Pattern Avoiding Colored Partitions

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1. History and Definitions

2. Colored Partitions and Avoidance

3. A Flavor of the Proofs

4. Summary and Future Ideas
Outline

1. History and Definitions
2. Colored Partitions and Avoidance
3. A Flavor of the Proofs
4. Summary and Future Ideas
Who and What

- Pattern Avoidance in Permutations. (Knuth [4], Simion and Schmidt [7], and Boom!)

- Pattern Avoidance in Colored Permutations. Done by considering the set $S_n \wr C_k$. (Mansour [5], Egge [2], and Sizzle!)

- Pattern Avoidance in Set Partitions. (Klazar [3], Sagan [6], Snap, Crackle)

The notion of signed set partitions was considered by Anders Björner and Michelle Wachs [1] from a poset and homological perspective.
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- The notion of signed set partitions was considered by Anders Björner and Michelle Wachs [1] from a poset and homological perspective.
- Now, we consider Pattern Avoidance in Colored Set Partitions. (Boom?)
Set Partition Definition

Definition

A partition $\pi$ of a set $S$, written $\pi \vdash S$, is a family of disjoint nonempty subsets $B_i \subseteq S$, called blocks, such that $\biguplus B_i = S$. 

Example
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We write

$$\pi = B_1/B_2/\ldots/B_k,$$

where

$$\min B_1 < \min B_2 < \cdots < \min B_k,$$
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Let

$$\Pi_n = \{\pi : \pi \vdash [n] = \{1, 2, \ldots, n\}\}, \text{ and } \Pi = \bigcup_n \Pi_n.$$
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Definition

*Given any word* $w \in [k]^n$ *we may canonize* $w$ *by replacing all occurrences of the first letter of* $w$ *by 1, all occurrences of the next occurring letter by 2, etc.*
Canonical Words

Definition

*Given any word* $w \in [k]^n$ *we may canonize* $w$ *by replacing all occurrences of the first letter of* $w$ *by 1, all occurrences of the next occurring letter by 2, etc.*

Example

*The canonized form of* $47411477$ *is* $12133122$. 
Canonical Words

Definition

Given any word $w \in [k]^n$ we may canonize $w$ by replacing all occurrences of the first letter of $w$ by 1, all occurrences of the next occurring letter by 2, etc.

Example

The canonized form of 47411477 is 12133122.

There is a bijection between all canonized words of length $n$ and partitions of $[n]$. 
Canonical Words

To each set partition is associated a canonical word $a_1a_2\ldots a_n$ where $a_i = j$ if $i \in B_j$. 
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Example

137/25/46 corresponds to 1213231.
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Example

$137/25/46$ corresponds to $1213231$.

We will say that a partition, $\pi$ is of length $n$, $\ell(\pi) = n$, if its associated canonical word has $n$ letters.

From now on we will refer to these canonical words as partitions.
Pattern Avoidance

Definition

Let $\sigma$ be a partition with $\ell(\sigma) = n$ and $\pi$ be a partition with $\ell(\pi) = k$. We say that $\sigma$ contains $\pi$ if there is a subsequence of $\sigma$ of length $k$ whose canonization is $\pi$. Otherwise we say that $\sigma$ avoids $\pi$.

Example

Let $\sigma = 1213431$. 
Definition

Let $\sigma$ be a partition with $\ell(\sigma) = n$ and $\pi$ be a partition with $\ell(\pi) = k$. We say that $\sigma$ contains $\pi$ if there is a subsequence of $\sigma$ of length $k$ whose canonization is $\pi$. Otherwise we say that $\sigma$ avoids $\pi$.

Example

Let $\sigma = 1213431$.

$\sigma$ contains a copy of 112 namely $\color{red}12\color{black}1343\color{red}1$ or $\color{red}1\\color{black}21343\color{red}1$. However, $\sigma$ avoids 1112.
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Colored Partitions

Definition

A \textit{colored partition} is a set partition where each element is given one of \( k \) colors.
Colored Partitions

**Definition**
A colored partition is a set partition where each element is given one of $k$ colors.

**Definition**
Denote the set of all $k$-colored set partitions of $[n]$ by $\Pi_n \wr C_k$.

**Example**
Consider $\sigma = 1213431 \in \Pi_7$ from the previous slide. We can make $\sigma$ an element of $\Pi_7 \wr C_3$ simply by choosing one of three colors for each of the elements. So $1211341 \in \Pi_7 \wr C_3$. 
Avoiding Colored Partitions

Definition

We say that $\sigma \in \Pi_n \wr C_k$ contains a copy of $\pi \in \Pi_m \wr C_j$ if

1. the uncolored version of $\sigma$ contains a copy of the uncolored version of $\pi$ and
Avoiding Colored Partitions

Definition

We say that $\sigma \in \Pi_n \wr C_k$ contains a copy of $\pi \in \Pi_m \wr C_j$ if

1. the uncolored version of $\sigma$ contains a copy of the uncolored version of $\pi$ and
2. the colors of this copy of $\pi$ equal the colors of $\pi$.

