Pattern avoidance in permutations

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Conventions and Definitions

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  e.g. $\pi = 7362541$ contains 231.

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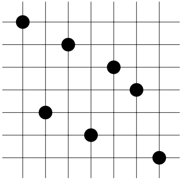
  $\pi = 7362541$ avoids 123.

Graphs of Permutations

Consider π as a discrete function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$.



Graph of $\pi = 7362541$

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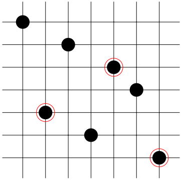

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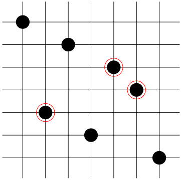

$\pi = 7362541$ contains 132.

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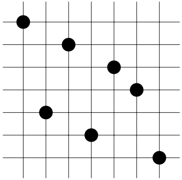


$\pi = 7362541$ avoids 123.

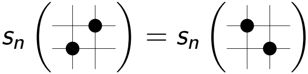
Counting Warmup

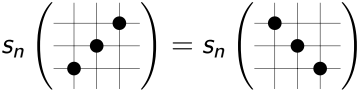
Notation

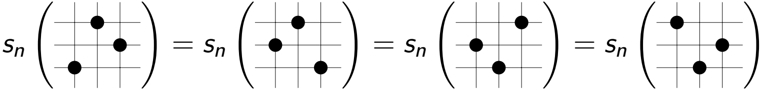
$\mathcal{S}_n(\rho) = \{\pi \in \mathcal{S}_n \mid \pi \text{ avoids } \rho\}$
$s_n(\rho) = |\mathcal{S}_n(\rho)|$

Observation

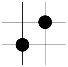
$s_n(\rho) = s_n(\rho^*)$ if the graph of ρ can be obtained by rotation and/or reflection of the graph of ρ^*.

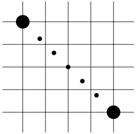




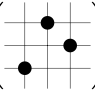


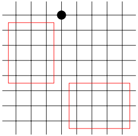
Let's Count...

$\mathcal{S}_n\Big($  $\Big)$



$s_n(12) = 1$

$\mathcal{S}_n\Big($  $\Big)$



$$s_n(132) = \sum_{i=1}^{n} s_{i-1}(132)s_{n-i}(132)$$

$s_n(\rho) = C_n$ for any $\rho \in \mathcal{S}_3$.

More Counting

n	1	2	3	4	5	6	7	exact enumeration
$s_n(1342)$	1	2	6	23	103	512	2740	Bóna (1997)
$s_n(1234)$	1	2	6	23	103	513	2761	Gessel (1990)
$s_n(1324)$	1	2	6	23	103	513	2762	unknown

Classical Pattern Sightings

Theorem (Erdős-Szekeres, 1935)

Given $a, b \in \mathbb{Z}^+$, if $n \geq (a-1)(b-1)+1$, then $s_n\left(12\cdots(a-1)a, b(b-1)\cdots 21\right) = 0$.

Theorem (Knuth, 1968)

A permutation is sortable after one pass through a stack if and only if it avoids 231.

Theorem (Lakshmibai and Sandhya, 1990)

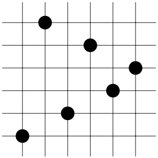
Schubert variety X_π is smooth if and only if π avoids 3412 and 4231.

Barred Pattern Notation

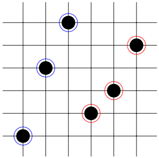
- A *barred* pattern is a permutation where each number may or may not have a bar over it.
- For a barred pattern ρ
  - $u(\rho)$ is the unbarred portion of ρ.
  - $e(\rho)$ is the entire permutation ρ (ignoring bars).
  - π *avoids* ρ if every copy of $u(\rho)$ extends to a copy of $e(\rho)$.

Example: $\rho = 1\bar{4}23$

- $u(\rho) = 123$
- $e(\rho) = 1423$



162534 avoids $1\bar{4}23$



146235 contains $1\bar{4}23$.

Barred Pattern Sightings

Theorem (West, 1990)

A permutation is sortable after two passes through a stack if and only if it avoids 2341 and $3\bar{5}241$.

Theorem (Bousquet-Mélou and Butler, 2007, proving a conjecture of Woo and Yong)

Schubert variety X_π is locally factorial if and only if π avoids 1324 and $21\bar{3}54$.

