Pattern avoidance in double lists

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joint work with
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Variations on Pattern Avoidance

- in words (Burstein 1998, and others)
- in centrosymmetric permutations (Egge 2007; Barnabei, Bonetti, Silimbani 2010)
- in centrosymmetric words (Ferrari 2011)
- in circular permutations (Callan 2002, Vella 2003)
  - Here, for example, $1324 = 3241 = 2413 = 4132$.

Our study: pattern avoidance in words with a special kind of symmetry/repetition.

Definitions/Notation

- $\mathcal{D}_n = \{\pi\pi \mid \pi \in \mathcal{S}_n\}$.

  $\mathcal{D}_3 = \{123123, 132132, 213213, 231231, 312312, 321321\}$.

- $\mathcal{D}_n(\rho) = \{\sigma \mid \sigma \in \mathcal{D}_n \text{ and } \sigma \text{ avoids } \rho\}$.

$$\mathcal{D}_3(12) = \emptyset.$$
$$\mathcal{D}_3(123) = \{321321\}.$$

Goal: Characterize $\mathcal{D}_n(\rho)$, compute $|\mathcal{D}_n(\rho)|$.

Warmup

- $|\mathcal{D}_n(\rho)| = |\mathcal{D}_n(\rho^r)| = |\mathcal{D}_n(\rho^c)|$.

  $|\mathcal{D}_n(123)| = |\mathcal{D}_n(321)|$.
  $|\mathcal{D}_n(132)| = |\mathcal{D}_n(213)| = |\mathcal{D}_n(231)| = |\mathcal{D}_n(312)|$.

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  $|\mathcal{D}_n(123)| = |\mathcal{D}_n(321)|$.
  $|\mathcal{D}_n(132)| = |\mathcal{D}_n(213)| = |\mathcal{D}_n(231)| = |\mathcal{D}_n(312)|$.

- $\mathcal{D}_n(123) = \{n \cdots 1 n \cdots 1\}$ for $n \geq 3$.
- $$\mathcal{D}_n(132) = \begin{cases} \{11\} & n = 1 \\ \{1212, 2121\} & n = 2 \\ \{231231\} & n = 3 \\ \emptyset & n \geq 4 \end{cases}.$$

Length 4 Trivial Wilf Classes

Pattern ρ	$\{\lvert\mathcal{D}_n(\rho)\rvert\}_{n=1}^{10}$
1342, 2431, 3124, 4213	1, 2, 6, 12, 15, 15, 15, 15, 15, 15
2143, 3412	1, 2, 6, 12, 13, 14, 16, 18, 20, 22
1423, 2314, 3241, 4132	1, 2, 6, 12, 17, 23, 27, 30, 33, 36
1432, 2341, 3214, 4123	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
1243, 2134, 3421, 4312	1, 2, 6, 12, 19, 25, 34, 44, 55, 67
2413, 3142	1, 2, 6, 12, 18, 29, 47, 76, 123, 199
1324, 4231	1, 2, 6, 12, 21, 38, 69, 126, 232, 427
1234, 4321	1, 2, 6, 12, 27, 58, 121, 248, 503, 1014

Contrast: For large n, $|\mathcal{S}_n(1342)| < |\mathcal{S}_n(1234)| < |\mathcal{S}_n(1324)|$.

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1342, 2431, 3124, 4213	1, 2, 6, 12, 15, 15, 15, 15, 15, 15
2143, 3412	1, 2, 6, 12, 13, 14, 16, 18, 20, 22
1423, 2314, 3241, 4132	1, 2, 6, 12, 17, 23, 27, 30, 33, 36
1432, 2341, 3214, 4123	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
1243, 2134, 3421, 4312	1, 2, 6, 12, 19, 25, 34, 44, 55, 67
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1234, 4321	1, 2, 6, 12, 27, 58, 121, 248, 503, 1014

