Patterns in Trees

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Problems for Undergraduate Research
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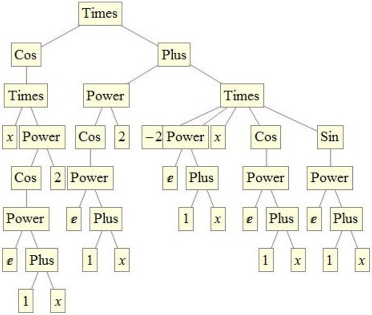
Why tree patterns?

compactly storing expressions in computer memory

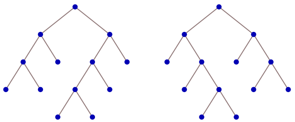
e.g. $\frac{d}{dx}\left(\sin(x\cos^2(e^{x+1}))\right) =$



Notation

Our trees are:

- rooted (root vertex at top, children below)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)



$\mathbb{T}_n$ is the set of n-leaf binary trees.

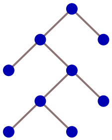
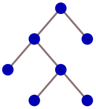
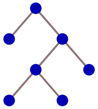
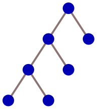
Question: How many trees in $\mathbb{T}_n$ avoid a given tree pattern?

Tree patterns

Contiguous tree pattern

Tree T contains tree t if and only if t is a contiguous rooted ordered subtree of T.

Example:

 contains  and  but avoids .

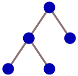
What is the number $a_t(n)$ of n-leaf binary trees avoiding t?

$t =$ 

$a_t(n) = 0$

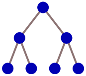
$t =$ 

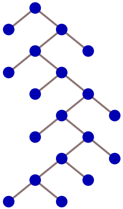
$$a_t(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

$t =$ 

$a_t(n) = 1$

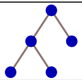
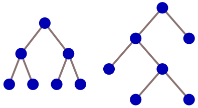
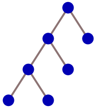
What is the number $a_t(n)$ of n-leaf binary trees avoiding t?

$t =$ 

"Typical" tree avoiding t: 

$$a_t(n) = \begin{cases} 1 & n = 1 \\ 2^{n-2} & n > 1 \end{cases}$$

Contiguous pattern enumeration data

t	$a_t(n)$
	0
	$\begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$
	1
	2^{n-2}
	M_{n-1} (Motzkin numbers)

Contiguous tree pattern results

- Rowland: contiguous pattern avoidance in binary trees
- Gabriel, Peske, P., Tay: extended Rowland's results to ternary trees

Contiguous tree pattern results

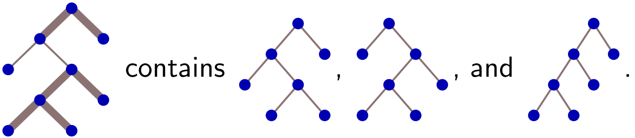
- Rowland: contiguous pattern avoidance in binary trees
  - Algorithm to determine $\sum_{n\geq 1} a_t(n)x^n$ for any binary tree pattern.
  - $\sum_{n\geq 1} a_t(n)x^n$ is always algebraic.
- Gabriel, Peske, P., Tay: extended Rowland's results to ternary trees
  - Counting results from avoiding ternary tree patterns include Catalan numbers, little Schröder numbers, and other known combinatorial sequences.

Tree patterns

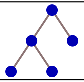
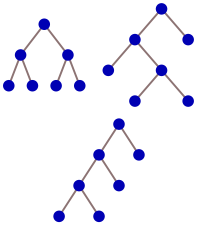
Noncontiguous tree pattern

Tree T contains tree t if and only if there exists a sequence of edge contractions of T (by pairs) that produces t.

Example:



Noncontiguous pattern enumeration data

Pattern t	Number of n-leaf trees avoiding t
	0
	$\begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$
	1
	2^{n-2}

The Main Theorem

Notation

- Let $\mathrm{av}_t(n)$ be the number trees in $\mathbb{T}_n$ that avoid t noncontiguously.
- Let $g_t(x) = \sum_{n=1}^{\infty} \mathrm{av}_t(n)x^n$.

Theorem (Dairyko, P., Tyner, & Wynn, 2011)

Fix $k \in \mathbb{Z}^+$. Let $t, s \in \mathbb{T}_k$. Then $g_t(x) = g_s(x)$.

Generating functions

k	$g_t(x)$, $t \in \mathbb{T}_k$	OEIS number
1	0	trivial
2	x	trivial
3	$\frac{x}{1-x}$	trivial
4	$\frac{x-x^2}{1-2x}$	A000079
5	$\frac{x-2x^2}{1-3x+x^2}$	A001519
6	$\frac{x-3x^2+x^3}{1-4x+3x^2}$	A007051
7	$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$	A080937
8	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$	A024175

Coefficient sightings...

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

Coefficient sightings...

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

Coefficient sightings...

