Avoiding an Ordered Partition of Length 3

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(joint work with Anant Godbole and Adam Goyt)

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Set Partitions

Definition

A *partition* p of set $\{1, \ldots, n\}$ is a collection of nonempty subsets $B_1, B_2, \ldots, B_k$ called *blocks* such that

1. $B_1 \cup B_2 \cup \cdots \cup B_k = \{1, \ldots, n\}$ and
2. $B_i \cap B_j = \emptyset$ for $i \neq j$.

Write $B_1/B_2/\cdots/B_k$.

Conventions:

- Write elements in each block in increasing order.
- Order blocks by increasing minimal elements.

Example: $15/2/346$ is a partition of $\{1, 2, 3, 4, 5, 6\}$.

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Pattern avoidance in set partitions:

Klazar (1996, 2000), Jelínek/Mansour (2008), Goyt (2008), Goyt/Sagan (2009), Sagan (2010)

Ordered Partitions

Definitions

An *ordered partition* is a set partition where the order of the blocks is important.
The number of blocks of p is the *length* of p.

Example: 15/2/346, 15/346/2, 2/15/346,
2/346/15, 346/15/2, and 346/2/15
are distinct ordered partitions of length 3
of the set $\{1, 2, 3, 4, 5, 6\}$.

Graphs of Ordered Partitions

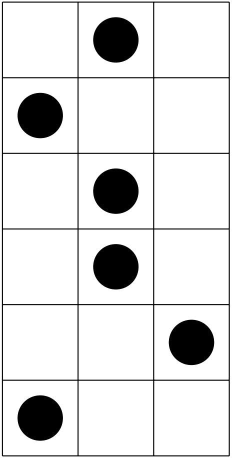
Plot the point (i, j) iff $j \in B_i$.

Note:

- May not have two points in same row.
- May have two points in same column.
- No empty rows or columns.

Example: $p = 15/346/2$

Graph of p:



Definition

Ordered partition p of length k *contains* permutation $\pi \in \mathcal{S}_n$ if there exist

- $1 \leq i_1 < i_2 < \cdots < i_n \leq k$
  and
- $b_j \in B_{i_j}$ for $1 \leq j \leq n$

such that $b_1 \cdots b_n$ is order-isomorphic to π.

Example: $p = 14/56/2/3$ contains $\pi = 312$.

$i_1 = 1, \ i_2 = 3, \ i_3 = 4$

$b_1 = 4, \ b_2 = 2, \ b_3 = 3$

... or as a graph

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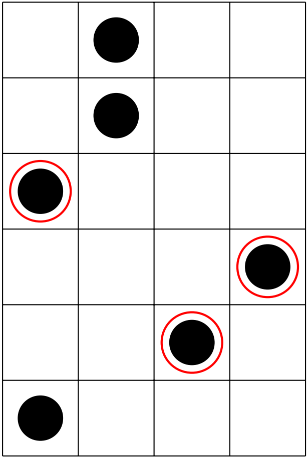

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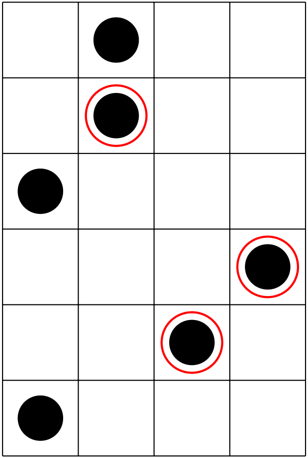

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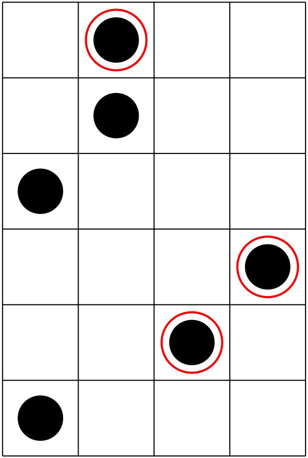

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- Note: $\mathrm{op}_{n,n}(\pi) = \mathrm{op}_{[\underbrace{1,\ldots,1}_{n}]}(\pi) = |\mathcal{S}_n(\pi)|$

Review?

