Enumeration Schemes for Permutations Avoiding Barred Patterns

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Bar Notation

- Consider *pairs* of permutations q_*, q^* such that q_* is contained in q^*.
- Choose one instance of q_* in q^*.
- Write q by taking the letters of q^* and putting a bar over letters *not* in q_*.

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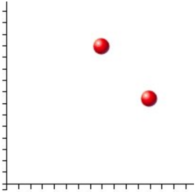

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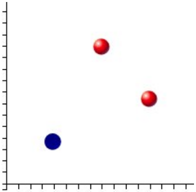

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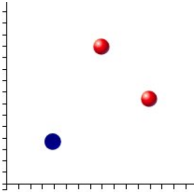

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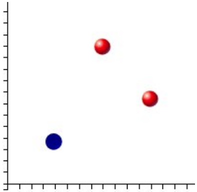

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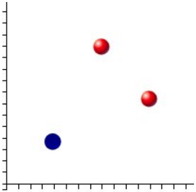

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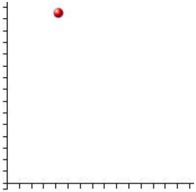

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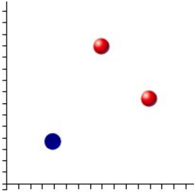

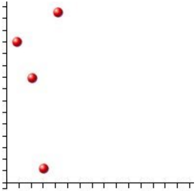

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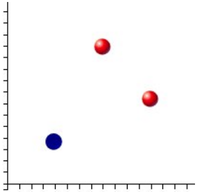

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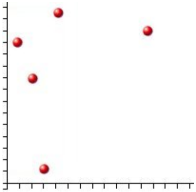

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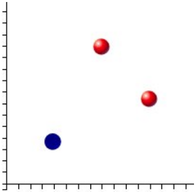

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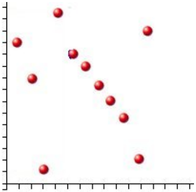

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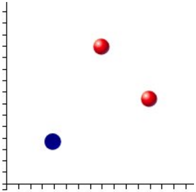

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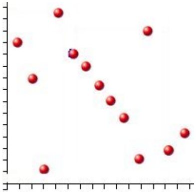

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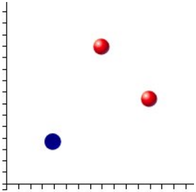

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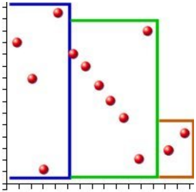

Results with Particular Barred Patterns

- (West, 1990) A permutation is 2-stack sortable if and only if it avoids 2341 and $3\bar{5}241$.
- (Bousquet-Melou and Butler, 2006) A permutation is forest-like if and only if it avoids 1324 and $21\bar{3}54$.
- (Claesson, Dukes, and Kitaev, 2008) $(2+2)$-free posets are in bijection with permutations which avoid $3\bar{1}52\bar{4}$.

Enumeration Results

Enumeration results:

- Permutations avoiding the sets of patterns involved in applications to stack sorting, forest-like permutations, and posets have been enumerated.
- (Callan, 2005): Permutations which avoid $3\bar{5}241$ are counted by OEIS Sequence A110447.
- (Callan, 2006): Permutations avoiding a pattern of length 4 with one bar give Catalan numbers, Bell numbers , OEIS Sequence A051295, or OEIS Sequence A137533.

Some Useful Observations

- You can have too many bars.

Lemma

If q is a pattern of length k with $k-1$ bars, then $S_n(q)=0$, $n\geq 1$.

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Lemma

If q is a pattern of length k with $k-1$ bars, then $S_n(q) = 0$, $n \geq 1$.

- You can place bars in such a way to make $S_n(q)$ degenerate to regular pattern avoidance.

Lemma

If q contains a symmetry of $\bar{i}(i+1)$, then $S_n(q)$ is equivalent to counting permutations avoiding some non-barred pattern.

Patterns of length ≤ 5

Length	Bars	No. Sequences	Possible Sequences
2	0	1	1
3	1	2	1, (n-1)!
4	2	2	1, (n-2)!
5	3	2	1, (n-3)!
3	0	1	Catalan
4	1	4	Catalan, Bell, A051295, & 1 more
5	2	17	A110447, A117106, & 15 more
4	0	3	A005802, A061552, A022558
5	1	13	A006789, A047970, A098569, A122993, & 9 more

Observations

Based on computation:

- Conjecture: If q is a barred pattern of length k with $k-2$ bars then either $S_n(q)=1$ or $S_n(q)=(n-(k-2))!$.
- Conjecture: $S_n(\overline{3}\overline{1}542)$ gives the number of ordered factorizations over the Gaussian polynomials. (OEIS A047970)
- Conjecture: $S_n(\overline{1}43\overline{5}2)$ has generating function $\prod_{n\geq 0}\frac{1}{(1-\frac{x}{(1-x)^n})^{(1/2)^{n+1}}}$ (OEIS A122993).
- There are at least 24 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.

