

Pattern avoidance in trees

Lara Pudwell

Introduction Brief history

Contiguous tree patterns Definition & examples

Noncontiguous patterns

Definition & examples Generating functions Sets of tree patterns

Connections to other objects OEIS hits Pattern-avoidin, permutations

# Pattern avoidance in trees

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# **Connections to other objects**

- OEIS hits
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### History of Tree Patterns: Labelled Trees

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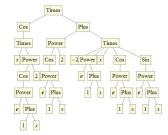
Noncontiguous patterns

Definition & examples Generating functions Sets of tree patterns

Connections to other objects OEIS hits Pattern-avoiding permutations • 1983: Flajolet and Steyaert

- focus on asymptotic probability of avoiding a given pattern
- 1990: Flajolet, Sipala, and Steyaert
  - every leaf of pattern must be matched by a leaf of the tree
  - motivated by compactly storing expressions in computer memory

• e.g. 
$$\frac{d}{dx}\left(\sin(x\cos^2(e^{x+1}))\right) =$$





## History of Tree Patterns: Labelled Trees

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# • 1983: Flajolet and Steyaert

- focus on asymptotic probability of avoiding a given pattern
- 1990: Flajolet, Sipala, and Steyaert
  - every leaf of pattern must be matched by a leaf of the tree
  - motivated by compactly storing expressions in computer memory
- 2012: Dotsenko
  - pattern may occur anywhere in tree
  - motivated by operad theory



# History of Tree Patterns: Unlabelled Trees

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- 2009: Rowland
  - contiguous pattern avoidance in binary trees
  - patterns can be anywhere, not just at leaves
- 2010: Gabriel, Peske, P., Tay
  - $\bullet\,$  extended Rowland's results to  $m\text{-}{\rm ary}$  trees
- 2011: Dairyko, P., Tyner, Wynn
  - non-contiguous pattern avoidance in binary trees
- 2012: P., Serrato, Scholten, Schrock
  - non-contiguous pattern containment in binary/m-ary trees



## **Key Question**

### Pattern avoidance in trees

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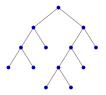
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Connections to other objects OEIS hits Pattern-avoiding permutations Today, our trees will be:

- rooted (root vertex at top)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)





## **Key Question**

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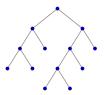
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Summary

Today, our trees will be:

- rooted (root vertex at top)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)



Question: How many trees with n leaves avoid a given tree pattern?



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### **Contiguous tree pattern**

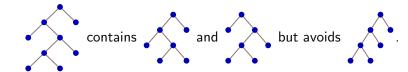
Tree T contains tree t if and only if T contains t as a contiguous rooted ordered subtree.

Example:

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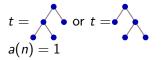
Definition & examples Generating functions Sets of tree patterns

Connections to other objects OEIS hits Pattern-avoiding permutations  $t = \bigwedge$ 

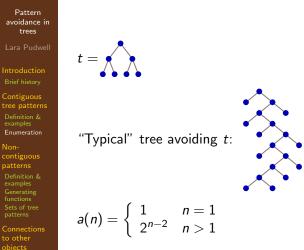
t =

a(n) = 0

$$a(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$











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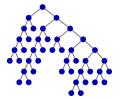
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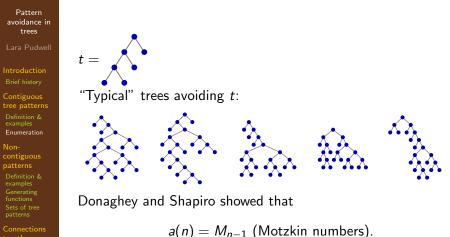


"Typical" tree avoiding t:

$$a(n) = \begin{cases} 1 & n=1\\ 2^{n-2} & n>1 \end{cases}$$







Summary

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### Contiguous pattern enumeration data



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t	a(n)
•	0
~	$\begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$
$\widehat{}$	1
$\widehat{\wedge} \widehat{\wedge}$	2 <sup><i>n</i>-2</sup>
	$M_{n-1}$ (Motzkin numbers)



### Contiguous tree pattern enumeration

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Connections to other objects OEIS hits Pattern-avoiding permutations Rowland

- Devised algorithm to find functional equation for avoidance generating function for any set of tree patterns.
  - Generating functions are always algebraic.
- Enumerated trees containing specified number of copies of a given tree pattern.
- Completely determined equivalence classes for tree patterns with at most 8 leaves.
  - For *n* = 1, 2, 3, ..., there are 1, 1, 1, 2, 3, 7, 15, 44, ... equivalence classes of *n*-leaf binary trees.



