# Pattern Avoiding Colored Partitions

Adam M. Goyt
Minnesota State University Moorhead
goytadam@mnstate.edu

Lara K. Pudwell
Valparaiso University
Lara.Pudwell@valpo.edu

Valparaiso University

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# Outline

1. History and Definitions
2. Colored Partitions and Avoidance
3. A Flavor of the Proofs
4. Summary and Future Ideas

# Who and What

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- The notion of signed set partitions was considered by Anders Björner and Michelle Wachs [1] from a poset and homological perspective.
- Now, we consider Pattern Avoidance in Colored Set Partitions. (Boom?)

# Set Partition Definition

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**Example**

$137/25/46 \vdash [7]$

# Canonical Words

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*The canonized form of 47411477 is 12133122.*

There is a bijection between all canonized words of length $n$ and partitions of $[n]$.

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137/25/46 *corresponds to* 1213231.

We will say that a partition, $\pi$ is of length $n$, $\ell(\pi) = n$, if its associated canonical word has $n$ letters.

From now on we will refer to these canonical words as partitions.

# Pattern Avoidance

**Definition**

*Let $\sigma$ be a partition with $\ell(\sigma) = n$ and $\pi$ be a partition with $\ell(\pi) = k$. We say that $\sigma$ contains $\pi$ if there is a subsequence of $\sigma$ of length $k$ whose canonization is $\pi$. Otherwise we say that $\sigma$ avoids $\pi$.*

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## Example

*Let* $\sigma = 1213431$.

$\sigma$ *contains a copy of* 112 *namely* 1213431 *or* 1213431. *However,* $\sigma$ *avoids* 1112.

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**Definition**

*Denote the set of all k-colored set partitions of* $[n]$ *by* $\Pi_n \wr C_k$.

**Example**

*Consider* $\sigma = 1213431 \in \Pi_7$ *from the previous slide. We can make* $\sigma$ *an element of* $\Pi_7 \wr C_3$ *simply by choosing one of three colors for each of the elements. So* $1211341 \in \Pi_7 \wr C_3$.

# Avoiding Colored Partitions

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*We say that $\sigma \in \Pi_n \wr C_k$ contains a copy of $\pi \in \Pi_m \wr C_j$ if*

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## Example

*Consider $\sigma = 1211341$. Then $\sigma$ contains a copy of 122, but $\sigma$ avoids 122.*

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*Otherwise we say that $\sigma$ avoids $\pi$.*

**Example**

*Consider $\sigma = 1211341$. Then $\sigma$ contains a copy of 122, but $\sigma$ avoids 122.*

**Definition**

*For a set of colored set partitions $S$ let $\Pi_n \wr C_k(S)$ be the set of partitions in $\Pi_n \wr C_k$ that avoid every pattern in $S$.*

# Outline

# Friendly Results

**Theorem**

*For* $n \geq 1$ *and* $c \geq 2$, $|\Pi_n \wr C_2(11, 11)| = \sum_{i=1}^{n} 2^i S(n, i)$
*(OEIS A001861)*

**Theorem**

*For* $n \geq 1$ *and* $c \geq 2$, $|\Pi_n \wr C_2(12)| = (B_{n+2} - B_{n+1}) - (B_{n+1} - B_n)$
*(OEIS A011965)*

# $|\Pi_n \wr C_k(112)|$

### Theorem

*For $n \geq 1$ and $c \geq 2$,* $|\Pi_n \wr C_k(112)| =$

$$B(n)(k-1)^n + \sum_{m=1}^{n} \sum_{j=1}^{m} \binom{n}{m}\binom{m}{j} B(n-j)(k-1)^{n-j} +$$
$$\sum_{1\leq i<j\leq n} \sum_{a,b} \sum_{d,e} \sum_{f,g} \sum_{m} \sum_{p,q} \sum_{\ell} \binom{i-1}{a,b}\binom{j-i-1}{d,e}\binom{n-j}{f,g}\cdot$$
$$\binom{i-a-b-1}{m}\binom{j-i-d-e-1}{p}\binom{n-a-b-f-g-j+i-m-1}{q}\cdot$$
$$S(p+q,\ell)m^{\ell}B(n-a-b-d-e-f-g-m-p-q-2)\cdot$$
$$((k-1)^b + b(k-1)^{b-1})((k-1)^a + a(k-1)^{a-1})(k-2)^{d+p}k^e\cdot$$
$$(k-1)^{n-a-b-d-e-m-p-2}.$$

Back

# Avoiding 112 – Sketch of Proof

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The Proof of this Theorem can be broken into 3 cases.

Case 1: No elements are colored blue.

Case 2: Exactly one block contains elements colored blue.

Case 3: At least two blocks contain elements colored blue.

# No elements are colored blue.

In this case there can't possibly be a copy of 112.

- $B(n)$ ($n^{th}$ Bell number) ways to partition the elements in $[n]$
- $(k-1)^n$ ways to color each element any color but blue.

Thus there are $B(n)(k-1)^n$ such 112 avoiding partitions in $\Pi_n \wr C_k$.

# Exactly one block contains elements colored blue.

In this case there can't possibly be a copy of 112.

- Select $m$ elements to be in the block with the elements that are colored blue.
- Select $j$ of these elements to be colored blue.
- Partition the remaining $n-m$ elements in $B(n-m)$ ways.
- Color the non-blue elements in $(k-1)^{n-j}$ ways.

Thus there are

$$\sum_{m=1}^{n}\sum_{j=1}^{m}\binom{n}{m}\binom{m}{j}B(n-m)(k-1)^{n-j}$$

such partitions avoiding 112.

# The Final Case

At least two blocks contain elements colored blue and there is no copy of 112.

SEE BOARD!

Jump to theorem

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- Set Partition Statistics

# Thank You

THANK YOU

- A. Björner, M. L. Wachs, Geometrically constructed bases for homology of partition lattices of type $A$, $B$ and $D$, Electron. J. Combin. 11 (2) (2004/06) Research Paper 3, 26.
  URL `http://www.combinatorics.org/Volume_11/Abstracts/v11i2r3.html`
- E. S. Egge, Restricted colored permutations and Chebyshev polynomials, Discrete Math. 307 (14) (2007) 1792–1800.
  URL `http://dx.doi.org/10.1016/j.disc.2006.09.027`
- M. Klazar, Counting pattern-free set partitions. I. A generalization of Stirling numbers of the second kind, European J. Combin. 21 (2000) 367–378.
- D. E. Knuth, The art of computer programming. Volume 3. Sorting and Searching, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1973.

- T. Mansour, Pattern avoidance in coloured permutations, Sém. Lothar. Combin. 46 (2001/02) Art. B46g, 12 pp. (electronic).
- B. E. Sagan, Pattern avoidance in set partitions, Ars Combin. 94 (2010) 79–96.
- R. Simion, F. W. Schmidt, Restricted permutations, European J. Combin. 6 (1985) 383–406.