How to Count Words Cleverly

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Outline

1. Introduction
   - Pattern Avoidance
   - Previous Work

2. Counting Pattern-Avoiding Words
   - Counting Concepts
   - Examples
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Given a string of letters $p = p_1 \ldots p_n$, the reduction of $p$ is the string obtained by replacing the $i^{th}$ smallest letter of $p$ with $i$. 
**Reduction**

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- For example, the reduction of $26745$ is $1_2 \ldots$.
Given a string of letters $p = p_1...p_n$, the reduction of $p$ is the string obtained by replacing the $i^{th}$ smallest letter of $p$ with $i$.

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For example, the reduction of $26745$ is $14523$. 
Pattern Avoidance in Permutations

Given $p \in S_n$ and $q \in S_k$, we say $p$ contains $q$ if there are $1 \leq i_1 < \cdots < i_k \leq n$ such that $p_{i_1} \cdots p_{i_k}$ reduces to $q$. Otherwise, $p$ avoids $q$. 

**Example:**

$p = 21354$ contains $132$ (since $21354$ reduces to $132$).

$p = 21354$ avoids $321$ (since $p$ has no decreasing subsequence of length 3.).
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Pattern Avoidance in Words

A word in $[k]^n$ is a string $w = w_1 \ldots w_n$ where $1 \leq w_i \leq k$ for each $i$. 
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  - \(w = 13531246 \in [6]^8\) contains 123 and 122.
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- Pattern avoidance in words is defined analogously for words.
  - $w = 13531246 \in [6]^8$ contains 123 and 122.  
    (13531246 reduces to 123.  
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  - $w = 13531246$ avoids 111 and 12345.  
    (There is no letter repeated 3 times, and there is no increasing subsequence of length 5.)
Key Question

For Permutations...

- Let $S_n(q) = |\{p \in S_n \mid p \text{ avoids } q\}|$. 

For Words...

Note: $A[a_1, \ldots, a_k](q) = |\{w \in [k]^{a_1 + \cdots + a_k} \mid w \text{ avoids } q, \text{ and } w \text{ has } a_i \text{'s}\}|$.

Given $q$, find a way to count $A[a_1, \ldots, a_k](q)$. 

Note: $A[1, \ldots, 1](q) = S_n(q)$. 

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  - Note: $A_{[1,\ldots,1]}(q) = S_n(q)$. 
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- 1998: Zeilberger’s *prefix enumeration schemes*, i.e. a system of recurrences to count $S_n(q)$. 
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- 1998: Zeilberger’s *prefix enumeration schemes*, i.e. a system of recurrences to count $S_n(q)$.
- 2005: Vatter’s modified schemes automate the enumeration of $S_n(q)$ for even more patterns $q$. 

Vatter and Zeilberger’s schemes can be modified to count pattern-avoiding words.
For Words

- Vatter and Zeilberger’s schemes can be modified to count pattern-avoiding words.
  - Advantage: One plan of attack for many different patterns.
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- Ideally, find a recurrence $A(n) = \sum_{i \in I} c(i) \cdot A(n - i)$.
- If not...
  1. Break $A(n)$ into disjoint union $\bigcup_{i \in I} B(n, i)$. (refinement according to parameter $i$)
  2. Look for a recurrence for each $B(n, i)$.
For pattern-avoiding words, refine according to prefixes.
Refinement for Words

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Given a prefix $p = p_1 ... p_t$, position $r$ is reversibly deletable if every possible bad pattern involving $p_r$ implies another bad pattern without $p_r$. 
Reversibly Deletable

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- For example, avoid $q = 123$, and let $p = 21...$

  $21...a...b$
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\[
\begin{align*}
21\ldots &a\ldots b \\
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For example, avoid $q = 123$, and let $p = 21...$

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If $p_r$ is reversibly deletable, and the role of $p_r$ is played by letter $j$, then

$A_{[a_1,...a_j,...a_n]}(q) = A_{[a_1,...a_{j-1},...a_n]}(q)$.
Consider words that avoid $q = 123$ and begin with prefix $p = 12$

- sorted prefix: 1 2
- letters involved in prefix: i j
- vector: <a, b, c>
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sorted word: $\cdots i \cdots j \cdots$
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Gap Vectors

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sorted prefix: \( 1 \ 2 \)

letters involved in prefix: \( i \ j \)

vector: \( <a, b, c> \)

sorted word: \( \begin{array}{cccc}
\cdot & \cdot & i & \cdot & \cdot & j & \cdot & \cdot \\
\geq a & \geq b - 1 & \geq c
\end{array} \)

\( \nu \) is a gap vector for \( p \) if there are no words avoiding \( q \) with prefix \( p \) and spacing \( \nu \).
Gap Vectors

Consider words that avoid $q = 123$ and begin with prefix $p = 12$

sorted prefix: 1 2
letters involved in prefix: $i$ $j$
vector: $<a, b, c>$

sorted word: $\cdot \cdot \cdot i \cdot \cdot \cdot j \cdot \cdot \cdot$
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$v$ is a gap vector for $p$ if there are no words avoiding $q$ with prefix $p$ and spacing $v$.

e.g. $v = <0, 1, 1>$ is a gap vector for $q = 123$, $p = 12$. 
Algorithm for finding an enumeration scheme:

1. Start with the empty prefix.
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3. Find gap vectors for all such refinements.
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5. Repeat steps 2-4 until all unrefined prefixes have reversibly deletable elements.
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Avoid(12)

Gap Vector: If a word avoids 12, the first letter must be the biggest letter.

Reversibly Deletable: The first (i.e., biggest) letter cannot be in a 12 pattern, so it can be deleted.
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Avoid(123)

- **Gap Vector**: If a word avoids 123, and has prefix 12, the role 2 must be played by the largest letter in the word.

- **Reversibly Deletable**:
  - The 2 in a 12 prefix cannot be part of a bad pattern if it is the largest letter.
  - If the 1st 1 in a 11 prefix is in a bad pattern, the other 1 is as well.
  - If the 2 in a 21 prefix is part of a bad pattern, the 1 is as well.
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Summary

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Future work
- Find closed formulas from recurrence relations given by schemes.
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