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# **Connections to other objects**

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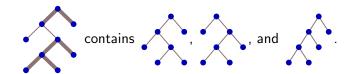
tree patterns

### Tree patterns

### Non-contiguous tree pattern

Tree T contains tree t if and only if there exists a sequence of edge contractions of T that produce  $T^*$  which contains t as a contiguous rooted ordered subtree.

### Example:



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functions



# Non-contiguous pattern enumeration data

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Pattern t	Number of $n$ -leaf trees avoiding $t$						
•	0						
•	$\int 1  n = 1$						
••	$iggl\{ 0  n>1 \ $						
	1						
	2 <sup><i>n</i>-2</sup>						
	2 <sup><i>n</i>-2</sup>						
	2 <sup><i>n</i>-2</sup>						



### The Main Theorem

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### Notation

Let av<sub>t</sub>(n) be the number of n-leaf trees that avoid t non-contiguously.

• Let 
$$g_t(x) = \sum_{n=1}^{\infty} \operatorname{av}_t(n) x^n$$
.



### The Main Theorem

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Connections to other objects OEIS hits Pattern-avoiding permutations Notation

Let av<sub>t</sub>(n) be the number of n-leaf trees that avoid t non-contiguously.

• Let 
$$g_t(x) = \sum_{n=1}^{\infty} \operatorname{av}_t(n) x^n$$
.

### Theorem

Fix  $k \in \mathbb{Z}^+$ . Let t and s be two k-leaf binary tree patterns. Then  $g_t(x) = g_s(x)$ .



### **Notation and Computation**

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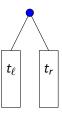
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Connections to other objects OEIS hits Pattern-avoiding permutations (More) Notation

## • Given tree *t*,

- let  $t_{\ell}$  be the subtree whose root is the left child of t's root.
- let *t<sub>r</sub>* be the subtree whose root is the right child of *t*'s root.





### **Notation and Computation**

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# (More) Notation • Given tree t,

- let  $t_{\ell}$  be the subtree whose root is the left child of t's root.
  - let *t<sub>r</sub>* be the subtree whose root is the right child of *t*'s root.

### Notice

$$g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)$$



### **Notation and Computation**

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# (More) Notation

• Given tree t,

- let  $t_{\ell}$  be the subtree whose root is the left child of t's root.
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### Notice

$$g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)$$

Solving...

$$g_t(x) = \frac{x - g_{t_\ell}(x) \cdot g_{t_r}(x)}{1 - g_{t_\ell}(x) - g_{t_r}(x)}.$$



### Proposition

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Connections to other objects OEIS hits Pattern-avoiding permutations  $g_t(x) = \frac{x - g_{t_\ell}(x) \cdot g_{t_r}(x)}{1 - g_{t_\ell}(x) - g_{t_r}(x)}.$ 

### Proposition

For any tree pattern t,  $g_t(x)$  is a rational function of x.



### A special case...

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Connections to other objects OEIS hits Pattern-avoiding permutations Let  $c_k$  be the k-leaf left comb (the unique k-leaf binary tree where every right child is a leaf).  $c_1 = \cdot, c_2 = \Lambda, c_3 = \Lambda, c_4 = \Lambda, c_5 = \Lambda, \text{etc.}$ 



### A special case...

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Connections to other objects OEIS hits Pattern-avoidin permutations Let  $c_k$  be the k-leaf left comb (the unique k-leaf binary tree where every right child is a leaf).  $c_1 = \cdot, c_2 = \Lambda, c_3 = \Lambda, c_4 = \Lambda, c_5 = \Lambda, c_5 = \Lambda, etc.$ If  $t = c_k$ , then  $t_\ell = c_{k-1}$  and  $t_r = \cdot$ .