Goals

Main Goal

Automate a method to count $S_n(Q)$.

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Enumeration Schemes are one such method that have been applied to regular pattern-avoiding permutations (Vatter, Zeilberger), and to pattern-avoiding words (P.).

Divide

Notation

$$S_n\left(Q; p_1 \cdots p_l\right) := \left\{ \pi \in S_n \;\middle|\; \begin{array}{l} \pi \text{ avoids } Q \\ \pi \text{ has prefix } p_1 \cdots p_l \end{array} \right\}$$

$$S_n\left(Q; \begin{matrix} p_1 \cdots p_l \\ i_1 \cdots i_l \end{matrix}\right) := \left\{ \pi \in S_n \;\middle|\; \begin{array}{l} \pi \text{ avoids } Q \\ \pi \text{ has prefix } p_1 \cdots p_l \\ \pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n \end{array} \right\}$$

For example,

$$S_3(\{132\}\,; 12) = \{123, 231\}$$

$$S_3\left(\{123\}\,; \begin{matrix} 12 \\ 23 \end{matrix}\right) = \{231\}$$

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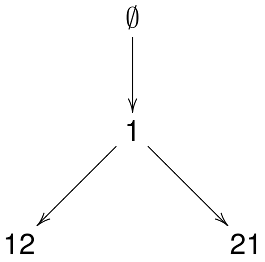
For *any* pattern set Q, we have

$$S_n(Q) = S_n(Q; 1)$$
$$= S_n(Q; 12) \cup S_n(Q; 21),$$

etc.



Conquer

Objectives

1. Given Q and p find r such that
$|S_n\,(Q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}\,(Q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$

$$
\begin{tikzcd}
 & \emptyset \arrow[d] & \\
 & 1 \arrow[dl] \arrow[dr] & \\
12 & & 21 \arrow[ul, dotted, bend right, "d_r"']
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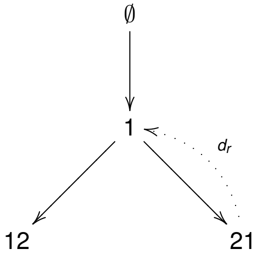
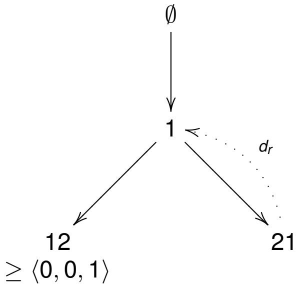
2. Given Q and p, find $i_1, \ldots, i_l$ such that

$$\left| S_n \left(Q; \begin{matrix} p_1 \cdots p_r \cdots p_l \\ i_1 \cdots i_r \cdots i_l \end{matrix} \right) \right| = 0$$





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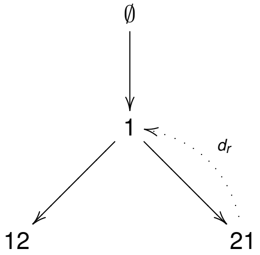
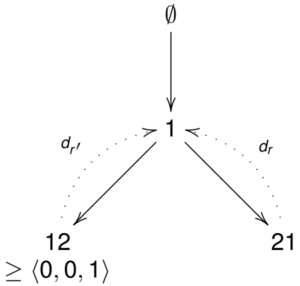




Reversibly Deletable

Objective

Given Q and p find r such that

$$|S_n(Q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(Q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$$

To find such a recurrence we must check that:

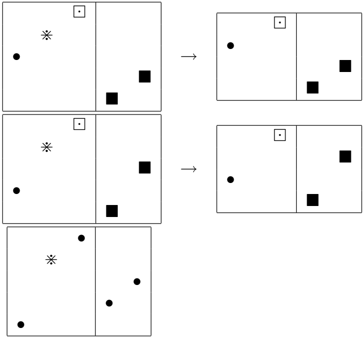
1. *inserting* p_r into a Q-avoiding permutation beginning with $p_1 \cdots p_{r-1} p_{r+1} \cdots p_l$ always produces a Q-avoiding permutation.
2. *deleting* p_r from a Q-avoiding permutation beginning with $p_1 \cdots p_l$ always produces a Q avoiding permutation.

