# Patterns in Parking Functions

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## Definition

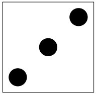
A permutation is a list where order matters.
$\mathcal{S}_n$ is the set of all permutations of $\{1, 2, \ldots, n\}$.

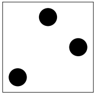
Examples:

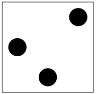
- $\mathcal{S}_1 = \{1\}$
- $\mathcal{S}_2 = \{12, 21\}$
- $\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$

$$|\mathcal{S}_n| = n!$$

Visualize $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ by plotting the points $(i, \pi_i)$ in the $xy$-plane.


123


132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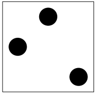
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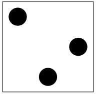
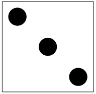
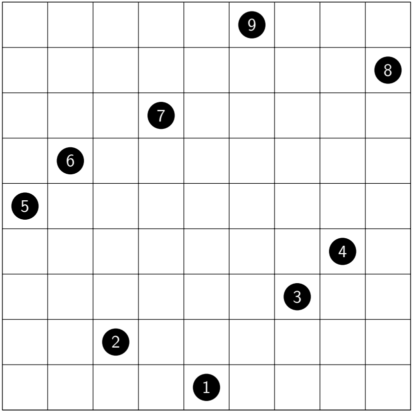
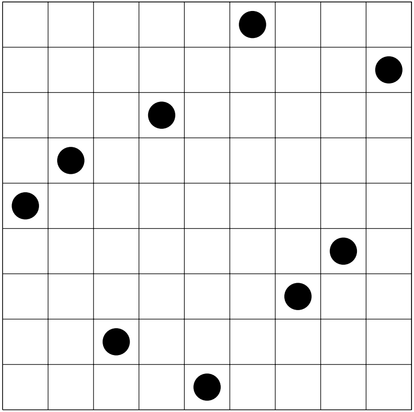
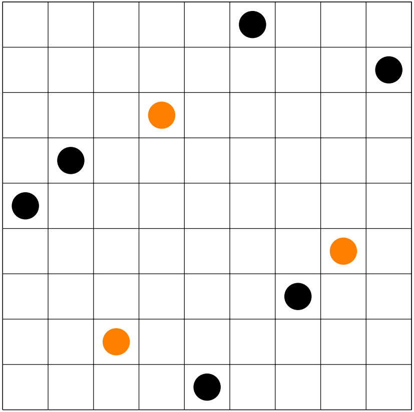

312

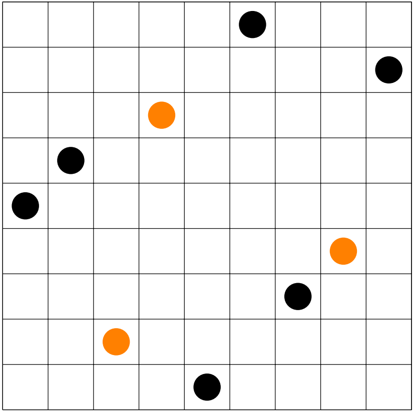

321

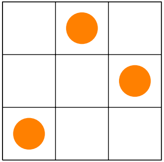



$$\pi = 562719348$$

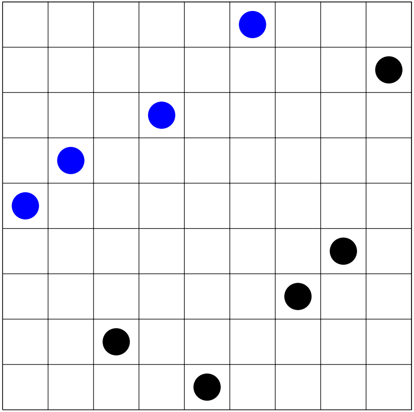


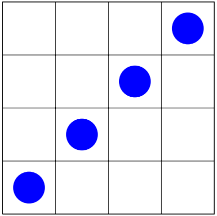




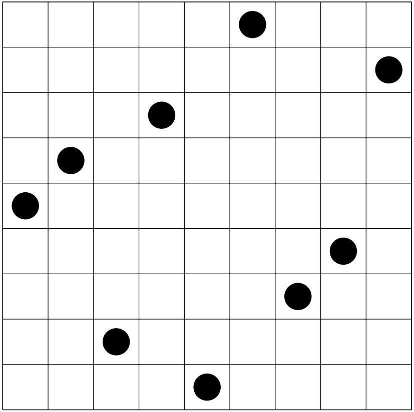


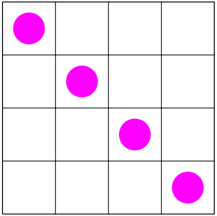
562719348 contains the pattern 132





562719348 contains the pattern 1234





562719348 avoids the pattern 4321

### Big question

How many permutations of length $n$ contain the pattern $\rho$?