Length 4 Trivial Wilf Classes

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Pattern ρ	
1342, 2431, 3124, 4213	constant
2143, 3412	1, 2, 6, 12, 13, 14, 16, 18, 20, 22
1423, 2314, 3241, 4132	1, 2, 6, 12, 17, 23, 27, 30, 33, 36
1432, 2341, 3214, 4123	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
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Length 4 Trivial Wilf Classes

Pattern ρ	
1342, 2431, 3124, 4213	constant
2143, 3412	linear
1423, 2314, 3241, 4132	linear
1432, 2341, 3214, 4123	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
1243, 2134, 3421, 4312	1, 2, 6, 12, 19, 25, 34, 44, 55, 67
2413, 3142	1, 2, 6, 12, 18, 29, 47, 76, 123, 199
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Length 4 Trivial Wilf Classes

Pattern ρ	
1342, 2431, 3124, 4213	constant
2143, 3412	linear
1423, 2314, 3241, 4132	linear
1432, 2341, 3214, 4123	quadratic
1243, 2134, 3421, 4312	quadratic
2413, 3142	1, 2, 6, 12, 18, 29, 47, 76, 123, 199
1324, 4231	1, 2, 6, 12, 21, 38, 69, 126, 232, 427
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Length 4 Trivial Wilf Classes

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1243, 2134, 3421, 4312	quadratic
2413, 3142	2nd-order linear recurrence
1324, 4231	1, 2, 6, 12, 21, 38, 69, 126, 232, 427
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Length 4 Trivial Wilf Classes

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1342, 2431, 3124, 4213	constant
2143, 3412	linear
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1432, 2341, 3214, 4123	quadratic
1243, 2134, 3421, 4312	quadratic
2413, 3142	2nd-order linear recurrence
1324, 4231	3rd-order linear recurrence
1234, 4321	exponential

Length 4 Trivial Wilf Classes

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Pattern ρ	
1342, 2431, 3124, 4213	1, 2, 6, 12, 15, 15, 15, 15, 15, 15
2143, 3412	linear
1423, 2314, 3241, 4132	linear
1432, 2341, 3214, 4123	quadratic
1243, 2134, 3421, 4312	quadratic
2413, 3142	2nd-order linear recurrence
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1234, 4321	1, 2, 6, 12, 27, 58, 121, 248, 503, 1014

$\mathcal{D}_n(1342)$

Let $\pi\pi \in \mathcal{D}_n(1342)$.

Erase n and $n-1$ to obtain $\pi'\pi' \in \mathcal{D}_{n-2}(1342)$.

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Observations:

- π' has at most one 12 pattern.

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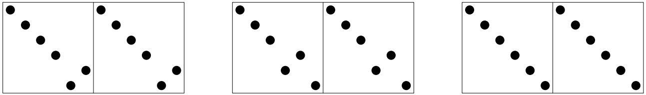
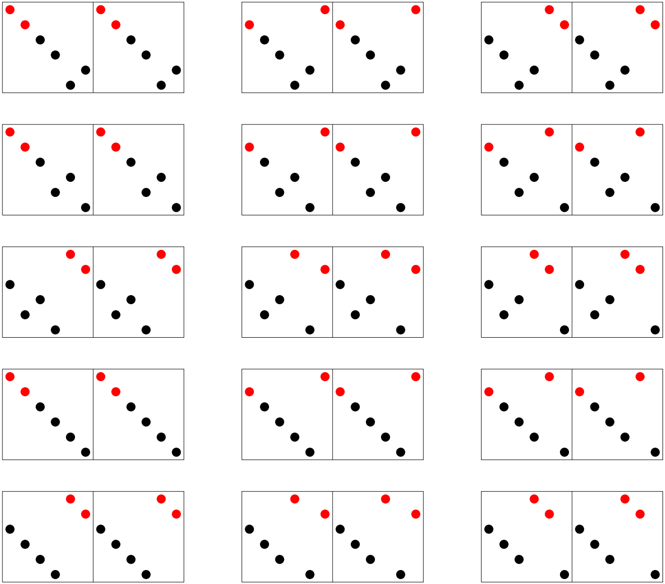
Observations:

- π' has at most one 12 pattern.
- If π' has a 12 pattern, it uses consecutive digits in adjacent positions.
- If π' has a 12 pattern it uses the digits 12 or 23.