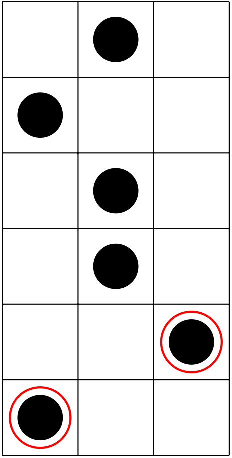
At PP2012, Goyt presented:

- initial counting results
- proof that $\mathrm{op}_{[b_1,\ldots,b_k]}(123) = \mathrm{op}_{[b_1,\ldots,b_k]}(132)$
- relationship between ordered partitions and pattern avoiding words in $[k]^n$.

Today...

- $\mathrm{op}_{n,k}(12)$
- $\mathrm{op}_{n,3}(123)$
- $\mathrm{op}_{n,n-1}(123)$
- $\mathrm{op}_{[2,\ldots,2]}(123)$

Avoiding 12



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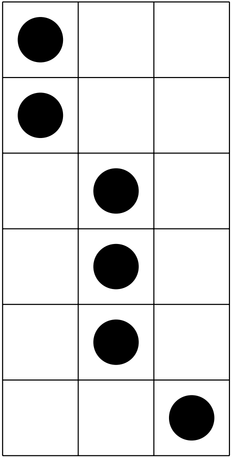

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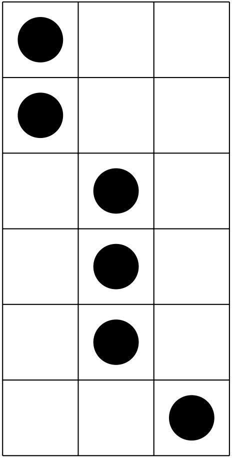

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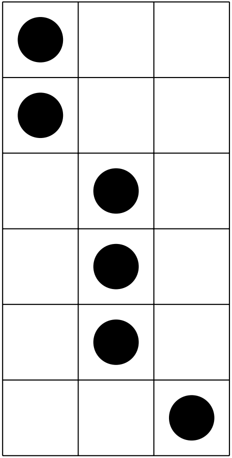


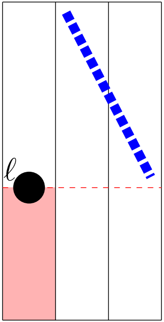
$\mathrm{op}_{n,k}(12) =$ number of integer compositions of n into k parts $= \binom{n-1}{k-1}$.

In as few blocks as possible

Theorem

$$\mathrm{op}_{n,3}(123) = \left(\frac{n^2}{8} + \frac{3n}{8} - 2\right) 2^n + 3$$

Sketch of Proof:



$b_1 := |B_1|$
$b_2 := |B_2|$
$\ell := min(B_1)$
$i := |\{j \in B_2 : j < \ell\}|$

Obtain

$$\sum_{b_1=1}^{n-2} \sum_{b_2=1}^{n-1-b_1} \sum_{\ell=1}^{n-b_1+1} \sum_{i=max(0,\ell-1-b_3)}^{min(\ell-1,b_2)} \binom{n-\ell}{b_1-1}\binom{\ell-1}{i}$$

and simplify using CAS.

In as many blocks as possible

Theorem

$$\mathrm{op}_{n,n-1}(123) = \frac{3(n-1)^2\binom{2n-2}{n-1}}{n(n+1)}$$

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Outline:

1. Pick $\pi \in \mathcal{S}_{n-1}(123)$ such that $\pi_1 = i$.
2. Treat π as if in $\mathcal{OP}_{n-1,n-1}(123)$ and insert a larger number into the first block.
3. Now have $\mathrm{op}_{[2,1,\ldots,1]}(123)$.
   Show that $\mathrm{op}_{[2,1,1,1,\ldots,1]}(123) = \mathrm{op}_{[1,2,1,1,\ldots,1]}(123) = \mathrm{op}_{[1,1,2,1,\ldots,1]}(123) = \cdots = \mathrm{op}_{[1,1,1,\ldots,1,2]}(123)$.

Step 1: Pick a permutation.

