Non-contiguous pattern avoidance in binary trees

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Key Question

How many permutations of length n avoid a given permutation pattern?

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How many binary trees with n leaves avoid a given tree pattern?

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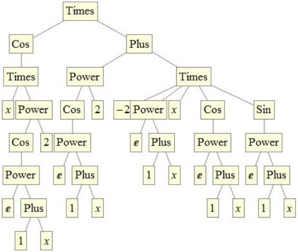
How many binary trees with n leaves avoid a given tree pattern?

Concerned with rooted, ordered, full binary trees
(each vertex has exactly 0 or 2 children)

History of Tree Patterns: Labelled Trees

- 1983: Flajolet and Steyaert
  - focus on asymptotic probability of avoiding a given pattern
- 1990: Flajolet, Sipala, and Steyaert
  - every leaf of pattern must be matched by a leaf of the tree
  - motivated by compactly storing expressions in computer memory
  - e.g. $\dfrac{d}{dx}\left(\sin(x\cos^2(e^{x+1}))\right) =$



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  - every leaf of pattern must be matched by a leaf of the tree
  - motivated by compactly storing expressions in computer memory
- 2012: Dotsenko
  - pattern may occur anywhere in tree
  - motivated by operad theory

History of Tree Patterns: Unlabelled Trees

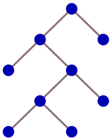
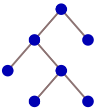
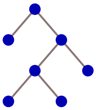
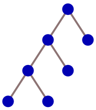
- 2009: Rowland
  - contiguous pattern avoidance in binary trees
  - patterns can be anywhere, not just at leaves
- 2010: Gabriel, Peske, P., Tay
  - extended Rowland's results to m-ary trees
- 2011: Dairyko, P., Tyner, Wynn
  - non-contiguous pattern avoidance in binary trees

Tree patterns

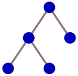
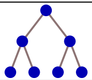
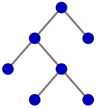
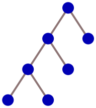
Contiguous tree pattern (Rowland)

Tree T contains tree t if and only if T contains t as a contiguous rooted ordered subtree.

Example:

 contains  and  but avoids .

Contiguous pattern enumeration data

Pattern t	Number of n leaf trees avoiding t
	0
	$\begin{cases} 1 & n=1 \\ 0 & n>1 \end{cases}$
	1
	2^{n-2}
	2^{n-2}
	M_n (Motzkin numbers)

Contiguous tree pattern enumeration

Rowland

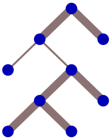
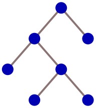
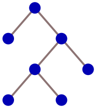
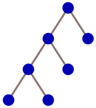
- Devised algorithm to find functional equation for avoidance generating function for any set of tree patterns.
- Generating functions are always algebraic.
- Enumerated trees containing specified number of copies of a given tree pattern.
- Completely determined Wilf classes for tree patterns with at most 8 leaves.

Tree patterns

Non-contiguous tree pattern (Dairyko, P., Tyner, Wynn)

Tree T contains tree t if and only if there exists a sequence of edge contractions of T that produce T^* which contains t as a contiguous rooted ordered subtree.

Example:

 contains , , and .

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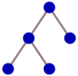
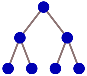
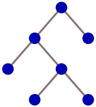
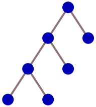
	1
	2^{n-2}
	2^{n-2}
	2^{n-2}

The Main Theorem

Notation

- Let $\text{av}_t(n)$ be the number of n-leaf trees that avoid t non-contiguously.
- Let $g_t(x) = \sum_{n=1}^{\infty} \text{av}_t(n) x^n$.

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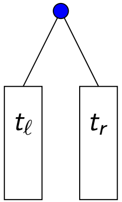
Theorem

Fix $k \in \mathbb{Z}^+$. Let t and s be two k-leaf binary tree patterns. Then $g_t(x) = g_s(x)$.

Notation and Computation

(More) Notation

- Given tree t,
  - let t_ℓ be the subtree whose root is the left child of t's root.
  - let t_r be the subtree whose root is the right child of t's root.



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Notice

$$g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)$$

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$$g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)$$

Solving...

$$g_t(x) = \frac{x - g_{t_\ell}(x) \cdot g_{t_r}(x)}{1 - g_{t_\ell}(x) - g_{t_r}(x)}.$$

Proposition

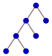
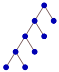
$$g_t(x) = \frac{x - g_{t_\ell}(x) \cdot g_{t_r}(x)}{1 - g_{t_\ell}(x) - g_{t_r}(x)}.$$

Proposition

For any tree pattern t, $g_t(x)$ is a rational function of x.

A special case...

