patterns are contained in $p$?
For example, $p = 1423$

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  - $1423 (1), 1423 (1), 1423 (1), 1423 (1)$
  - contains 1.

- **length 2**
  - $1423 (12), 1423 (12), 1423 (12), 1423 (21), 1423 (21), 1423 (12)$
  - contains 12, 21.

- **length 3**
  - $1423 (132), 1423 (132), 1423 (132), 1423 (123), 1423 (312)$
  - contains 123, 132, and 312; avoids 213, 231, and 321.
**Pattern Avoidance in Permutations**

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For example, $p = 1423$

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  - contains 12, 21.

- **length 3**
  - $1423$ (132), $1423$ (132), $1423$ (123), $1423$ (312)
  - contains 123, 132, and 312; avoids 213, 231, and 321.

- **length 4**
  - $1423$
Easier Question: Fix $p$. What patterns are contained in $p$?
For example, $p = 1423$

- **length 1**
  - 1423 (1), 1423 (1), 1423 (1), 1423 (1)
  - contains 1.

- **length 2**
  - 1423 (12), 1423 (12), 1423 (12),
    1423 (21), 1423 (21), 1423 (12)
  - contains 12, 21.

- **length 3**
  - 1423 (132), 1423 (132), 1423 (123), 1423 (312)
  - contains 123, 132, and 312; avoids 213, 231, and 321.

- **length 4**
  - 1423 (1423)
  - contains 1423; avoids all 23 other patterns of length 4.
**Harder Question:** Fix a pattern $q$. Enumerate $S_n(q) := \{ p \in S_n \mid p \text{ avoids } q \}$. 
Avoiding a Pattern of Length 1

There is one pattern of length 1:
1.

Question: What is $|S_n(1)|$?
Avoiding a Pattern of Length 1

There is one pattern of length 1:
1.

Question: What is $|S_n(1)|$?

Answer: $|S_n(1)| = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$
Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

Questions: What is $|S_n(12)|$? What is $|S_n(21)|$?
Avoiding a Pattern of Length 2

There are two patterns of length 2: 12, 21.

Questions: What is $|S_n(12)|$? What is $|S_n(21)|$?

Answer: $|S_n(12)| = |S_n(21)| = 1$ (for $n \geq 0$).
Useful Observation (Wilf Equivalence)

For any pattern $p$, we have:

$$|S_n(q)| = |S_n(q^r)| = |S_n(q^c)| = |S_n(q^{-1})|$$
There are six patterns of length 3: 123, 132, 213, 231, 312, 321.

Using the useful observation, we have

\[ |S_n(123)| = |S_n(321)| \quad \text{and} \quad |S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|. \]
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\[ |S_n(123)| = |S_n(132)| \] (Simion and Schmidt, 1985).
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\[ |S_n(123)| = |S_n(321)| \quad \text{and} \quad |S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|. \]

Question: What is \(|S_n(132)|\)?
Avoiding a Pattern of Length 3

There are six patterns of length 3: 123, 132, 213, 231, 312, 321.

Using the useful observation, we have
\[ |S_n(123)| = |S_n(321)| \text{ and } |S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|. \]

| \( S_n(123) \) | = | \( S_n(132) \) | (Simion and Schmidt, 1985).

Question: What is | \( S_n(132) \) |?

Answer: | \( S_n(132) \) | = \( \frac{2^n n}{n+1} \) = \( C_n \) (for \( n \geq 0 \)).
Avoiding a Pattern of Length 4

There are 24 patterns of length 4.

Using the useful observation and similar bijections, we can narrow our work to 3 cases: $S_n(1342)$, $S_n(1234)$, and $S_n(1324)$. 
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Using the useful observation and similar bijections, we can narrow our work to 3 cases: $S_n(1342)$, $S_n(1234)$, and $S_n(1324)$.

<table>
<thead>
<tr>
<th>Pattern</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_n(1342)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>23</td>
<td>103</td>
<td>512</td>
<td>2740</td>
<td>15485</td>
</tr>
<tr>
<td>$S_n(1234)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>23</td>
<td>103</td>
<td>513</td>
<td>2761</td>
<td>15767</td>
</tr>
<tr>
<td>$S_n(1324)$</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>23</td>
<td>103</td>
<td>513</td>
<td>2762</td>
<td>15793</td>
</tr>
</tbody>
</table>

$\sim 8^n$  
$\sim 9^n$  
$\sim 9.3^n$
A **stack** is a *last in, first out* data structure.

Question: What permutations can be sorted by passing them through the stack exactly one time?
A stack is a last in, first out data structure.