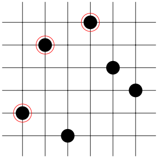
Theorem (Bousquet-Mélou, Claesson, Dukes, and Kitaev, 2010)

Self-modifying ascent sequences are in bijection with permutations which avoid $3\bar{1}52\bar{4}$.

Vincular pattern notation

- A *vincular* pattern is a permutation where each pair of consecutive entries may or may not have a dash between them.
- If ρ is a vincular pattern...
  - ρ_i-ρ_{i+1} indicates ρ_i and ρ_{i+1} may appear arbitrarily far apart.
  - $\rho_i\rho_{i+1}$ indicates ρ_i and ρ_{i+1} must appear consecutively.

Example:



251643 contains 1-2-3 and 12-3 but avoids 1-23.

Vincular Pattern Sightings

(Babson and Steingrímsson, 2000)

Given vincular pattern β, write $\beta(\pi)$ for the number of copies of β in π. Nearly all known Mahonian permutation statistics can be expressed in terms of vincular patterns.

Examples:

- $\text{inv}(\pi) = \text{2-1}(\pi)$
- $\text{maj}(\pi) = \text{1-32}(\pi) + \text{2-31}(\pi) + \text{3-21}(\pi) + \text{21}(\pi)$.

First half recap

- Permutation patterns appear in a variety of applications (computer science, algebraic geometry, and more).
- Permutation patterns provide a plethora of enumeration problems.
- Enumeration is generally *hard* and requires arguments depending on the particular pattern.

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- Permutation patterns appear in a variety of applications (computer science, algebraic geometry, and more).
- Permutation patterns provide a plethora of enumeration problems.
- Enumeration is generally *hard* and requires arguments depending on the particular pattern.
- Goal: Design an algorithm
  - Input: Set of permutation patterns Q to be avoided
  - Output: Recurrence to compute $s_n(Q)$

Enumeration Schemes

Definition (informal)

An *enumeration scheme* is an encoding for a family of recurrence relations enumerating members of a family of sets.

Find enumeration schemes via a divide-and-conquer algorithm.

3 pieces:

- Divide:
  - Refinement
- Conquer:
  - Reversibly Deletable Elements
  - Gap Vectors

Refinement

Notation: Prefix Pattern

$\mathcal{S}_n(Q)[p] = \{\pi \in \mathcal{S}_n(Q) \mid \pi_1 \cdots \pi_{|p|} \sim p\}$
$s_n(Q)[p] = |\mathcal{S}_n(Q)[p]|$

Note

For any set of patterns Q

$$\mathcal{S}_n(Q) = \mathcal{S}_n(Q)[1] = \mathcal{S}_n(Q)[12] \cup \mathcal{S}_n(Q)[21] = \cdots .$$

Therefore

$$s_n(Q) = s_n(Q)[1] = s_n(Q)[12] + s_n(Q)[21] = \cdots .$$

Running Example:
$s_n(132) = s_n(132)[12] + s_n(132)[21]$

Refinement

Notation: Prefix Pattern with Specified Letters

$\mathcal{S}_n(Q)[p; w] = \{\pi \in \mathcal{S}_n(Q)[p] \mid \pi_1 \cdots \pi_{|p|} = w\}$
$s_n(Q)[p; w] = |\mathcal{S}_n(Q)[p; w]|$

Note

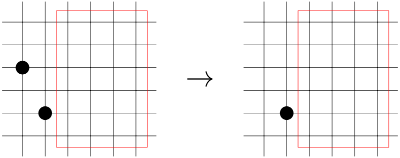
For any set of patterns Q
$s_n(Q)[p] = \sum_{w^* \sim p} s_n(Q)[p; w^*]$

Running Example:

$$s_n(132) = s_n(132)[12] + s_n(132)[21]$$
$$= \sum_{1 \leq i < j \leq n} s_n(132)[12; ij] + \sum_{1 \leq j < i \leq n} s_n(132)[21; ij]$$

Reversibly Deletable Elements

Consider $\pi \in \mathcal{S}_n(132)[21]$



So deleting π_1 produces $\pi^* \in \mathcal{S}_{n-1}(132)[1]$.