For  $k \geq 2$ , we have

$$g_{c_k}(x) = \frac{x - g_{c_{k-1}}(x) \cdot g_{\bullet}(x)}{1 - g_{c_{k-1}}(x) - g_{\bullet}(x)} = \frac{x}{1 - g_{c_{k-1}}(x)}.$$



### Back to the main result

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Connections to other objects OEIS hits Pattern-avoiding permutations Theorem Fix  $k \in \mathbb{Z}^+$ . Let t and s b

Fix  $k \in \mathbb{Z}^+$ . Let t and s be two k-leaf binary tree patterns. Then  $g_t(x) = g_s(x)$ .

### Proof sketch

Inductive step:

- Assume the theorem holds for tree patterns with  $\ell$  leaves where  $\ell < k$ .
- Then any  $\ell$ -leaf tree has avoidance generating function  $g_{c_{\ell}}(x)$ .
- Consider tree t with  $\ell$  leaves to the left of its root and tree s with  $\ell 1$  leaves to the left of its root.
- Do algebra with previous work to show that  $g_t(x) = g_s(x)$ .



Patte avoidan tree

tree pati

Generatin functions

# **Generating functions**

ern	k	$g_{c_k}(x)$	OEIS number
nce in es		$\mathcal{B}_k(\wedge)$	
	1	0	trivial
	2	x	trivial
ous terns	3	$\frac{x}{1-x}$	trivial
n & s ition	4	$\frac{x-x^2}{1-2x}$	A000079
	5	$\frac{x-2x^2}{1-3x+x^2}$	A001519
n & s ng s	6	$\frac{x-3x^2+x^3}{1-4x+3x^2}$	A007051
ree tions	7	$\frac{x - 4x^2 + 3x^3}{1 - 5x + 6x^2 - x^3}$	A080937
	8	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$	A024175
avoiding tions 'Y	9	$\frac{x-6x^2+10x^3-4x^4}{1-7x+15x^2-10x^3+x^4}$	A080938



# **Coefficient sightings...**

Pattern avoidance in trees											
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Introduction	1										
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Contiguous	1	1									
tree patterns	1	2	1								
Definition & examples	-	_	-	-							
Enumeration	1	3	3	1							
Non-	1	4	6	4	1						
contiguous	1	5	10	10	5	1					
patterns Definition &	Т	5	10	10	5	T					
examples	1	6	15	20	15	6	1				
Generating functions	1	7	21	35	35	21	7	1			
Sets of tree	T	1	<b>Z</b> 1	35	35	<b>Z</b> 1	1	T			
patterns	1	8	28	56	70	56	28	8	1		
Connections											



# **Coefficient sightings...**

Pattern avoidance in trees										$\frac{x}{1}$
Lara Pudwell										$\frac{x}{1-x}$
Introduction Brief history Contiguous	1 1	1								$\frac{x-x^2}{1-2x}$
tree patterns Definition & examples Enumeration	1 1	2 3	1 3	1						$\frac{x-2x^2}{1-3x+x^2}$
Non- contiguous patterns	1 1	4 5	6 10	4 10	1 5	1				$\frac{x-3x^2+x^3}{1-4x+3x^2}$
Definition & examples Generating functions Sets of tree	1 1	6 7	15 21	20 35	15 35	6 21	1 7	1		$\frac{x - 4x^2 + 3x^3}{1 - 5x + 6x^2 - x^3}$
patterns Connections to other objects	1	8	28	56	70	56	28	8	1	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$
OEIS hits Pattern-avoiding permutations										$\frac{x - 6x^2 + 10x^3 - 4x^4}{1 - 7x + 15x^2 - 10x^3 + x^4}$
Summary										$1 - ix + 15x^2 - 10x^3 + x^4$



# **Coefficient sightings...**

Pattern avoidance in trees										<u>×</u> 1
Lara Pudwell										$\frac{x}{1-x}$
Introduction Brief history	1 1	1								$\frac{x-x^2}{1-2x}$
Contiguous tree patterns Definition &	1	1 2	1							
examples Enumeration	1	3	3	1						$\frac{x-2x^2}{1-3x+x^2}$
Non- contiguous patterns	1 1	4 5	6 10	4 10	1 5	1				$\frac{x-3x^2+x^3}{1-4x+3x^2}$
Definition & examples Generating	1	6	15	20	15	6	1			
functions Sets of tree patterns	1 1	7 8	21 28	35 56	35 70	21 56	7 28	1 8	1	$\frac{x - 4x^2 + 3x^3}{1 - 5x + 6x^2 - x^3}$
Connections to other objects										$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$