Reversibly Deletable: Insertion

As with non-barred patterns, check that *every* possible instance of a forbidden pattern involving p_r implies the existence of a forbidden pattern without p_r.

Example:

$Q = \{\bar{1}423\}$, and $p = 123$. Check p_2.



Reversibly Deletable: Insertion

Non-Example:
$Q = \{134\overline{2}\}$, and $p = 21$. Check p_1.

p_1 can be involved in a 123 pattern in precisely one way (“2” $< a < b$).

Reversibly Deletable: Insertion

Non-Example:
$Q = \{134\bar{2}\}$, and $p = 21$. Check p_1.

p_1 can be involved in a 123 pattern in precisely one way (“2” $< a < b$).

But what about π beginning with $21abc$? (e.g. 31452)

Observation: Must look at scenarios with extra letters, depending on how many bars are in forbidden patterns.

Reversibly Deletable: Deletion

No longer non-trivial, as with unbarred patterns.

Check that *every* possible instance of a forbidden pattern without p_r implies the existence of a forbidden pattern with p_r.

Requires similar case analysis to checking for insertion.

(Partial) Algorithm

Given Q, a set of forbidden patterns, we can find an enumeration scheme E for $S_n(Q)$ in the following way.

1. Let $N = \{\emptyset\}$, and let $E = \{[\emptyset, \emptyset]\}$.
2. Let $N2 = \{\text{children of } n \in N\}$, $E2 = \{[n_i, R_i]\}$, where for $n_i \in N2$, R_i is the corresponding set of reversibly deletable elements.
3. If $R_i \neq \emptyset$ for all $n_i \in N2$, then return $E \cup E2$. Otherwise, let $E = E \cup E2$, $N = \{n_i \in N2 | R_i = \emptyset\}$, and return to step 2.

Spacing Vectors

Spacing Vectors

Given Q and p (of length l) let v be a vector in $\mathbb{N}^{l+1}$. Then, $S_n(Q; p; v)$ denotes the set of permutations of length n, avoiding Q, beginning with prefix p with exactly v_1 letters smaller than "1", v_j letters greater than "j-1" and smaller than "j", and exactly v_{l+1} letters greater than "l".

For example,

$$S_5(\{132\}; 12; \langle 2, 0, 1\rangle) =$$

$$\{34125, 34215, 34251, 34512, 34521\}$$

but

$$S_5(\{132\}; 12; \langle 0, 1, 0\rangle) = \{\}\,.$$

Gap Vectors

A spacing vector v is a *gap vector* for $[Q, p]$ if there are no permutations avoiding Q with prefix p and spacing vector $\geq v$ (componentwise).

To check if v is a gap vector for $[Q, p]$,

- Let S consist of v_1 fractional letters between 0 and 1, ..., v_{l+1} fractional letters between l and $l + 1$.
- Let S^* be the set of all $\|v\|!$ permutations of the elements of S.
- Consider all permutations formed by appending an element of S^* to p. If each of these permutations contains a forbidden pattern, then v is a gap vector.

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This is almost true...

Gap Vector Considerations

The standard algorithm for finding gap vectors fails when $q = q_1 \cdots q_i \overline{q_{i+1}} \cdots \overline{q_k}$.

Theorem

Let $q \in \overline{S}_m$ such that $q_* = q_1 \cdots q_{m-1}$. Then there are *no* gap vectors for $[\{q\}, p]$ for any prefix p.

(Partial) Algorithm

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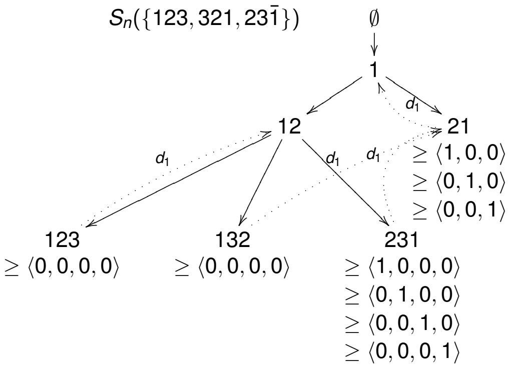
One more consideration



gives the sequence $1, 1, 2, 1, 0, \ldots$, but we expect $1, 1, 1, 1, 0, \ldots$. What goes wrong?

Stop Points

With barred patterns, there may be *no* permutations of length n that avoid Q and begin with p, but there may be such permutations of longer length.

Given Q and p, we say $s \geq |p|$ is a *stop point* for $[Q, p]$ if there are no permutations of length $\leq s$ that avoid Q and begin with prefix p.