Proposition

The number of 123-avoiding permutations of length n that begin with i is given by

$$c_{n,i} = \frac{(n-2+i)!(n-i+1)}{(i-1)!n!}.$$

(OEIS A0009766: Catalan's triangle read by rows)

e.g. $c_{3,1} = 1$ (132)

$c_{3,2} = 2$ (213, 231)

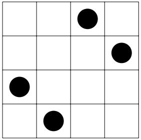
$c_{3,3} = 2$ (312, 321)

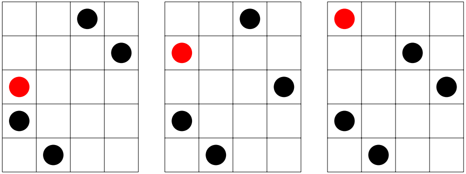
Step 2: Add a dot.

Proposition

If $\pi \in \mathcal{S}_{n-1}(123)$ has $\pi_1 = i$, then there are $n - i$ ways to insert an element larger than i into B_1.

And inserting a larger element into B_1 cannot introduce a 123-pattern.





Halfway there...

We've just shown:

$$\mathrm{op}_{[2,\underbrace{1,\ldots,1}_{n-2}]}(123) = \sum_{i=1}^{n-1}(n-i)c_{n-1,i} = \sum_{i=1}^{n-1}(n-i)\frac{(n-3+i)!(n-i)}{(i-1)!(n-1)!}.$$

Or, after simplifying....

Lemma

For $n \geq 2$,

$$\mathrm{op}_{[2,\underbrace{1,\ldots,1}_{n-2}]}(123) = \frac{3(n-1)\binom{2n-2}{n-1}}{n(n+1)}.$$

(OEIS A000245)

Reassessing...

Goal: $\mathrm{op}_{n,n-1}(123)$

Have: $\mathrm{op}_{[2,\underbrace{1,\ldots,1}_{n-2}]}(123)$

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In fact....

Proposition (generalized!)

$\mathrm{op}_{[b_1,\ldots,b_k]}(123) = \mathrm{op}_{[c_1,\ldots,c_k]}(123)$
where $c_1 \cdots c_k$ is any permutation of $b_1 \cdots b_k$.

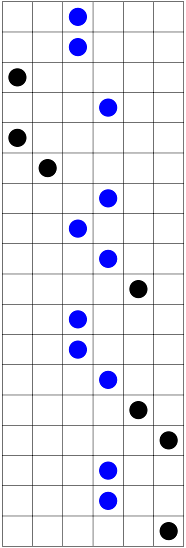
Bijection

$$\mathcal{OP}_{[b_1,\ldots,b_i,b_{i+1},\ldots,b_k]}(123) \leftrightarrow \mathcal{OP}_{[b_1,\ldots,b_{i+1},b_i,\ldots,b_k]}(123)$$

For every $j \in B_i \cup B_{i+1}$ exactly one of the following is true:

1. j is not involved in a 12 pattern outside of $B_i \cup B_{i+1}$.
2. j plays the role of a 1 in a 12 pattern involving elements outside of $B_i \cup B_{i+1}$.
3. j plays the role of a 2 in a 12 pattern involving elements outside of $B_i \cup B_{i+1}$.

Example



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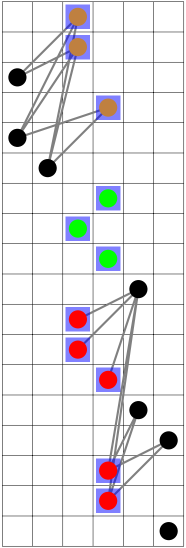

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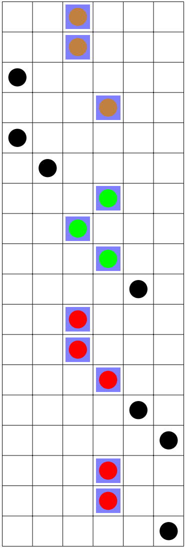

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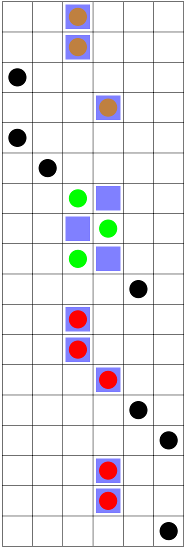