OEIS hits Pattern-avoidir

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 $\frac{x-6x^2+10x^3-4x^4}{1-7x+15x^2-10x^3+x^4}$ 



### An explicit formula

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Connections to other objects OEIS hits Pattern-avoiding permutations

### Theorem

Let  $k \in \mathbb{Z}^+$  and let t be a binary tree pattern with k leaves. Then

$$g_t(x) = \frac{\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot {\binom{k-(i+2)}{i}} \cdot x^{i+1}}{\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot {\binom{k-(i+1)}{i}} \cdot x^i}.$$



### Avoiding multiple tree patterns

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Connections to other objects OEIS hits Pattern-avoiding permutations Methods extend naturally to trees avoiding multiple tree patterns simultaneously:

- Generating functions are still rational.
- No longer one equivalence class per size of tree pattern



# Equivalence classes for avoiding a 4 leaf and a 5 leaf tree pattern

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Pattern representatives	OEIS
$\left[ \left\{ \left( \begin{array}{c} \left( \left( \begin{array}{c} \left( \left( \begin{array}{c} \left( \left( \left( \begin{array}{c} \left( $	0 for $n \ge 11$
$\left\{ \left( \bigwedge^{h}, \bigwedge^{h} \right) \right\}$	A016777
	(3k + 1)
$\left  - \left\{ \bigwedge^{\circ}, \bigwedge^{\circ} \right\} \right $	A152947
	$(rac{(k-2)\cdot(k-1)+1}{2})$
$\left\{ \bigwedge^{h}, \bigwedge^{h} \right\}$	A000071
	(Fibonacci numbers -1)
$\left[\left\{ \left( \begin{array}{c} \left( \left( \begin{array}{c} \left( \left( \begin{array}{c} \left( \left( \left( \begin{array}{c} \left( $	A000073
	(Tribonacci Numbers)



### Avoiding multiple tree patterns

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Connections to other objects OEIS hits Pattern-avoiding permutations Methods extend naturally to trees avoiding multiple tree patterns simultaneously:

- Generating functions are still rational.
- No longer one equivalence class per size of tree pattern (Open: Find a combinatorial characterization of when two sets of tree patterns are enumeratively equivalent.)



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### **Contiguous patterns**

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Summary

For binary patterns...

- A001006: Motzkin numbers
- A011782: Powers of 2
- A036765: Number of Dyck n-paths that avoid UUUU
- A086581: Number of Dyck n-paths that avoid DDUU.
- A036766: Number of Dyck n-paths that avoid UUUUU
- A005773: Number of n-permutations avoiding 1-23-4 and 1-3-2



### **Contiguous patterns**

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For binary patterns...

- A001006: Motzkin numbers
- A011782: Powers of 2
- A036765: Number of Dyck n-paths that avoid UUUU
- A086581: Number of Dyck n-paths that avoid DDUU.
- A036766: Number of Dyck n-paths that avoid UUUUU
- A005773: Number of n-permutations avoiding 1-23-4 and 1-3-2

For ternary patterns...

- A000108: Catalan numbers
- A001003: Little Schroeder numbers
- A107264: Counts colored Motzkin paths, where H(1,0) and U(1,1) each have 3 colors and D(1,-1) one color.
- A006605: Number of modes of connections of 2n points. (under Baxter's generalization of the Temperley-Lieb operators)



## First things first...

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Connections to other objects OEIS hits Pattern-avoiding permutations Notation

 $S_n$  is the set of all permutations of length n written in one-line notation.

Examples:  $S_2 = \{12, 21\}$  $S_3 = \{123, 132, 213, 231, 312, 321\}$ 

### Definition

Let  $w \in [k]^n$ . The *reduction* of w, red(w), is the string obtained by replacing the *i*th smallest letter(s) of w with *i*.

Examples: red(1534) = 1423red(72884) = 31442red(4231) = 4231



### **Permutation patterns**

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Let  $\pi \in S_n$  and  $\rho \in S_k$ .  $\pi$  contains  $\rho$  if there exist  $1 \leq i_1 < \cdots < i_k \leq n$  such that  $\operatorname{red}(\pi_{i_1} \cdots \pi_{i_k}) = \rho$ . If  $\pi$  does not contain  $\rho$ , then  $\pi$  avoids  $\rho$ 

Examples: 7245631 contains 132 (e.g. 243, 253, 263) contains 4321 (e.g. 7631, 7531, 7431) avoids 54321



### **Permutation patterns**

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### Notation

 $s_n(\rho)$  is the number of permutations of length *n* that avoid  $\rho$ .