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

An explicit formula

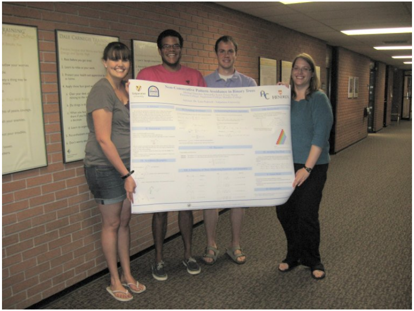
Theorem (Dairyko, P., Tyner, & Wynn, 2011)

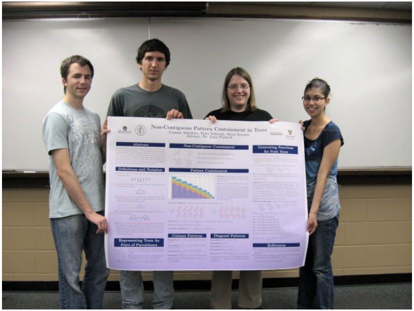
Let $k \in \mathbb{Z}^+$ and let $t \in \mathbb{T}_k$. Then

$$g_t(x) = \frac{\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot \binom{k-(i+2)}{i} \cdot x^{i+1}}{\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot \binom{k-(i+1)}{i} \cdot x^i}.$$

Noncontiguous tree pattern results

- Dairyko, P., Tyner, Wynn: **noncontiguous** pattern avoidance in binary trees
- P., Serrato, Scholten, Schrock: noncontiguous pattern **containment** in *m*-ary trees





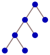
Contiguous Containment

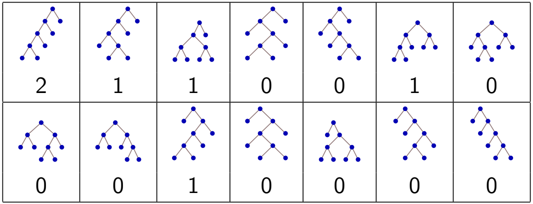
Theorem (Flajolet & Steyaert, 1983)

The number of copies of a k-leaf tree in the set of all n-leaf trees is independent of the tree pattern and is $\binom{2n-k}{n-k}$.

Example: Any 4-leaf tree is contained in the set of 5-leaf trees $\binom{2\cdot 5-4}{5-4} = \binom{6}{1} = 6$ times.

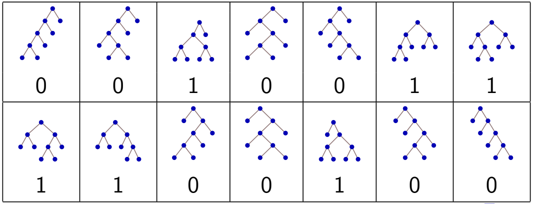
Contiguous Containment Example

6 copies of  in 5-leaf trees



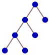
2	1	1	0	0	1	0
0	0	1	0	0	0	0

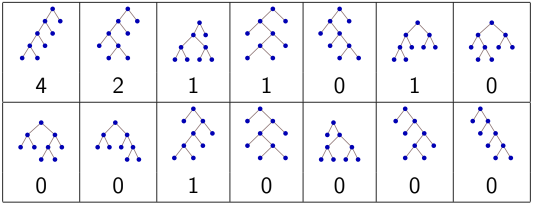
6 copies of  in 5-leaf trees



0	0	1	0	0	1	1
1	1	0	0	1	0	0

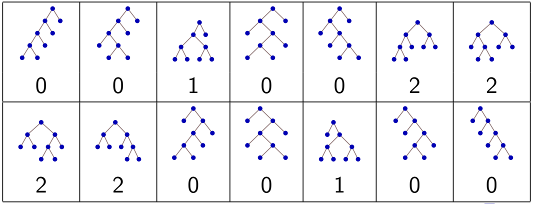
Noncontiguous Containment Example

10 copies of  in 5-leaf trees



4	2	1	1	0	1	0
0	0	1	0	0	0	0

10 copies of  in 5-leaf trees



0	0	1	0	0	2	2
2	2	0	0	1	0	0

Noncontiguous Containment

Theorem (P., Scholten, Schrock, Serrato, 2012)

If $\mathrm{occ}_k(n)$ is the number of noncontiguous occurrences of $t \in \mathbb{T}_k$ in $\mathbb{T}_n$, then $\mathrm{occ}_k(n)$ is independent of t and

$$\sum_{n\geq 1}\sum_{k\geq 1}\mathrm{occ}_k(n)x^n y^k = \frac{\sqrt{1-4x}(1-\sqrt{1-4x})y}{(y+2)\sqrt{1-4x}-y}.$$

Expansion:

$$\frac{\sqrt{1-4x}(1-\sqrt{1-4x})y}{(y+2)\sqrt{1-4x}-y} = xy + x^2(y+y^2) + x^3(2y+4y^2+y^3)+$$

$$x^4(5y+15y^2+7y^3+y^4) + x^5(14y+56y^2+37y^3+\mathbf{10}y^4+y^5)+\cdots$$

Recap

- Rowland: How many *binary* trees *avoid* a given *contiguous* tree pattern?
- Variation 1: How many *ternary* trees *avoid* a given *contiguous* tree pattern?
- Variation 2: How many *binary* trees *avoid* a given *noncontiguous* tree pattern?
- Variation 3: How many *m-ary* trees *contain* a given *noncontiguous* tree pattern?

Where next?

- non-ordered trees
- non-rooted trees
- non-full trees
- semi-contiguous tree patterns
  (some parts must be contiguous; other parts not)

For more details...

- M. Dairyko, L. Pudwell, S. Tyner, and C. Wynn, Non-contiguous pattern avoidance in binary trees, *Electron. J. Combin.* **19** (3) (2012), P22.
- P. Flajolet and J. M. Steyaert, Patterns and pattern-matching in trees: an analysis. *Info. Control* **58** (1983), 19–58.
- N. Gabriel, K. Peske, L. Pudwell, and S. Tay, Pattern avoidance in ternary trees, *J. Integer Seq.* **15** (2012), 12.1.5.
- L. Pudwell, C. Scholten, T. Schrock, and A. Serrato, Non-contiguous pattern containment in binary trees, *ISRN Combinatorics* vol. 2014, Article ID 316535, 8 pages, 2014.
- E. S. Rowland, Pattern avoidance in binary trees, *J. Combin. Theory, Ser. A* **117** (2010), 741–758.

- slides at **faculty.valpo.edu/lpudwell**

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Thanks for listening!