Counting Pattern-Avoiding Permutations

Lara Pudwell

Trinity REU
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Outline

Reduction

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- $S_1 = \{1\}$
- $S_2 = \{12, 21\}$
- $S_3 = \{123, 132, 213, 231, 312, 321\}$
- $S_4 = \{1234, 1243, 1342, 1324, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, 4321\}$

Pattern Avoidance/Containment

Given permutations $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$,

- p **contains** q as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $p_{i_1} \cdots p_{i_m}$ reduces to q;
- otherwise p **avoids** q.

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For example,

- 4576213 contains 312 (4576213).
- 4576213 avoids 1234.

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  - contains 1.
- length 2
  - 1423 , 1423 , 1423 ,
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- length 2
  - 1423 (12), 1423 (12), 1423 (12), 1423 (21), 1423 (21), 1423 (12)
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- length 3
  - 1423 , 1423 , 1423 , 1423

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- length 3
  - 1423 (132), 1423 (132), 1423 (123), 1423 (312)
  - contains 123, 132, and 312; avoids 213, 231, and 321.

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  - 1423 (132), 1423 (132), 1423 (123), 1423 (312)
  - contains 123, 132, and 312; avoids 213, 231, and 321.
- length 4
  - 1423

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  - 1423 (12), 1423 (12), 1423 (12), 1423 (21), 1423 (21), 1423 (12)
  - contains 12, 21.
- length 3
  - 1423 (132), 1423 (132), 1423 (123), 1423 (312)
  - contains 123, 132, and 312; avoids 213, 231, and 321.
- length 4
  - 1423 (1423)
  - contains 1423; avoids all 23 other patterns of length 4.

Pattern Avoidance in Permutations

Harder Question: Fix a pattern q.
Enumerate $S_n(q) := \{p \in S_n \mid p \text{ avoids } q\}$.

Avoiding a Pattern of Length 1

There is one pattern of length 1:
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Question: What is $|S_n(1)|$?

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Question: What is $|S_n(1)|$?

Answer: $|S_n(1)| = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1. \end{cases}$

Avoiding a Pattern of Length 2

There are two patterns of length 2:
12, 21.

Questions: What is $|S_n(12)|$? What is $|S_n(21)|$?

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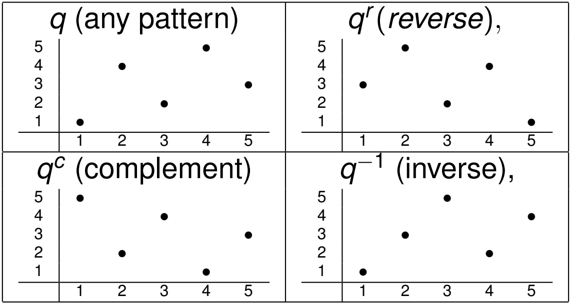
Questions: What is $|S_n(12)|$? What is $|S_n(21)|$?

Answer: $|S_n(12)| = |S_n(21)| = 1$ (for $n \geq 0$).

Useful Observation (Wilf Equivalence)



For any pattern p, we have:

$$|S_n(q)| = |S_n(q^r)| = |S_n(q^c)| = \left|S_n(q^{-1})\right|$$

Avoiding a Pattern of Length 3

There are six patterns of length 3:
123, 132, 213, 231, 312, 321.

Using the useful observation, we have
$|S_n(123)| = |S_n(321)|$ and
$|S_n(132)| = |S_n(231)| = |S_n(213)| = |S_n(312)|$.

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$|S_n(123)| = |S_n(132)|$ (Simion and Schmidt, 1985).

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$|S_n(123)| = |S_n(132)|$ (Simion and Schmidt, 1985).

Question: What is $|S_n(132)|$?

Answer: $|S_n(132)| = \frac{\binom{2n}{n}}{n+1} = C_n$ (for $n \geq 0$).

Avoiding a Pattern of Length 4

There are 24 patterns of length 4.

Using the useful observation and similar bijections, we can narrow our work to 3 cases:
$S_n(1342), S_n(1234),$ and $S_n(1324)$.

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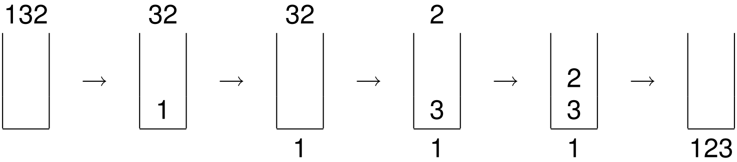
	1	2	3	4	5	6	7	8	
$\lvert S_n(1342)\rvert$	1	2	6	23	103	512	2740	15485	$\sim 8^n$
$\lvert S_n(1234)\rvert$	1	2	6	23	103	513	2761	15767	$\sim 9^n$
$\lvert S_n(1324)\rvert$	1	2	6	23	103	513	2762	15793	$\sim 9.3^n$

Stack Sorting

A stack is a *last in, first out* data structure.