Or, alternatively...

### Big question

How many permutations of length $n$ avoid the pattern $\rho$?

(depends on what $\rho$ is!)

### Notation

$\mathcal{S}_n(\rho)$ is the set of permutations of length $n$ *avoiding* $\rho$.

### Definition

A parking function is a sequence $a_1 \cdots a_n \in [n]^n$ such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1 \cdots a_n$ then $b_i \leq i$ for all $1 \leq i \leq n$.

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Examples: 11111, 32123, 45312
11111, 12233, 12345

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22222, 12445, 11155

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11111, 12233, 12345

Nonexamples: 22222, 51244, 15151
22222, 12445, 11155

## Observations

- There are $(n+1)^{n-1}$ parking functions of size $n$.
- Every permutation of size $n$ is a parking function of size $n$.

# History

Jelínek and Mansour (2009)

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- Determined all equivalence classes of patterns of length at most 5

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- Each Dyck path is associated with a permutation (many-to-one correspondence)
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Current project:

- Extend Remmel and Qiu's work
- Count parking functions avoiding a subset of $\mathcal{S}_3$.

## Parking functions of size 2

| | | | |
|---|---|---|---|
| Sequences: | 11 | 12 | 21 |

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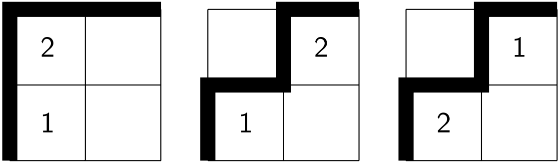
Dyck paths:



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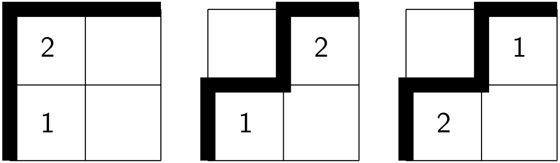
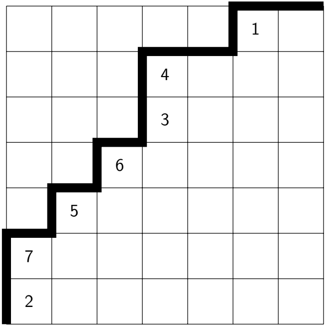



| | | | |
|---|---|---|---|
| Associated permutations: | 12 | 12 | 21 |

Parking function: 6144231

Blocks: $\{2,7\},\{5\},\{6\},\{3,4\},\emptyset,\{1\},\emptyset$

Dyck path:




Associated permutation: 2756341

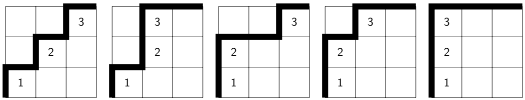
# Warmup

## Notation

Let $\mathrm{pf}_n(\rho)$ be the number of parking functions of size $n$ whose associated permutations avoid $\rho$.

## Proposition

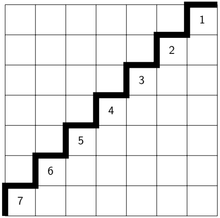
$\mathrm{pf}_n(21) = C_n$ ($n$th Catalan number)



# Warmup

## Proposition

$\mathrm{pf}_n(12) = 1$



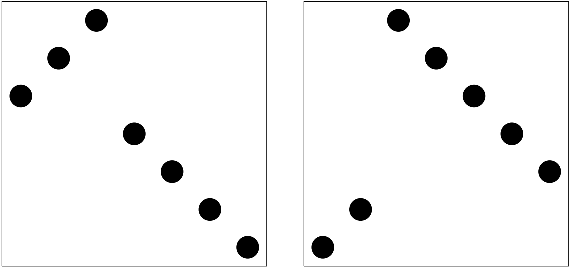
| Patterns $P$ | $\mathrm{pf}_n(P)$, $1 \leq n \leq 6$ | OEIS |
|---|---|---|
| 123, 132, 231 | 1, 3, 5, 7, 9, 11 | A005408 |
| 123, 132, 312<br>123, 213, 231<br>123, 231, 312 | 1, 3, 6, 10, 15, 21 | A000217 |
| 123, 213, 312 | 1, 3, 7, 13, 21, 31 | A002061 |
| 123, 132, 213 | 1, 3, 6, 17, 43, 123 | A143363 |
| 132, 213, 231<br>132, 231, 312 | 1, 3, 8, 22, 64, 196 | A014138 |
| 132, 213, 312<br>213, 231, 312 | 1, 3, 9, 28, 90, 297 | A000245 |
| 132, 231, 321 | 1, 3, 9, 29, 98, 342 | A077587 |
| 132, 213, 321<br>132, 312, 321<br>213, 231, 321 | 1, 3, 10, 35, 126, 462 | A001700 |
| 213, 312, 321 | 1, 3, 11, 41, 154, 582 | A076540 |
| 231, 312, 321 | 1, 3, 10, 38, 154, 654 | A001002 |