Otherwise we say that $\sigma$ avoids $\pi$. 
Avoiding Colored Partitions

Definition

We say that $\sigma \in \Pi_n \wr C_k$ contains a copy of $\pi \in \Pi_m \wr C_j$ if

1. the uncolored version of $\sigma$ contains a copy of the uncolored version of $\pi$ and
2. the colors of this copy of $\pi$ equal the colors of $\pi$.

Otherwise we say that $\sigma$ avoids $\pi$.

Example

Consider $\sigma = 1211341$. Then $\sigma$ contains a copy of $122$, but $\sigma$ avoids $122$. 
Avoiding Colored Partitions

Definition

We say that $\sigma \in \Pi_n \wr C_k$ contains a copy of $\pi \in \Pi_m \wr C_j$ if

1. the uncolored version of $\sigma$ contains a copy of the uncolored version of $\pi$ and
2. the colors of this copy of $\pi$ equal the colors of $\pi$.

Otherwise we say that $\sigma$ avoids $\pi$.

Example

Consider $\sigma = 1211341$. Then $\sigma$ contains a copy of 122, but $\sigma$ avoids 122.

Definition

For a set of colored set partitions $S$ let $\Pi_n \wr C_k(S)$ be the set of partitions in $\Pi_n \wr C_k$ that avoid every pattern in $S$. 
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Friendly Results

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \land C_2(11, 11)| = \sum_{i=1}^{n} 2^i S(n, i)$

(OEIS A001861)

Theorem

For $n \geq 1$ and $c \geq 2$, $|\Pi_n \land C_2(12)| = (B_{n+2} - B_{n+1}) - (B_{n+1} - B_n)$

(OEIS A011965)
Theorem

For \( n \geq 1 \) and \( c \geq 2 \), \(|\Pi_n \wr C_k(112)| =

\[B(n)(k - 1)^n + \sum_{m=1}^{n} \sum_{j=1}^{m} \binom{n}{m} \binom{m}{j} B(n - j)(k - 1)^{n-j} + \]
\[
\sum_{1 \leq i < j \leq n} \sum_{a,b} \sum_{d,e} \sum_{f,g} \sum_{m} \sum_{p,q} \sum_{\ell} \binom{i-1}{a,b} \binom{j-i-1}{d,e} \binom{n-j}{f,g} \cdot
\]
\[
\binom{i-a-b-1}{m} \binom{j-i-d-e-1}{p} \binom{n-a-b-f-g-j+i-m-1}{q} \cdot
\]
\[
S(p+q, \ell) m^\ell B(n - a - b - d - e - f - g - m - p - q - 2) \cdot
\]
\[
((k - 1)^b + b(k - 1)^{b-1})((k - 1)^a + a(k - 1)^{a-1})(k - 2)^{d+p} k^e \cdot
\]
\[
(k - 1)^{n-a-b-d-e-m-p-2}.
\]
The Proof of this Theorem can be broken into 3 cases.

Case 1: No elements are colored blue.

Case 2: Exactly one block contains elements colored blue.

Case 3: At least two blocks contain elements colored blue.
Avoiding 112 – Sketch of Proof

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The Proof of this Theorem can be broken into 3 cases.

Case 1: No elements are colored blue.

Case 2: Exactly one block contains elements colored blue.

Case 3: At least two blocks contain elements colored blue.
No elements are colored blue.

In this case there can’t possibly be a copy of 112.

- $B(n)$ ($n^{th}$ Bell number) ways to partition the elements in $[n]$
- $(k - 1)^n$ ways to color each element any color but blue.

Thus there are $B(n)(k - 1)^n$ such 112 avoiding partitions in $\Pi_n \wr C_k$. 
Exactly one block contains elements colored blue.

In this case there can’t possibly be a copy of 112.

- Select \( m \) elements to be in the block with the elements that are colored blue.
- Select \( j \) of these elements to be colored blue.
- Partition the remaining \( n - m \) elements in \( B(n - m) \) ways.
- Color the non-blue elements in \( (k - 1)^{n-j} \) ways.

Thus there are

\[
\sum_{m=1}^{n} \sum_{j=1}^{m} \binom{n}{m} \binom{m}{j} B(n - m)(k - 1)^{n-j}
\]

such partitions avoiding 112.
The Final Case

At least two blocks contain elements colored blue and there is no copy of 112.

SEE BOARD!
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Where Do We Go from Here?

- Generating Functions?
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- connections via OEIS
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- Wilf Classes
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- Set Partition Statistics
Thank You

THANK YOU