Let c_k be the k-leaf left comb
(the unique k-leaf binary tree where every right child is a leaf).

$c_1 =$ , $c_2 =$ , $c_3 =$ , $c_4 =$ , $c_5 =$ ,etc.

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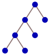
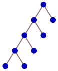
$c_1 = \bullet$, $c_2 =$ , $c_3 =$ , $c_4 =$ , $c_5 =$ ,etc.

If $t = c_k$, then $t_\ell = c_{k-1}$ and $t_r = \bullet$.

For $k \geq 2$, we have

$$g_{c_k}(x) = \frac{x - g_{c_{k-1}}(x) \cdot g_{\bullet}(x)}{1 - g_{c_{k-1}}(x) - g_{\bullet}(x)} = \frac{x}{1 - g_{c_{k-1}}(x)}.$$

Back to the main result

Theorem

Fix $k \in \mathbb{Z}^+$. Let t and s be two k-leaf binary tree patterns. Then $g_t(x) = g_s(x)$.

Proof sketch

Inductive step:

- Assume the theorem holds for tree patterns with ℓ leaves where $\ell < k$.
- Then any ℓ-leaf tree has avoidance generating function $g_{c_\ell}(x)$.
- Consider tree t with ℓ leaves to the left of its root and tree s with $\ell + 1$ leaves to the left of its root.
- Do algebra with previous work to show that $gf_t(x) = gf_s(x)$.

Generating functions

k	$g_{c_k}(x)$	OEIS number
1	0	trivial
2	x	trivial
3	$\frac{x}{1-x}$	trivial
4	$\frac{x-x^2}{1-2x}$	A000079
5	$\frac{x-2x^2}{1-3x+x^2}$	A001519
6	$\frac{x-3x^2+x^3}{1-4x+3x^2}$	A007051
7	$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$	A080937
8	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$	A024175
9	$\frac{x-6x^2+10x^3-4x^4}{1-7x+15x^2-10x^3+x^4}$	A080938

An explicit formula

Theorem

Let $k \in \mathbb{Z}^+$ and let t be a binary tree pattern with k leaves. Then

$$g_t(x) = \frac{\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot \binom{k-(i+2)}{i} \cdot x^{i+1}}{\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot \binom{k-(i+1)}{i} \cdot x^i}.$$

...and permutations

We know that the Catalan numbers count:

- the number of binary trees
- the number of 231-avoiding permutations

Can we say more?

...and permutations

We know that the Catalan numbers count:

- the number of binary trees
- the number of 231-avoiding permutations

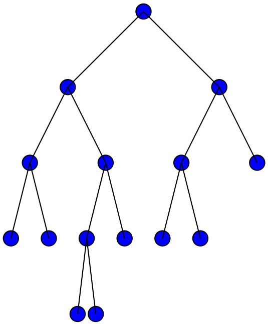
Can we say more?

Theorem

Let t be any binary tree pattern with $k \geq 2$ leaves. Then

$$\mathsf{av}_t(n) = \mathsf{s}_{n-1}(231, (k-1)(k-2)\cdots 21).$$

Example



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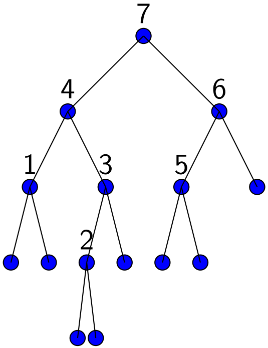

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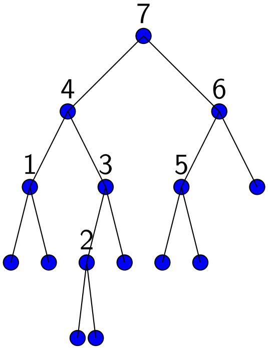

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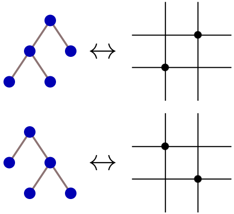
Main theorem revisited....

Theorem

Fix $k \in \mathbb{Z}^+$. Let t and s be two k-leaf binary tree patterns. Then $g_t(x) = g_s(x)$.

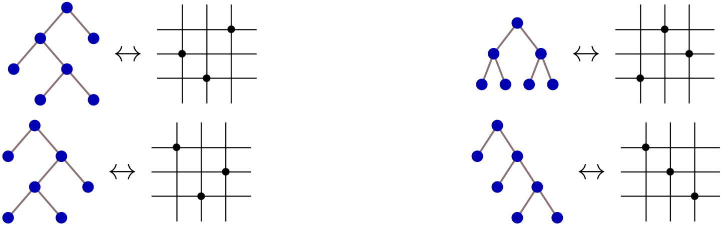
Under the tree $\leftrightarrow$ 231-avoiding permutation bijection, this theorem translates into a set of Wilf-equivalances for permutations too!