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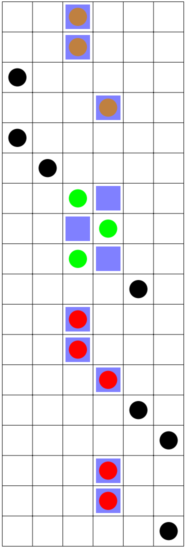

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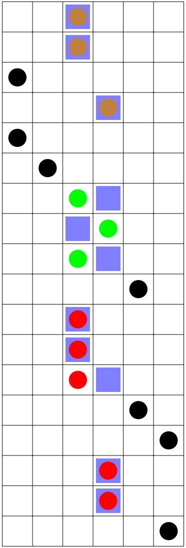

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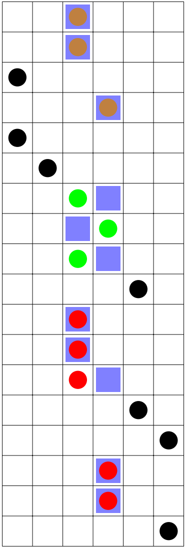

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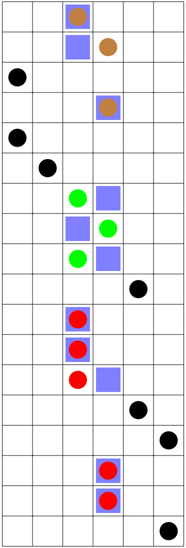

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Upshot

Theorem

For $n \geq 2$,

$$\mathrm{op}_{n,n-1}(123) = (n-1)\mathrm{op}_{[2,\underbrace{1,\ldots,1}_{n-2}]}(123) = \frac{3(n-1)^2\binom{2n-2}{n-1}}{n(n+1)}.$$

And more generally...

Theorem

For $n > p \geq 1$, the number of 123-avoiding ordered partitions of n into $n-p+1$ parts where there is one part of size p and $n-p$ parts of size 1 is given by

$$\frac{(n-p+1)(p+1)\binom{2n-p}{n-p}}{(n+1)}.$$

Credit to Shalosh B. Ekhad...

We didn't notice the nice formula for $\mathrm{op}_{n,n-1}(123)$ until Zeilberger's *FindRec* package predicted

$$\mathrm{op}_{n,n-1}(123) = \begin{cases} 1 & n = 2 \\ \dfrac{(4n-6)(n-1)^2}{(n-2)^2(n+1)}\mathrm{op}_{n-1,n-2}(123) & n > 2. \end{cases}$$

(The solution to this recurrence is indeed the formula from the previous slide.)

Zeilberger's *FindRec* package also predicted:

$$\mathrm{op}_{[\underbrace{2,\ldots,2}_{k}]}(123) =$$

$$\frac{329k^3-749k^2+514k-96}{2k(2k+1)(7k-9)}\mathrm{op}_{[\underbrace{2,\ldots,2}_{k-1}]}(123)+\frac{3(14k^3-39k^2+31k-6)}{k(2k+1)(7k-9)}\mathrm{op}_{[\underbrace{2,\ldots,2}_{k-2}]}(123).$$

More fun with Shalosh

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(Proved by Chen, Dai, and Zhou using generating function techniques, preprint on arXiv)

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- Characterize when two permutations π and σ have $\mathrm{op}_{[b_1,\ldots,b_k]}(\pi) = \mathrm{op}_{[b_1,\ldots,b_k]}(\sigma)$ for any choice of $b_1, \ldots, b_k$.
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- Asymptotics of $\mathrm{op}_{[b_1,\ldots,b_k]}(\pi)$ and $\mathrm{op}_{n,k}(\pi)$
  Stay tuned for Adam Goyt's talk next!

References

- W. Chen, A. Dai, R. Zhou, Ordered Partitions Avoiding a Permutation of Length 3, arXiv:1304.3187.
- A. Godbole, A. Goyt, J. Herdan, L. Pudwell, Pattern Avoidance in Ordered Set Partitions, arXiv:1212.2530, submitted.

References

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Thanks for listening!