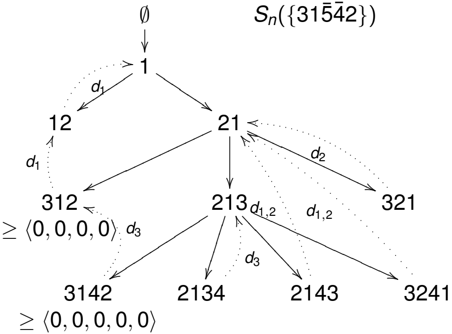
Observation: If p is an interior scheme prefix, the set of stop points for $[Q, p]$ is finite.

Algorithm

Given Q, a set of forbidden patterns, we can find an enumeration scheme E for $S_n(Q)$ in the following way.

1. Let $N = \{\emptyset\}$, and let $E = \{[\emptyset, \emptyset, \emptyset, \emptyset]\}$.
2. Let $N2 = \{\text{children of } n \in N\}$, $E2 = \{[n_i, G_i, R_i, S_i]\}$, where for $n_i \in N2$, G_i is the corresponding set of gap vectors, R_i is the corresponding set of reversibly deletable elements, and S_i is the corresponding set of stop points.
3. If $R_i \neq \emptyset$ for all $n_i \in N2$, then return $E \cup E2$. Otherwise, let $E = E \cup E2$, $N = \{n_i \in N2 | R_i = \emptyset\}$, and return to step 2.

Summary of Extra Considerations



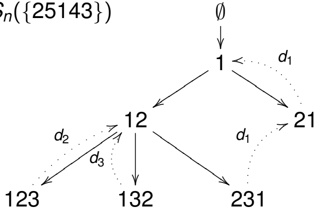
- More complicated to test for recurrences between subsets. (Deletion is no longer trivial, bars require more cases in analysis)
- May need to find recurrences that delete multiple letters at once.
- Gap vectors may be tricky to find depending on the structure of the forbidden patterns.
- More work to determine base cases of recurrence.

Success Rate

Pattern Lengths	Success Rate	Pattern Lengths	Success Rate
[2,1]	1/1 (100%)	[3,0],[3,0],[3,1]	43/45 (95.6%)
[2,1],[2,0]	2/2 (100%)	[3,0],[3,0],[3,2]	45/45 (100%)
[2,1],[2,1]	2/2 (100%)	[3,0],[3,1],[3,1]	135/138 (97.8%)
		[3,0],[3,1],[3,2]	280/280 (100%)
[3,1]	4/4 (100%)	[3,0],[3,2],[3,2]	138/138 (100%)
[3,2]	4/4 (100%)	[3,1],[3,1],[3,1]	115/118 (97.5%)
[3,0],[3,1]	18/20 (90%)	[3,1],[3,1],[3,2]	378/378 (100%)
[3,0],[3,2]	20/20 (100%)	[3,1],[3,2],[3,2]	378/378 (100%)
[3,1],[3,1]	27/28 (96.4%)	[3,2],[3,2],[3,2]	118/118 (100%)
[3,1],[3,2]	50/50 (100%)		
[3,2],[3,2]	28/28 (100%)	[4,1]	12/16 (75%)
		[4,2]	25/26 (96.2%)
[3,1],[4,0]	59/71 (83.1%)	[4,3]	16/16 (100%)
[3,1],[4,1]	229/240 (95.4%)		
[3,1],[4,2]	355/364 (97.5%)	[5,1]	15/89 (16.9%)
[3,0],[4,1]	84/88 (95.5%)	[5,2]	(in progress)
[3,0],[4,2]	133/136 (97.8%)		
[4,0],[5,1]	(in progress)		

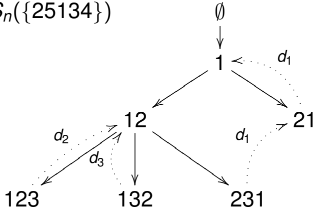
New Results: Length 5 with 1 Bar

$S_n(\{25\bar{1}43\})$



$\geq \langle 0, 1, 0, 0 \rangle$

$S_n(\{25\bar{1}34\})$

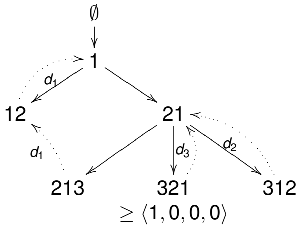


$\geq \langle 0, 0, 1, 0 \rangle$

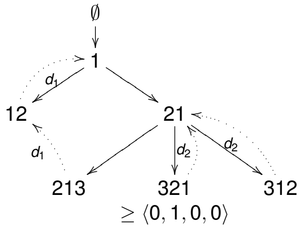
both give the sequence
1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810, 6325151, 49448313, 405298482, 3470885747, 30965656442
for $1 \leq n \leq 15$.