3-leaf tree equivalence class



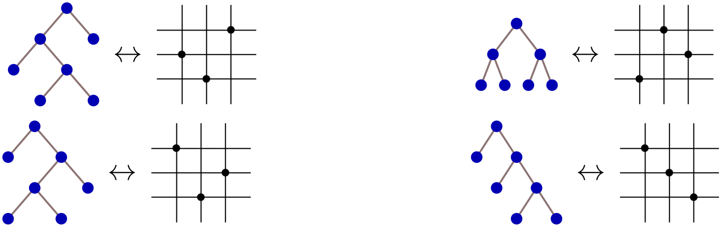
So $s_n(231, 12) = s_n(231, 21)$ (or, really $s_n(12) = s_n(21)$).

4-leaf tree equivalence class



So $s_n(231, 213) = s_n(231, 132) = s_n(231, 312) = s_n(231, 321)$

4-leaf tree equivalence class



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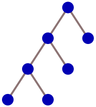
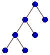
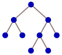


$\leftrightarrow$?

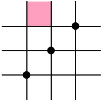
The permutation pattern 123

123 may appear in a binary tree in two ways:

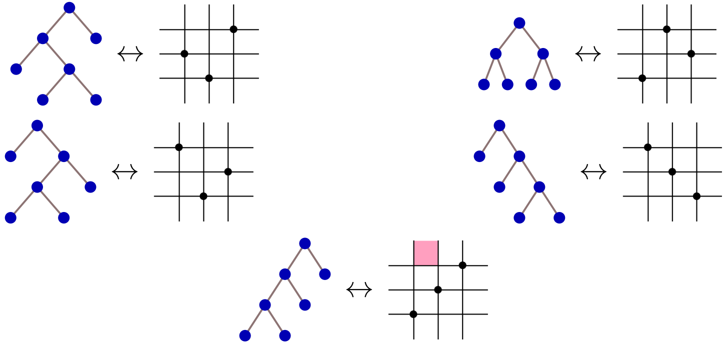




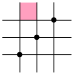
We are interested in the first type, rather than the second type.

Therefore, use the *mesh* pattern  .

4-leaf tree equivalence class

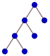
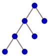
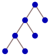


So $s_n(231, 213) = s_n(231, 132) = s_n(231, 312) =$

$s_n(231, 321) = s_n\left(231, \right.$  $\left.\right)$

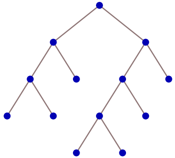
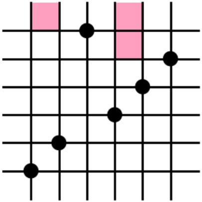
Corollary: Wilf classes for permutations

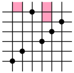
Let $f(t)$ be the classical permutation corresponding to t under the tree $\leftrightarrow$ 231-avoiding permutation bijection.
Define π^t as follows:

- If t avoids , then $\pi^t = f(t)$ is a classical pattern.
- If t contains , then π^t is a mesh pattern whose underlying pattern is $f(t)$, but for every 123 pattern ijk in $f(t)$ corresponding to a copy of , place a mesh restriction between i and j and above k.

Then the permutation pattern sets $\{231, \pi^t\}$ and $\{231, \pi^s\}$ are Wilf-equivalent for any two n-leaf trees t and s.

Larger example

 $\leftrightarrow$ 

So, $s_n\Big(231,$  $\Big) = s_n(231, 654321).$

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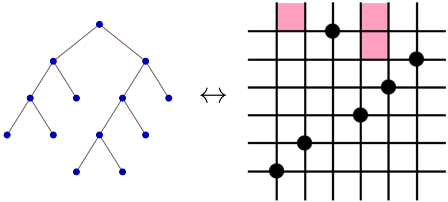
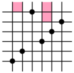


So, $s_n\Big(231,$  $\Big) = s_n(231, 654321)$.

Corollary

For any $n \in \mathbb{Z}^+$, there Catalan many Wilf-equivalent pattern sets of the form $\{231, \pi\}$ where π is a mesh pattern of length n.

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Summary

- $g_t(x)$ is rational and of a very nice form for any non-contiguous tree pattern t.
- Only one Wilf class for each number of leaves!
- Trees avoiding a k-leaf tree pattern are in bijection with permutations avoiding 231 and $(k-1)(k-2)\cdots 1$.
- For any $n \in \mathbb{Z}^+$, there are at least Catalan-many Wilf equivalent pattern sets of the form $\{231, \pi\}$ where π is a mesh pattern of length n.

Thank You!

References

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