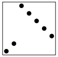
| Patterns $P$ | $\mathrm{pf}_n(P)$, $1 \leq n \leq 6$ | OEIS |
|---|---|---|
| 123, 231 | 1, 3, 8, 17, 31, 51 | A105163 |
| 123, 312 | 1, 3, 9, 21, 41, 71 | A064999 |
| 123, 132 | 1, 3, 8, 24, 75, 243 | A000958 |
| 123, 213 | 1, 3, 9, 28, 90, 297 | A000245 |
| 132, 231 | 1, 3, 10, 36, 137, 543 | A002212 |
| 132, 213<br>132, 312<br>213, 231<br>231, 312 | 1, 3, 11, 45, 197, 903 | A001003 |
| 132, 321 | 1, 3, 12, 52, 229, 1006 | new |
| 213, 321 | 1, 3, 13, 60, 275, 1238 | new |
| 213, 312 | 1, 3, 12, 54, 259, 1293 | new |
| 231, 321 | 1, 3, 12, 55, 273, 1428 | A001764 |
| 312, 321 | 1, 3, 13, 63, 324, 1736 | new |

### Theorem

$$\mathrm{pf}_n(132, 213, 312) = \mathrm{pf}_n(213, 231, 312) = \frac{3(2n)!}{(n+2)!(n-1)!} = C_{n+1} - C_n$$

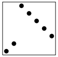
$\mathcal{S}_n(132, 213, 312)$ $\mathcal{S}_n(213, 231, 312)$



$$\mathcal{S}_n(213, 231, 312)$$ 

Let $a(n, k)$ be the number of size $n$ parking functions whose associated permutation begins with $k - 1$ ascents.

- $a(n, 1) = 1$
- $a(n, n) = C_n$

$\mathcal{S}_n(213, 231, 312)$ 

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1. Last block has one element
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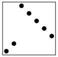
$\mathcal{S}_n(213, 231, 312)$ 

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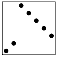
1. Last block has one element ($a(n-1,k)$)
2. Last block is empty

Case 1? Deleting/reinserting last block (and standardizing) is bijection

$$\{1,2\}, \emptyset, \{7\}, \{6\}, \{5\}, \{4\}, \cancel{\{3\}} \leftrightarrow \{1,2\}, \emptyset, \{6\}, \{5\}, \{4\}, \{3\}$$

$\mathcal{S}_n(213, 231, 312)$ 

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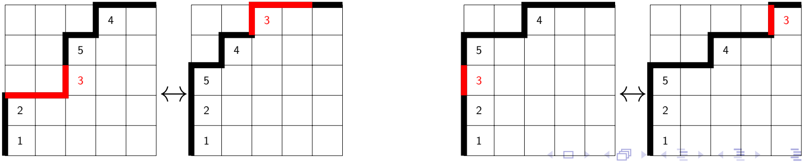
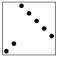
Case 2? Bijection via moving last element before decreasing run.



$\mathcal{S}_n(213, 231, 312)$ 

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- $a(n,1) = 1$
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Two cases:

1. Last block has one element $(a(n-1,k))$
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In general:

$$a(n,k) = a(n-1,k) + a(n,k-1).$$

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$$\mathrm{pf}_n(132, 213, 312) = \mathrm{pf}_n(213, 231, 312) = \sum_{k=1}^{n} a(n,k) = C_{n+1} - C_n$$

## Theorem

$$\mathrm{pf}_n(231, 312, 321) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n+1} \binom{2n-k}{n+k} \binom{n+k}{k} \quad \text{(OEIS A001002)}$$

(number of dissections of a convex $(n+2)$-gon into triangles and quadrilaterals)

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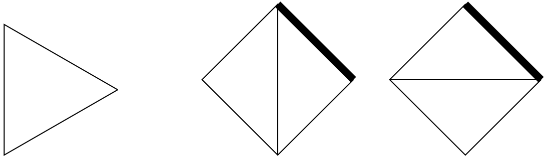
(number of dissections of a convex $(n+2)$-gon into triangles and quadrilaterals)

Catalan object 1: dissections of a convex $(n+2)$-gon into triangles
Catalan object 2: 21-avoiding parking functions

Small cases:



$\{1\}$ $\{1\},\{2\}$ $\{1,2\},\emptyset$

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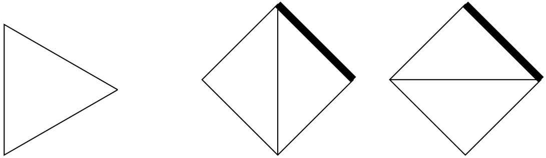