Question: What permutations can be sorted by passing them through the stack exactly one time?

Answer: (Knuth, 1973) Exactly the permutations that avoid 231.
1900: Hilbert’s 15th problem is to find a "rigorous foundation of Schubert's enumerative calculus".

1900s: In algebraic geometry, Schubert varieties are the most commonly studied type of singular variety.

Question: Which Schubert varieties are smooth?
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1900s: In algebraic geometry, Schubert varieties are the most commonly studied type of singular variety.

Question: Which Schubert varieties are smooth?

Answer: (Lakshmibai and Sandhya, 1990): Exactly the varieties whose indexing permutation avoids 4231 and 3412.
The techniques to count $|S_n(q)|$ usually depend on $q$.

Goal: Find an algorithm to count $|S_n(q)|$ that works well regardless of what $q$ is.
We may consider permutations which simultaneously avoid more than one pattern:

\[ S_n(Q) := \{ p \in S_n \mid p \text{ avoids } q \text{ for every } q \in Q \}. \]

Some particularly nice results include:

- \( S_n(\{123, 132\}) = 2^{n-1} \)
- \( S_n(\{132, 213, 321\}) = n \)
- \( S_n(\{123, 132, 213\}) = F_n \) (Fibonacci numbers)
Consider patterns where there may or may not be a dash between each pair of letters.
E.g. $3 - 12$

A dash indicates those two letters can be arbitrarily far apart, no dash indicates they must be adjacent.
E.g. $241653$ contains $12 - 3$ ($241653$), but not $1 - 23$

Further, we can specify *exact* distances between letters of a pattern.
Consider *pairs* of patterns $q$ contained in $\overline{q}$. Then $q = \overline{q}$ with a bar over each element *not* in $q$.

E.g. $15342$ contains reduction$(134) = 123$.
$q = 1\overline{5342}, \overline{q} = 15342, q = \text{reduction}(134) = 123$.

$p$ avoids $q$ if every instance of $q$ in $p$ is part of an instance of $\overline{q}$ in $p$. 
Consider *pairs* of patterns $q$ contained in $\overline{q}$. Then $q = \overline{q}$ with a bar over each element *not* in $q$.

E.g. 15342 contains reduction$(134) = 123$.
$q = 1\overline{5}3\overline{4}2$, $\overline{q} = 15342$, $\overline{q} =$ reduction$(134) = 123$.

$p$ avoids $q$ if every instance of $q$ in $p$ is part of an instance of $\overline{q}$ in $p$.

Two friendly examples:
- $S_n(\overline{132}) = (n - 1)!$
- $S_n(\overline{1423}) = B_n$ (Bell numbers)
Barred Pattern Avoidance

Two Theorems involving Barred Patterns:

1. (West, 1990) A permutation is 2-stack sortable if and only if it avoids 2341 and 35241.
2. (Woo and Yong, 2006) A Schubert variety $X_w$ is locally factorial if and only if $w$ avoids the patterns 1324 and 21354.
Barred Pattern Avoidance

Two Theorems involving Barred Patterns:

- (West, 1990) A permutation is 2-stack sortable if and only if it avoids 2341 and 3$\bar{5}$241.

- (Woo and Yong, 2006) A Schubert variety $X_w$ is locally factorial if and only if $w$ avoids the patterns 1324 and 21$\bar{3}$54.

- (Claesson, Dukes, and Kitaev, 2008) $(2 + 2)$-free posets are in bijection with permutations which avoid 3$\bar{1}$52$\bar{4}$. 

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Counting Pattern-Avoiding Permutations
Open Problems

• Enumeration
  • What is $S_n(1324)$?, $S_n(q)$, where $|q| \geq 5$?
  • What can you say about permutations avoiding generalized or barred patterns of length $\geq 5$?
Open Problems 2

- **Algebra**
  - There are Wilf equivalences other than the symmetries of the square. What other equivalences can you find?
  - What Wilf equivalences carry over to subsets or subgroups of $S_n$?

- **Asymptotics**
  - Given any pattern $q$, there exists a constant $c_q$ such that
    
    $$ \lim_{n \to \infty} |S_n(q)| \to c_q^n. $$

    What values of $c_q$ are possible?
Open Problems 3

Applications

- (Stack Sorting) What is a characterization for 3-stack-sortable permutations?
- (Schubert calculus, etc.) $S_n$ is an example of a Coxeter group (a group generated by reflections). What can you say about pattern avoidance in other Coxeter groups?
- (Experimental Mathematics) Can you find a single method that efficiently counts $|S_n(Q)|$ for many different examples of $Q$?
Thank You!