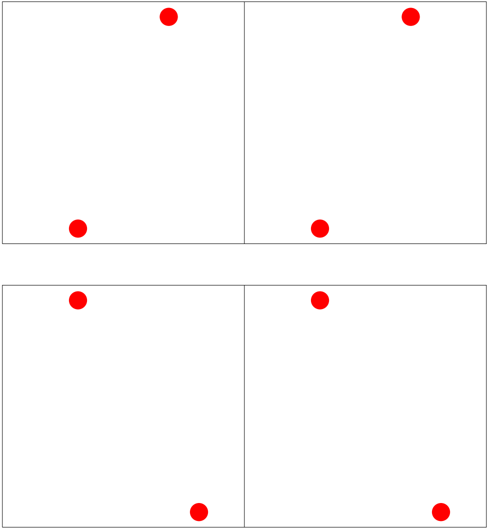
3 possible choices for $\pi'\pi'$:



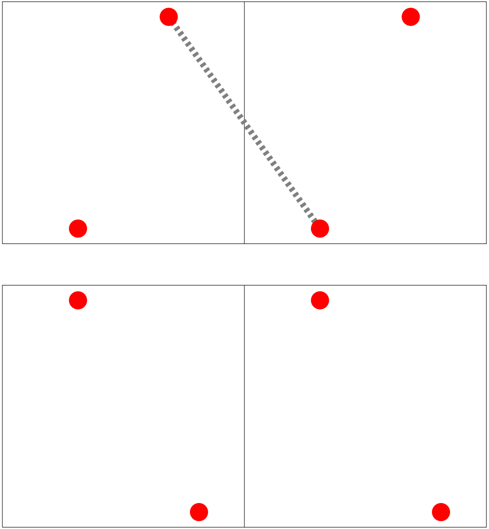
$\mathcal{D}_6(1342)$



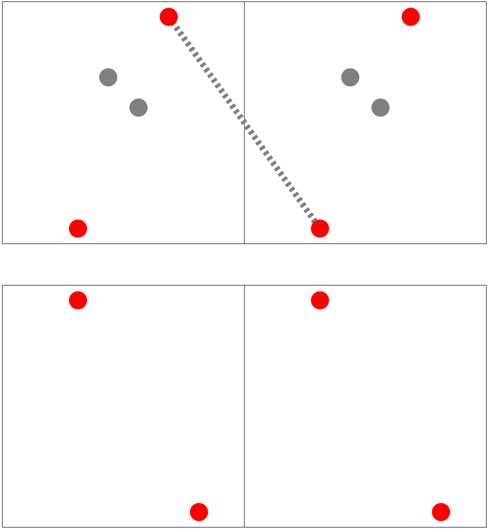
$\mathcal{D}_n(1234)$



$\mathcal{D}_n(1234)$



$\mathcal{D}_n(1234)$



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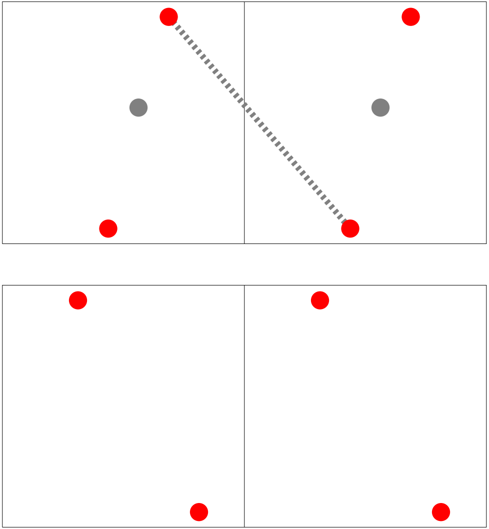