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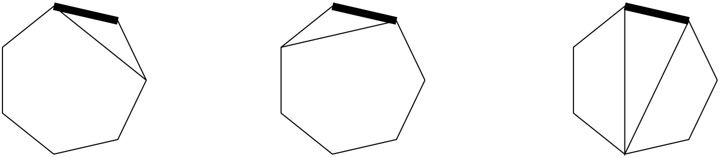

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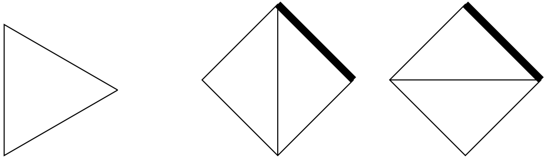

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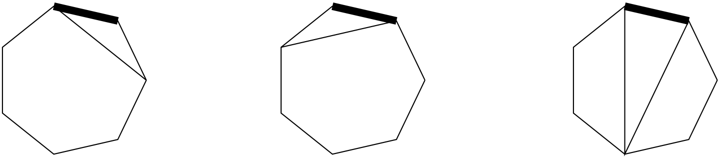
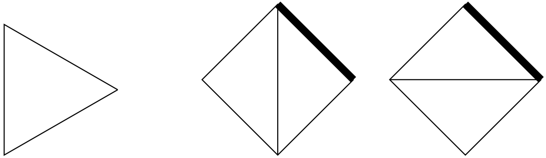


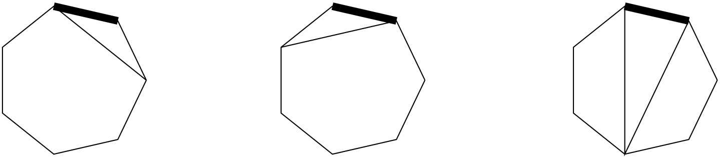
$\{n-1\},\{n\}$ $\{n-1,n\},\emptyset$

Small cases:



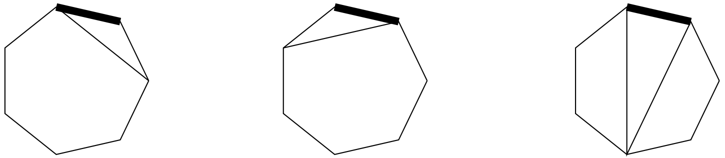
$\{1\}$ $\quad$ $\{1\},\{2\}$ $\quad$ $\{1,2\},\emptyset$

General cases:



$\{n-1\},\{n\}$ $\quad$ $\{n-1,n\},\emptyset$ $\quad$ $\{n-1\},\emptyset,\{n\}$

General case:



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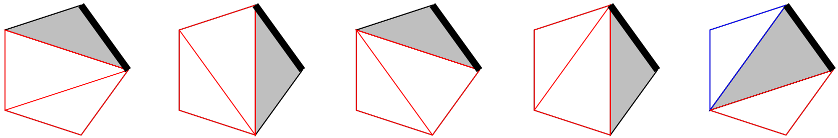
$n=3$:



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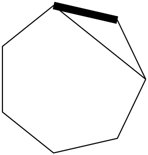
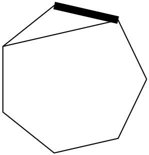
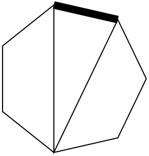


$\{n-1\}, \{n\}$

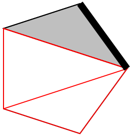


$\{n-1, n\}, \emptyset$

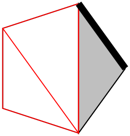


$\{n-1\}, \emptyset, \{n\}$

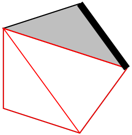
$n = 3$:

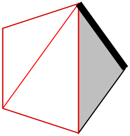


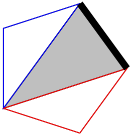
$\{1, 2, 3\}, \emptyset, \emptyset$



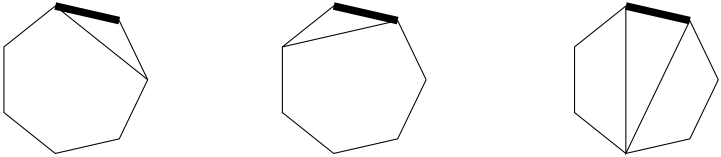
$\{1, 2\}, \{3\}, \emptyset$





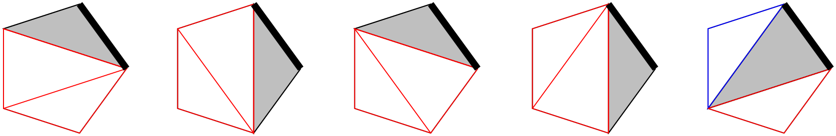


General case:



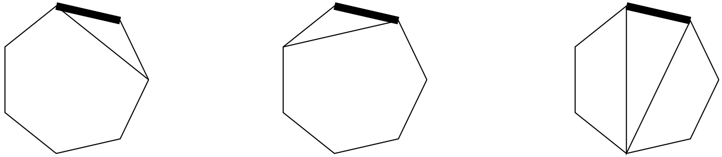
$\{n-1\}, \{n\}$ $\qquad$ $\{n-1, n\}, \emptyset$ $\qquad$ $\{n-1\}, \emptyset, \{n\}$

$n = 3$:



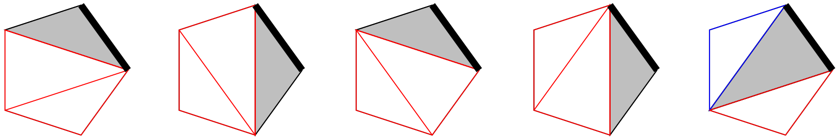
$\{1, 2, 3\}, \emptyset, \emptyset$ $\qquad$ $\{1, 2\}, \{3\}, \emptyset$ $\qquad$ $\{1\}, \{2, 3\}, \emptyset$ $\qquad$ $\{1\}, \{2\}, \{3\}$

General case:



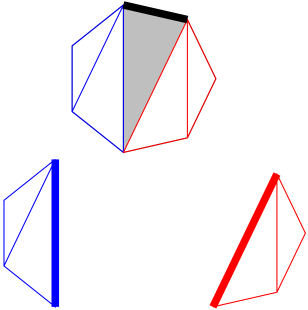
$\{n-1\},\{n\}$ $\{n-1,n\},\emptyset$ $\{n-1\},\emptyset,\{n\}$

$n=3$:



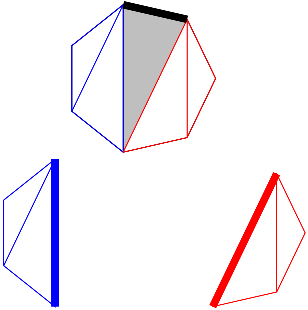
$\{1,2,3\},\emptyset,\emptyset$ $\{1,2\},\{3\},\emptyset$ $\{1\},\{2,3\},\emptyset$ $\{1\},\{2\},\{3\}$ $\{1,2\},\emptyset,\{3\}$

# Larger example



$\{3\},\{4\}$ $\{1,2\},\emptyset$

# Larger example



$\{3\},\{4\}$ $\{1,2\},\emptyset$

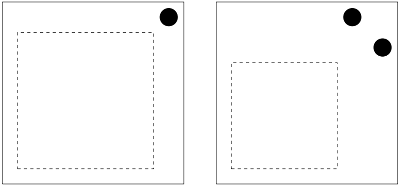
$\{1,2,3\},\{4\},\emptyset,\{5\},\emptyset$

## Theorem

$$\mathrm{pf}_n(231, 312, 321) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n+1} \binom{2n-k}{n+k} \binom{n+k}{k} \quad \text{(OEIS A001002)}$$

(number of dissections of a convex $(n+2)$-gon into triangles and quadrilaterals)

Avoiding $\{231, 312, 321\}$



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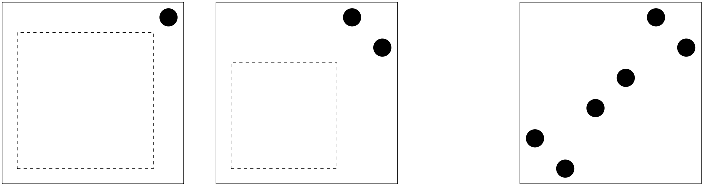

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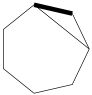
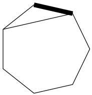
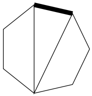


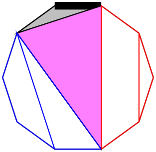
$\{n-1\},\{n\}$



$\{n-1,n\},\emptyset$



$\{n-1\},\emptyset,\{n\}$



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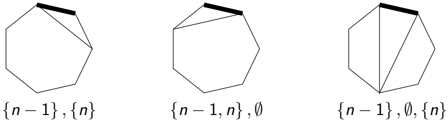