### Preliminary results:

• 
$$s_n(1) = \begin{cases} 1 & n=0\\ 0 & n>0 \end{cases}$$

• 
$$s_n(12) = s_n(21) = 1$$

• 
$$s_n(123) = s_n(132) = s_n(213) = s_n(231) = s_n(312) =$$
  
 $s_n(321) = \frac{\binom{2n}{n}}{(n+1)} = C_n \text{ (Catalan numbers)}$ 



### ...and permutations

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- the number of binary trees
- the number of 231-avoiding permutations

Can we say more?



### ...and permutations

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Connections to other objects OEIS hits Pattern-avoiding permutations We know that the Catalan numbers count:

- the number of binary trees
- the number of 231-avoiding permutations

Can we say more?

### Theorem

Let t be any non-contiguous binary tree pattern with  $k\geq 2$  leaves. Then

$$av_t(n) = s_{n-1}(231, (k-1)(k-2)\cdots 21).$$



### Example

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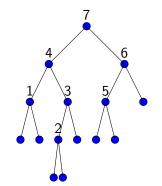
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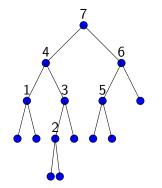
### Example



Pattern-avoiding permutations



### Example



1423756

Pattern-avoiding permutations



### Main theorem revisited....

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Connections to other objects OEIS hits Pattern-avoiding permutations Theorem

Fix  $k \in \mathbb{Z}^+$ . Let t and s be two k-leaf binary tree patterns. Then  $g_t(x) = g_s(x)$ .

Under the tree  $\leftrightarrow$  231-avoiding permutation bijection, this theorem translates into a set of enumeration-equivalances for permutations too!





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Connections to other objects OEIS hits Pattern-avoiding permutations  $\bigwedge^{\leftrightarrow 12} \leftrightarrow 21$ 

So  $s_n(231, 12) = s_n(231, 21)$  (or, really  $s_n(12) = s_n(21)$ ).





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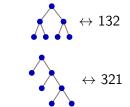
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Connections to other objects OEIS hits Pattern-avoiding permutations  $\leftrightarrow 213$   $\leftrightarrow 312$ 



So  $s_n(231, 213) = s_n(231, 132) = s_n(231, 312) = s_n(231, 321)$ 





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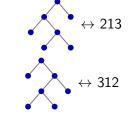
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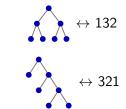
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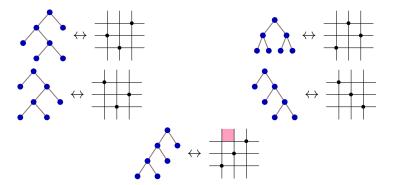
So  $s_n(231, 213) = s_n(231, 132) = s_n(231, 312) = s_n(231, 321)$ 







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- Enumeration

## **Non-contiguous patterns**

- Definition & examples
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- Sets of tree patterns



# **Connections to other objects**

- OEIS hits
- Pattern-avoiding permutations



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- Tree patterns have a rich history ranging from data storage considerations to non-associative algebra.
- Both contiguous and non-contiguous tree patterns yield nice enumeration sequences.
  - For contiguous tree patterns,  $g_t(x)$  is algebraic.
  - For non-contiguous tree patterns,  $g_t(x)$  is rational and has a nice closed form.
  - Open:
    - Equivalence classes for contiguous tree patterns with 9 or more leaves.
    - Equivalence classes for trees avoiding sets of tree patterns.



# Summary (continued)

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- Trees avoiding a non-contiguous k-leaf tree pattern are in bijection with permutations avoiding 231 and (k-1)(k-2)···1.
  - For any n ∈ Z<sup>+</sup>, there are at least Catalan-many enumeration equivalent pattern sets of the form {231, π} where π is a mesh pattern of length n.



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Connections to other objects OEIS hits Pattern-avoiding permutations Thank You!



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