How to Count Words Cleverly

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Outline

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Pattern Avoidance in Permutations

- Given $p \in S_n$ and $q \in S_k$, we say p contains q if there are $1 \leq i_1 < \cdots < i_k \leq n$ such that $p_{i_1} ... p_{i_k}$ reduces to q. Otherwise, p avoids q.

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- $p = 21354$ *contains* 132.
  (since **2**1**354** reduces to 132.)

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- $p = 21354$ *contains* 132.
  (since **2**13**54** reduces to 132.)
- $p = 21354$ *avoids* 321.
  (since p has no decreasing subsequence of length 3.)

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  - $w = 13531246 \in [6]^8$ *contains* 123 and 122.
    (13531**246** reduces to 123.
    13531246 reduces to 122.)
  - $w = 13531246$ *avoids* 111 and 12345.
    (There is no letter repeated 3 times, and
    there is no increasing subsequence of length 5.)

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  - Note: $A_{[1,\ldots,1]}(q) = S_n(q)$.

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- 1998: Zeilberger's *prefix enumeration schemes*, i.e. a system of recurrences to count $S_n(q)$.
- 2005: Vatter's modified schemes automate the enumeration of $S_n(q)$ for even more patterns q.

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- Vatter and Zeilberger's schemes can be modified to count pattern-avoiding words.
  - Advantage: One plan of attack for many different patterns.

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   - Pattern Avoidance
   - Previous Work
2. Counting Pattern-Avoiding Words
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   - Examples

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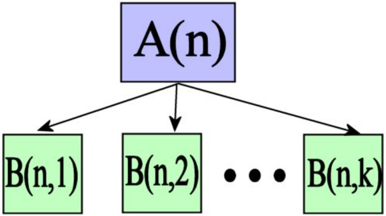
  1. Break $A(n)$ into disjoint union $\bigcup_{i \in I} B(n,i)$. (refinement according to parameter i)
  2. Look for a recurrence for each $B(n,i)$.



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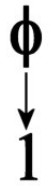

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$$\phi \downarrow 1$$

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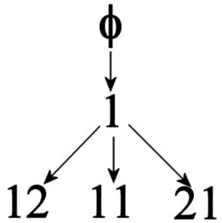

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- If p_r is reversibly deletable, and the role of p_r is played by letter j, then $A_{[a_1,...a_j,...a_n]}(q) = A_{[a_1,...a_j-1,...a_n]}(q)$.

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e.g. $v =< 0, 1, 1 >$ is a gap vector for $q = 123$, $p = 12$.

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4. Find reversibly deletable positions with respect to gap vectors.
5. Repeat steps 2-4 until all unrefined prefixes have reversibly deletable elements.

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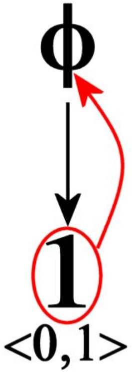

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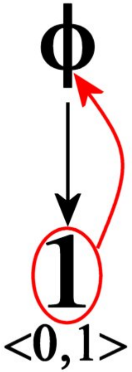

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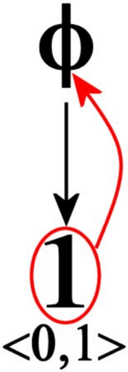
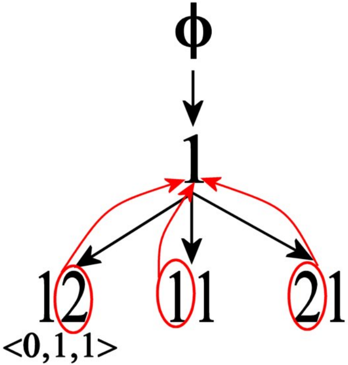


- Gap Vector: If a word avoids 12, the first letter must be the biggest letter.
- Reversibly Deletable: The first (i.e. biggest) letter cannot be in a 12 pattern, so it can be deleted.

Avoid(123)



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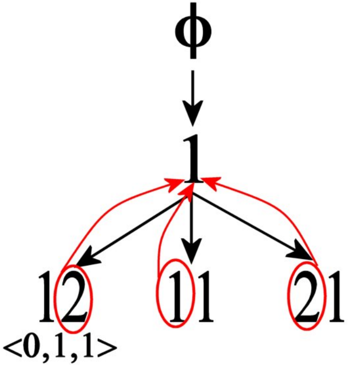

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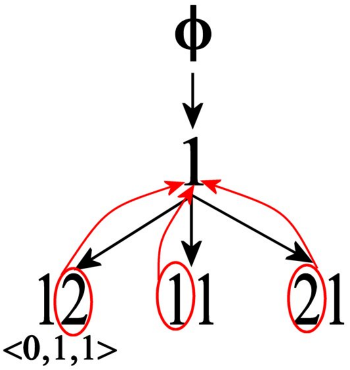
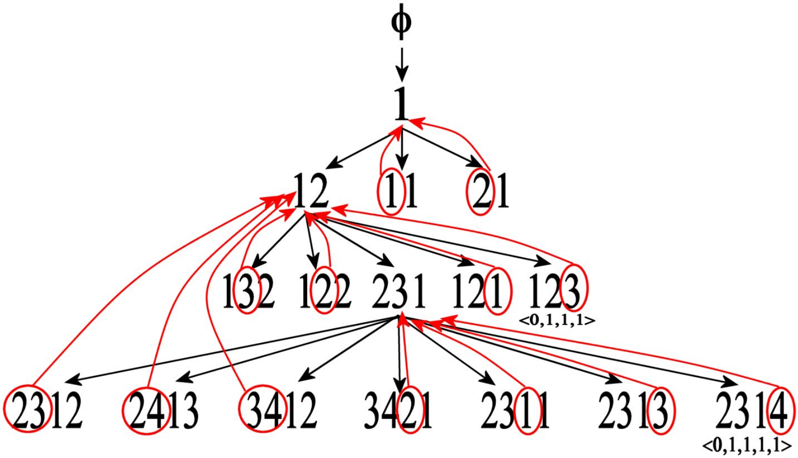
Avoid(123)



- Gap Vector: If a word avoids 123, and has prefix 12, the role 2 must be played by the largest letter in the word.
- Reversibly Deletable:
  - The 2 in a 12 prefix cannot be part of a bad pattern if it is the largest letter.
  - If the 1st 1 in a 11 prefix is in a bad pattern, the other 1 is as well.
  - If the 2 in a 21 prefix is part of a bad pattern, the 1 is as well.

Avoid(1234)


ϕ
1
12
11
21
132
122
231
121
123
<0,1,1,1>
2312
2413
3412
3421
2311
2313
2314
<0,1,1,1,1>

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