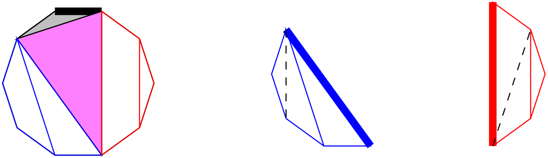

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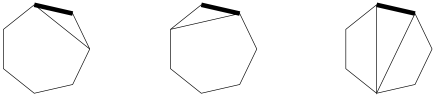
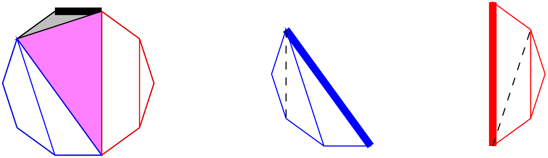
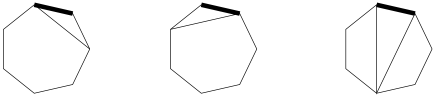


$\{n-1\},\{n\}$ $\{n-1,n\},\emptyset$ $\{n-1\},\emptyset,\{n\}$

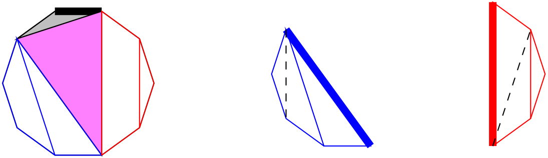


$\{\underline{5}\},\{\underline{4}\},\{6\}$ $\{1,\underline{3}\},\{\underline{2}\},\emptyset$

# Larger example



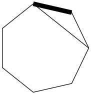
$\{n-1\},\{n\}$ $\{n-1,n\},\emptyset$ $\{n-1\},\emptyset,\{n\}$



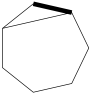
$\{\underline{5}\},\{\underline{4}\},\{6\}$ $\{1,\underline{3}\},\{\underline{2}\},\emptyset$

$\{1,3\},\{2,5\},\{4\},\{6\},\emptyset,\{7\},\emptyset$

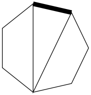
# Larger example



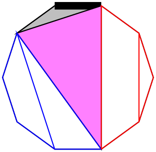
$\{n-1\},\{n\}$

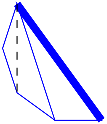


$\{n-1,n\},\emptyset$

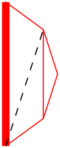


$\{n-1\},\emptyset,\{n\}$





$\{\underline{5}\},\{\underline{4}\},\{6\}$



$\{1,\underline{3}\},\{\underline{2}\},\emptyset$

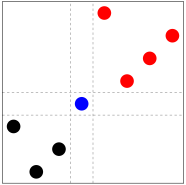
$$\{1,3\},\{2,5\},\{4\},\{6\},\emptyset,\{7\},\emptyset$$

$$\{1,3\},\{2,5\},\{4\},\{6\},\emptyset,\{7,8\},\emptyset,\emptyset$$

## Theorem

$$\mathrm{pf}_n(231, 321) = \frac{\binom{3n}{n}}{2n+1} \quad \text{(OEIS A001764)}$$

$\mathcal{S}_n(231, 321)$



$\frac{\binom{3n}{n}}{2n+1}$ counts

- ternary trees
- non-crossing trees

Strategy for $\mathrm{pf}_n(231, 321)$

1. bijection between Dyck paths and rooted ordered trees
2. bijection between parking functions and non-crossing trees via...
   - labeling Dyck paths
   - arranging tree vertices on circle

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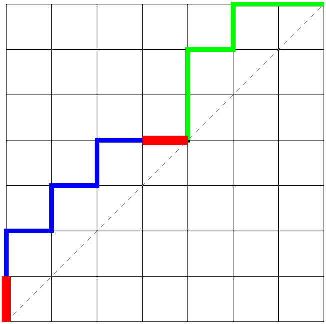
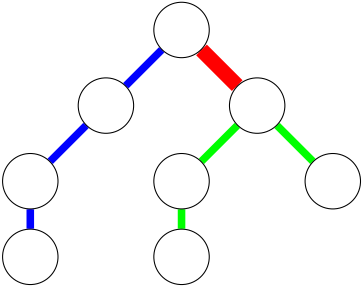
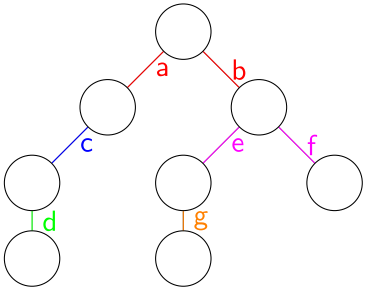
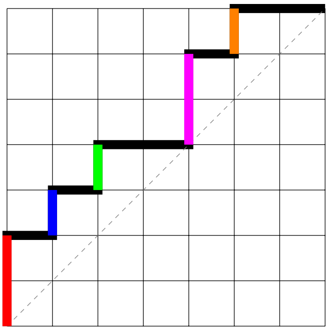




# Reframing the Dyck path/tree bijection

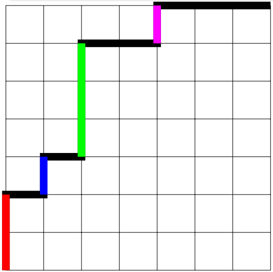


# Labelling the Dyck path to avoid $\{231, 321\}$

## Characterization of $\{231, 321\}$-avoiding permutations

The digit $d$ must be either first or second among the digits $\{d, d+1, \ldots, n\}$.