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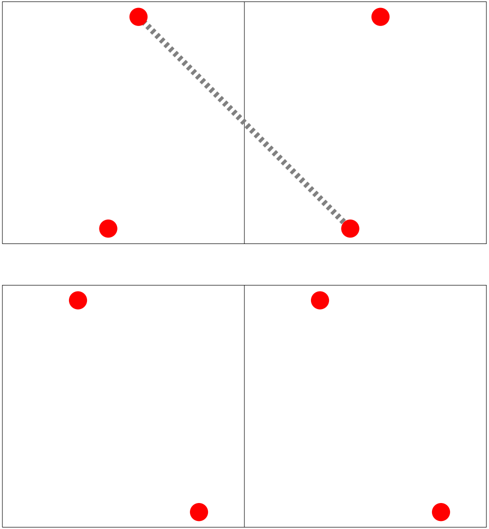

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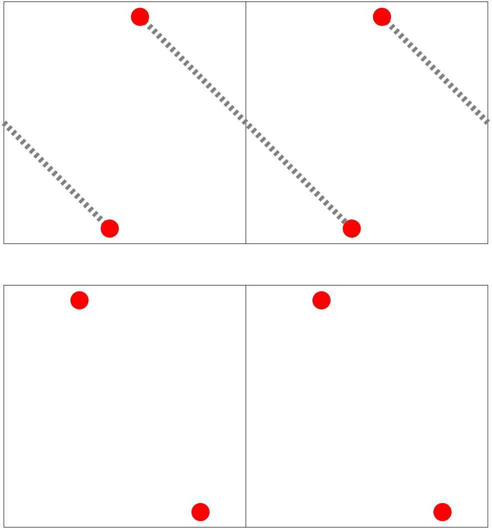

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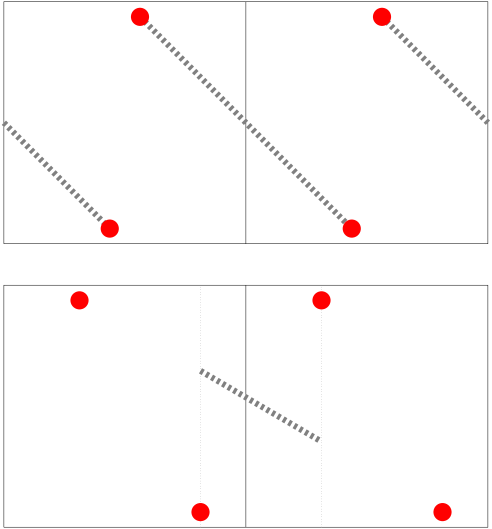

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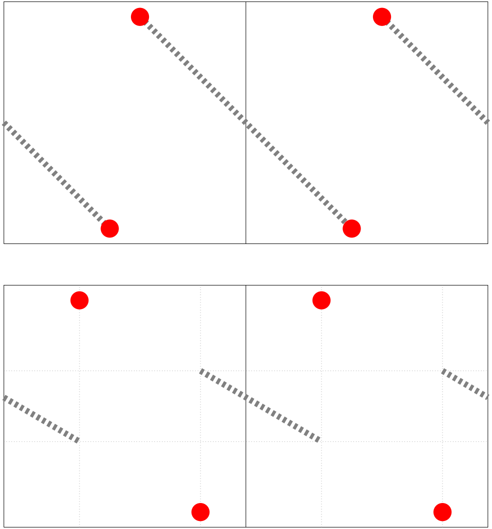