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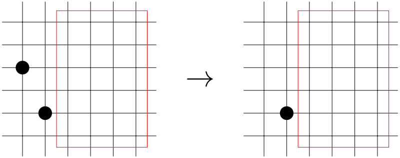

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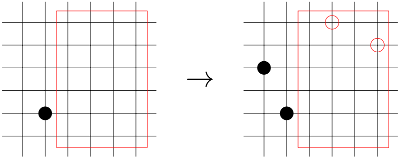

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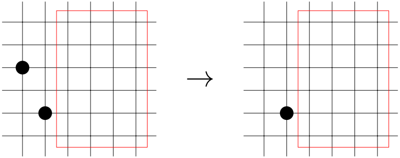

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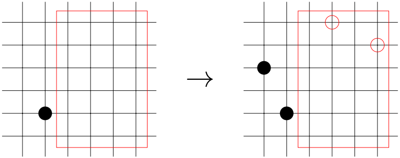

So reinserting π_1 into π^* produces $\pi \in \mathcal{S}_n(132)[21]$.

$s_n(132)[21; ij] = s_{n-1}(132)[1; j]$.

Running example

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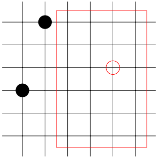
$$
\begin{aligned}
s_n(132) &= s_n(132)[12] + s_n(132)[21] \\
&= \sum_{1 \le i < j \le n} s_n(132)[12; ij] + \sum_{1 \le j < i \le n} s_n(132)[21; ij] \\
&= \sum_{1 \le i < j \le n} s_n(132)[12; ij] + \sum_{1 \le j < i \le n} s_{n-1}(132)[1; j]
\end{aligned}
$$

Gap Vectors

Consider $\pi \in \mathcal{S}_n(132)[12]$



$$s_n(132)[12; ij] = \begin{cases} 0 & j > i+1 \\ ? & j = i+1 \end{cases}$$

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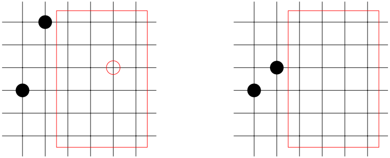

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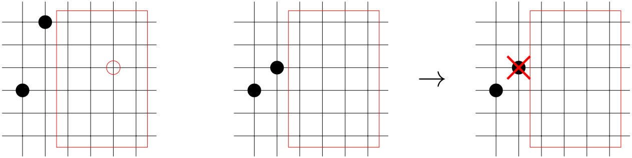

$$s_n(132)[12; ij] = \begin{cases} 0 & j > i+1 \\ s_{n-1}(132)[1; i] & j = i+1 \end{cases}$$

Running example

$$\begin{aligned}
s_n(132) &= s_n(132)[12] + s_n(132)[21] \\
&= \sum_{1\leq i<j\leq n} s_n(132)[12; ij] + \sum_{1\leq j<i\leq n} s_{n-1}(132)[1; j] \\
&= \sum_{1\leq i<n} s_n(132)[1; i(i+1)] + \sum_{1\leq j<i\leq n} s_{n-1}(132)[1; j] \\
&= \sum_{1\leq i\leq n-1} s_{n-1}(132)[1; i] + \sum_{1\leq j<i\leq n} s_{n-1}(132)[1; j]
\end{aligned}$$

Running example

$$\begin{aligned} s_n(132) &= s_n(132)[12] + s_n(132)[21] \\ &= \sum_{1 \le i < j \le n} s_n(132)[12; ij] + \sum_{1 \le j < i \le n} s_{n-1}(132)[1; j] \\ &= \sum_{1 \le i < n} s_n(132)[1; i(i+1)] + \sum_{1 \le j < i \le n} s_{n-1}(132)[1; j] \\ &= \sum_{1 \le i \le n-1} s_{n-1}(132)[1; i] + \sum_{1 \le j < i \le n} s_{n-1}(132)[1; j] \end{aligned}$$

So, $s_1(132) = s_1(132)[1; 1] = 1$, and for $n \geq 2$

$$\begin{aligned} s_n(132) &= \sum_{i=1}^{n} s_n(132)[1; i] \\ &= \sum_{i=1}^{n} \sum_{j=1}^{i} s_{n-1}(132)[1; j] \end{aligned}$$

History of Enumeration Schemes

- Schemes for classical patterns (Zeilberger 1998/Vatter 2005)
- Schemes for barred patterns (P. 2010)
- Schemes for vincular patterns (Baxter, P., 2012)

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Each algorithm can

- discover new enumeration results.
- help determine when $s_n(\rho) = s_n(\rho^*)$ for pairs of patterns ρ and ρ^*.
- along with OEIS, help conjecture relationships with other combinatorial objects.
- q-count pattern-avoiding permutations according to other permutation statistics.