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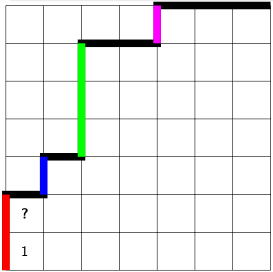

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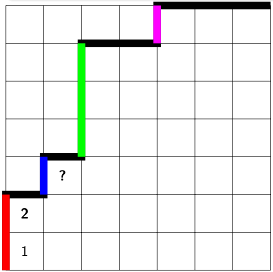


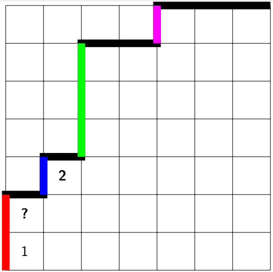

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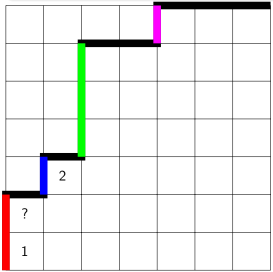

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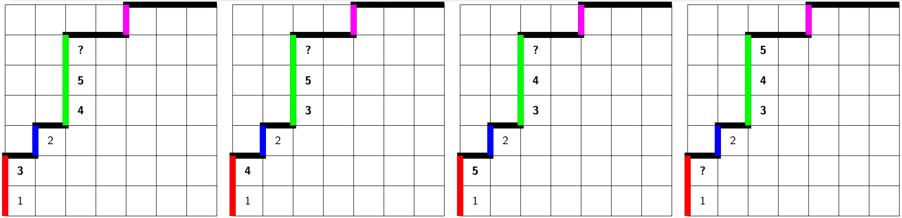

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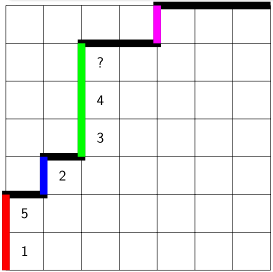

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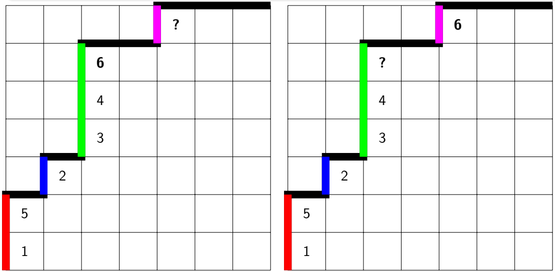

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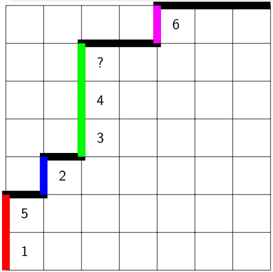

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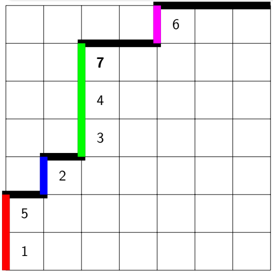

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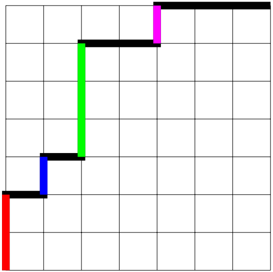
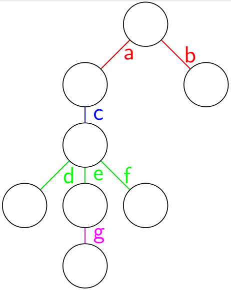
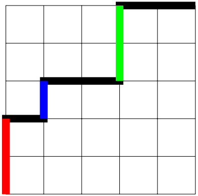
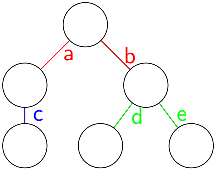