New Results: Length 5 with 1 Bar

$S_n(\{43\bar{5}21\})$



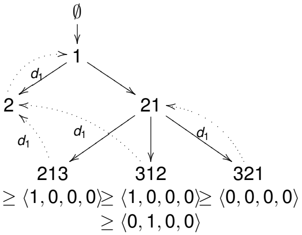
$S_n(\{43\bar{5}12\})$



both also give the sequence
1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810, 6325151, 49448313, 405298482, 3470885747, 30965656442
for $1 \leq n \leq 15$.

New Results: Length 5 with 2 Bars

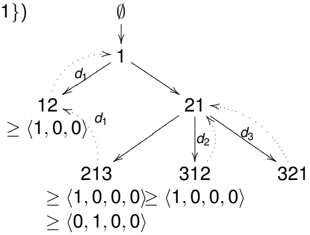
$S_n(\{5\bar{1}\bar{2}43\})$



gives the new sequence
1, 2, 5, 14, 43, 143, 511, 1950, 7903, 33848, 152529, 720466, 3555715, 18285538, 97752779

for $1 \leq n \leq 15$.

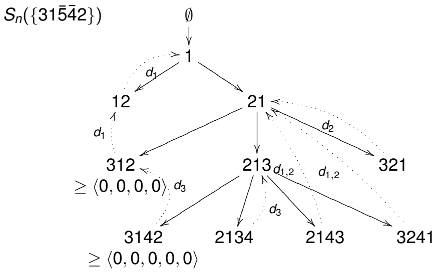
$S_n(\{\bar{5}\bar{4}231\})$



gives the new sequence
1, 2, 5, 14, 43, 146, 561, 2518, 13563, 88354, 686137, 6191526, 63330147, 720314930, 8985750097

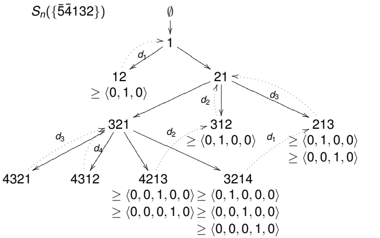
for $1 \leq n \leq 15$.

New Results: Length 5 with 2 Bars



gives the new sequence
1, 1, 2, 5, 14, 43, 144, 522, 2030, 8398, 36714, 168793, 813112, 4091735, 21451972, 116891160

for $1 \leq n \leq 15$.



gives the new sequence
1, 1, 2, 5, 14, 43, 147, 575, 2648, 14617, 96696, 754585, 6794015, 69116493, 781266266, 9688636317

for $1 \leq n \leq 15$.

Summary

- The method of enumeration schemes confirms many known results for barred patterns and generates new results for $S_n(25\bar{1}43)$, $S_n(25\bar{1}34)$, $S_n(43\bar{5}21)$, $S_n(43\bar{5}12)$, $S_n(5\bar{1}\bar{2}43)$, $S_n(\bar{5}\bar{4}231)$, $S_n(31\bar{5}\bar{4}2)$, $S_n(\bar{5}\bar{4}132)$.
- It remains to find other ways to count permutations avoiding barred patterns.

Summary

Based on computation:

- Conjecture: If q is a barred pattern of length k with $k-2$ bars then either $S_n(q)=1$ or $S_n(q)=(n-(k-2))!$.
- Conjecture: $S_n(\overline{3}\overline{1}542)$ gives the number of ordered factorizations over the Gaussian polynomials. (OEIS A047970)
- Conjecture: $S_n(\overline{1}43\overline{5}2)$ has generating function $\prod_{n\geq 0}\frac{1}{(1-\frac{x}{(1-x)^n})^{(1/2)^{n+1}}}$ (OEIS A122993).
- There are at least 24 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.

Summary

Based on computation:

- Conjecture: If q is a barred pattern of length k with $k-2$ bars then either $S_n(q) = 1$ or $S_n(q) = (n-(k-2))!$.
- Conjecture: $S_n(\overline{3}\overline{1}542)$ gives the number of ordered factorizations over the Gaussian polynomials. (OEIS A047970)
- Conjecture: $S_n(\overline{1}43\overline{5}2)$ has generating function $\prod_{n\geq 0} \frac{1}{(1-\frac{x}{(1-x)^n})^{(1/2)^{n+1}}}$ (OEIS A122993).
- There are at least 19 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.