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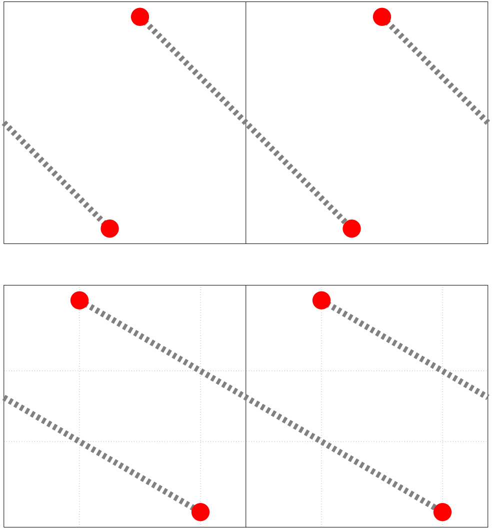
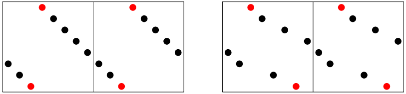


$\mathcal{D}_n(1234)$

Picture of 1234-avoiding double lists:



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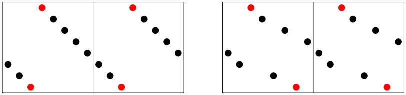
Picture of 1234-avoiding double lists:



1. Begin with ordered deck $n \cdots 1$.
2. Cut and shuffle.
3. Each card either comes from upper or lower partial deck.

$\mathcal{D}_n(1234)$

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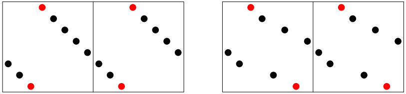

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3. Each card either comes from upper or lower partial deck.

- 2^n strings on $\{U, L\}^n$
- $(n+1)$ strings of the form $U \cdots UL \cdots L$ are all equivalent to $n \cdots 1$.

$$|\mathcal{D}_n(1234)| = 2^n - n \text{ for } n \geq 4.$$

Summary

Pattern ρ	$\lvert\mathcal{D}_n(\rho)\rvert$	
1342, 2431, 3124, 4213	15	$(n \geq 5)$
2143, 3412	$2n+2$	$(n \geq 6)$
1423, 2314, 3241, 4132	$3n+6$	$(n \geq 7)$
1432, 2341, 3214, 4123	$\frac{1}{2}n^2 + \frac{3}{2}n - 4$	$(n \geq 6)$
1243, 2134, 3421, 4312	$\frac{1}{2}n^2 + \frac{5}{2}n - 8$	$(n \geq 6)$
2413, 3142	L_{n+2}	$(n \geq 5)$
1324, 4231	$\lvert\mathcal{D}_{n-1}(\rho)\rvert + \lvert\mathcal{D}_{n-2}(\rho)\rvert + \lvert\mathcal{D}_{n-3}(\rho)\rvert$	$(n \geq 10)$
1234, 4321	$2^n - n$	$(n \geq 4)$

Patterns with repeated letters

Pattern ρ	Conjectured Enumeration for $\lvert\mathcal{D}_n(\rho)\rvert$
132142	$\lvert\mathcal{S}_n(1342, 3142)\rvert$ large Schröder numbers
123142	$\lvert\mathcal{S}_n(2314, 3142)\rvert$
124132	$\lvert\mathcal{S}_n(1243, 4132)\rvert$
124312	$\lvert\mathcal{S}_n(1243, 2431, 4312)\rvert$
14123	$\lvert\mathcal{S}_n(1423, 2413, 4123, 4213)\rvert$ central binomial coefficients
12134	$\lvert\mathcal{S}_n(1234, 1324, 1342, 2134)\rvert$ perms where every 321 pattern is actually a 3-21 pattern
41321	$\lvert\mathcal{S}_n(2413, 2431, 3142, 3241, 4132, 4213)\rvert$ perms sortable through 2 pop-stacks in series

Future Work

- Is $1 \cdots m$ always the easiest permutation to avoid? What is the hardest pattern in $\mathcal{S}_m$ to avoid?
- Under what conditions does $\{|\mathcal{D}_n(\rho)|\}$ have a rational generating function?
- Systematically study $\mathcal{D}_n(\rho)$ where ρ has repeated digits.
- Consider words on $\{1, 1, 2, 2, \ldots, n, n\}$ with other structure.

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Thanks for listening!

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