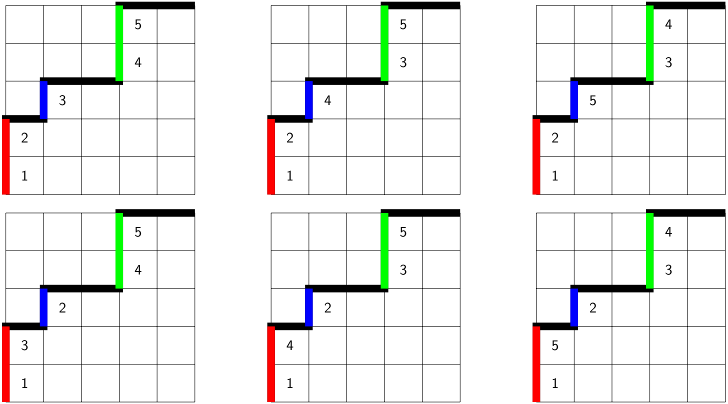
corresponds to
$2 \cdot 4 \cdot 2 = 16$ different
$\{231, 321\}$-avoiding
parking functions.





corresponds to
$2 \cdot 3 = 6$ different
$\{231, 321\}$-avoiding
parking functions.

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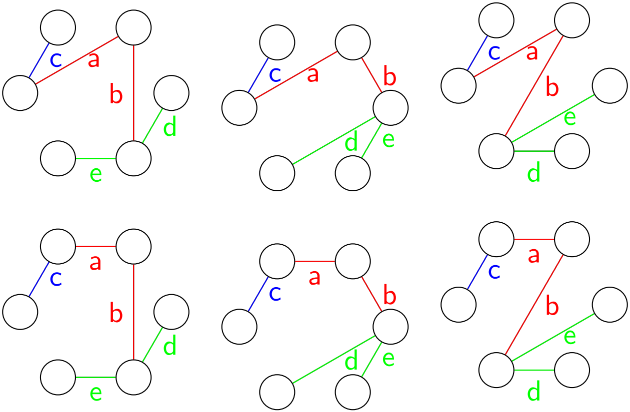
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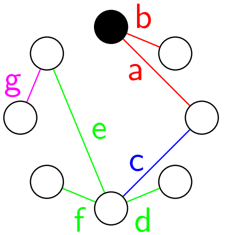


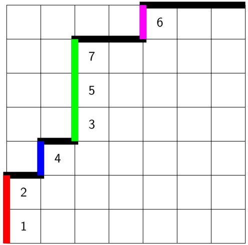


$\{1, ?\}$

*a* is left of 1 subtree, so ? is replaced with smallest remaining number.

$\{1, 2\}, \{?\}$

*c* is left of 2 subtrees, so ? is replaced with 2nd smallest remaining number.

$\{1, 2\}, \{4\}, \{3, 5, ?\}$

*e* is left of 0 subtrees, so ? remains.

$\{1, 2\}, \{4\}, \{3, 5, ?\}, \{6\}$

## Recap

- $\mathrm{pf}_n(132, 213, 312) = \mathrm{pf}_n(213, 231, 312) = C_{n+1} - C_n$
- $\mathrm{pf}_n(231, 312, 321) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{n+1} \binom{2n-k}{n+k} \binom{n+k}{k}$
- $\mathrm{pf}_n(231, 321) = \dfrac{\binom{3n}{n}}{2n+1}$

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- $\mathrm{pf}_n(231, 321) = \dfrac{\binom{3n}{n}}{2n+1}$

Forthcoming:
results for avoiding any set of 2 or more patterns in $\mathcal{S}_3$

| Pattern $P$ | $\mathrm{pf}_n(P)$, $1 \leq n \leq 6$ | OEIS |
|---|---|---|
| 123 | 1, 3, 11, 48, 232, 1207 | new (Remmel & Qiu) |
| 132<br>231 | 1, 3, 13, 69, 417, 2759 | A243688* |
| 213<br>312 | 1, 3, 14, 81, 533, 3822 | new |
| 321 | 1, 3, 15, 97, 728, 6024 | new |

* "Number of Sylvester classes of 1-multiparking functions of length $n$."

# For further reading...


- A. Adeniran and L. Pudwell, Pattern Avoidance in Parking Functions, `arXiv:2209.04068`
- A.M. Garsia and M. Haiman, A Remarkable $q,t$-Catalan Sequence and $q$-Lagrange Inversion, *J. Algebraic Combin.* **5** (1996), 191–244.
- V. Jelínek and T. Mansour, Wilf-equivalence on $k$-ary words, compositions, and parking functions, *Electron. J. Combin.* **16** (2009), #R58, 9pp.
- J. Remmel and D. Qiu, Patterns in ordered set partitions and parking functions, Permutation Patterns 2016 (slides), available electronically at `https://www.math.ucsd.edu/~duqiu/files/PP16.pdf`.
- Richard Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 2001.
- The On-Line Encyclopedia of Integer Sequences at `oeis.org`.


# Thanks for listening!

slides at `faculty.valpo.edu/lpudwell`

email: `Lara.Pudwell@valpo.edu`