Question: What permutations can be sorted by passing them through the stack exactly one time?

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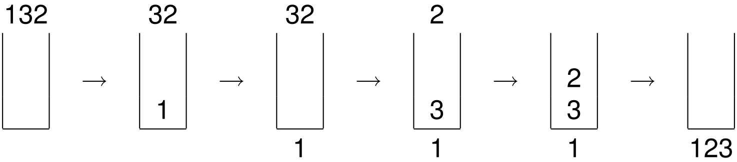

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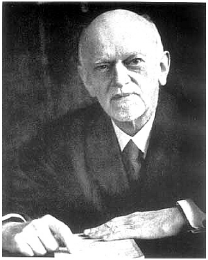
Answer: (Knuth, 1973) Exactly the permutations that avoid 231.

Schubert Varieties



- 1900: Hilbert's 15th problem is to find a "rigorous foundation of Schubert's enumerative calculus".
- 1900s: In algebraic geometry, Schubert varieties are the most commonly studied type of singular variety.

Question: Which Schubert varieties are smooth?

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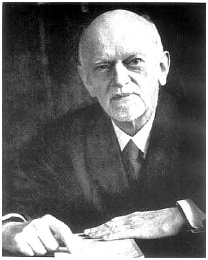

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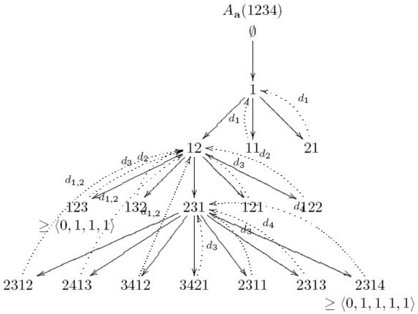
Answer: (Lakshmibai and Sandhya, 1990): Exactly the varieties whose indexing permutation avoids 4231 and 3412.

Experimental Mathematics

- The techniques to count $|S_n(q)|$ usually depend on q.
- Goal: Find an algorithm to count $|S_n(q)|$ that works well regardless of what q is.



Avoiding Sets of Patterns

We may consider permutations which simultaneously avoid more than one pattern:

$$S_n(Q) := \{p \in S_n \mid p \text{ avoids } q \text{ for every } q \in Q\}.$$

Some particularly nice results include:

- $S_n(\{123, 132\}) = 2^{n-1}$
- $S_n(\{132, 213, 321\}) = n$
- $S_n(\{123, 132, 213\}) = F_n$ (Fibonacci numbers)

Generalized and Distanced Patterns

Consider patterns where there may or may not be a dash between each pair of letters.
E.g. $3-12$

A dash indicates those two letters can be arbitrarily far apart, no dash indicates they must be adjacent.
E.g. 241653 contains $12-3$ (241653), but not $1-23$

Further, we can specify *exact* distances between letters of a pattern.

Barred Pattern Avoidance

Consider *pairs* of patterns $\underline{q}$ contained in $\overline{q}$. Then $q = \overline{q}$ with a bar over each element *not* in $\underline{q}$.

E.g. 15342 contains *reduction*$(134) = 123$.
$q = 1\bar{5}34\bar{2}, \overline{q} = 15342, \underline{q} = \textit{reduction}(134) = 123.$

p avoids q if *every* instance of $\underline{q}$ in p is part of an instance of $\overline{q}$ in p.

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Two friendly examples:

- $S_n(\overline{1}32) = (n-1)!$
- $S_n(1\overline{4}23) = B_n$ (Bell numbers)

Barred Pattern Avoidance

Two Theorems involving Barred Patterns:

- (West, 1990) A permutation is 2-stack sortable if and only if it avoids 2341 and $3\bar{5}241$.
- (Woo and Yong, 2006) A Schubert variety X_w is locally factorial if and only if w avoids the patterns 1324 and $21\bar{3}54$.

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- (Claesson, Dukes, and Kitaev, 2008) $(2+2)$-free posets are in bijection with permutations which avoid $3\bar{1}52\bar{4}$.

Open Problems

- Enumeration
  - What is $S_n(1324)$?, $S_n(q)$, where $|q| \geq 5$?
  - What can you say about permutations avoiding generalized or barred patterns of length ≥ 5?

Open Problems 2

- Algebra
  - There are Wilf equivalences other than the symmetries of the square. What other equivalences can you find?
  - What Wilf equivalences carry over to subsets or subgroups of S_n?
- Asymptotics
  - Given any pattern q, there exists a constant c_q such that

$$\lim_{n \to \infty} |S_n(q)| \to c_q^n.$$

    What values of c_q are possible?

Open Problems 3

- Applications
  - (Stack Sorting) What is a characterization for 3-stack-sortable permutations?
  - (Schubert calculus, etc.) S_n is an example of a Coxeter group (a group generated by reflections). What can you say about pattern avoidance in other Coxeter groups?
  - (Experimental Mathematics) Can you find a single method that efficiently counts $|S_n(Q)|$ for many different examples of Q?

Thank You!