Results for Classical Pattern Sets

Pattern Lengths	Success Rate
$\{3\}$	2/2 (100%)
$\{3,3\}$	5/5 (100%)
$\{3,3,3\}$	5/5 (100%)
$\{3,3,3,3\}$	5/5 (100%)
$\{4\}$	2/7 (29%)
$\{4,4\}$	9/56 (16%)
$\{4,4,4\}$	116/317 (37%)
$\{3,4\}$	22/30 (73%)
$\{3,3,4\}$	66/66 (100%)
$\{3,4,4\}$	179/268 (67%)
$\{3,5\}$	15/118 (13%)

Results for Barred Pattern Sets

Pattern Lengths	Success Rate	Pattern Lengths	Success Rate
[2, 1]	1/1 (100%)	[3, 0], [3, 0], [3, 1]	43/45 (95.6%)
[2, 0], [2, 1]	2/2 (100%)	[3, 0], [3, 0], [3, 2]	45/45 (100%)
[2, 1], [2, 1]	2/2 (100%)	[3, 0], [3, 1], [3, 1]	135/138 (97.8%)
		[3, 0], [3, 1], [3, 2]	280/280 (100%)
[3, 1]	4/4 (100%)	[3, 0], [3, 2], [3, 2]	138/138 (100%)
[3, 2]	4/4 (100%)	[3, 1], [3, 1], [3, 1]	115/118 (97.5%)
[3, 0], [3, 1]	18/20 (90%)	[3, 1], [3, 1], [3, 2]	378/378 (100%)
[3, 0], [3, 2]	20/20 (100%)	[3, 1], [3, 2], [3, 2]	378/378 (100%)
[3, 1], [3, 1]	27/28 (96.4%)	[3, 2], [3, 2], [3, 2]	118/118 (100%)
[3, 1], [3, 2]	50/50 (100%)		
[3, 2], [3, 2]	28/28 (100%)	[4, 1]	12/16 (75%)
		[4, 2]	25/26 (96.2%)
[3, 1], [4, 0]	59/71 (83.1%)	[4, 3]	16/16 (100%)
[3, 1], [4, 1]	229/240 (95.4%)		
[3, 1], [4, 2]	355/364 (97.5%)	[5, 1]	15/89 (16.9%)
[3, 0], [4, 1]	84/88 (95.5%)	[5, 2]	136/172 (79.1%)
[3, 0], [4, 2]	133/136 (97.8%)	[5, 3]	168/172 (97.7%)

Results for Vincular Pattern Sets

Pattern Lengths	Success Rate
$\{2\}$	2/2 (100%)
$\{2, 2\}$	3/3 (100%)
$\{2, 3\}$	11/11 (100%)
$\{3\}$	7/7 (100%)
$\{3, 3\}$	68/70 (97.1%)
$\{3, 3, 3\}$	354/358 (98.9%)
$\{4\}$	35/55 (63.6%)
$\{4, 4\}$	1600/4624 (34.6%)
$\{5\}$	144/479 (30.1%)
$\{3, 4\}$	639/914 (69.9%)
$\{3, 5\}$	2465/7411 (33.3%)

Guaranteed Schemes

Theorem 1 – Consecutive Patterns

If ρ is a dashless pattern of length t, then $\{\rho\}$ has a finite enumeration scheme of depth t.

Theorem 2 – "Nearly Consecutive" Patterns

If ρ is a pattern of length t where only the last two numbers have a dash between them, then $\{\rho\}$ has a finite enumeration scheme of depth $t-1$.

Theorem 3

If the finite set Q contains only consecutive and "nearly consecutive" patterns, then Q has a finite enumeration scheme.

Dashes vs. Depth

Question

Is number of dashes related to scheme depth?

ρ	**Depth**
1234	4
123–4	3
12–34	4
1–234	4
12–3–4	4
1–23–4	3
1–2–34	5
1–2–3–4	4

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Is number of dashes related to scheme depth?

ρ	**Depth**
1234	4
123–4	3
12–34	4
1–234	4
12–3–4	4
1–23–4	3
1–2–34	5
1–2–3–4	4

Answer: Maybe, but not monotonically.

Success Rate by Block Type

Block Type

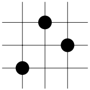
The block type of a dashed pattern is a vector describing the number of letters between each dash.

Block type	Success Rate
(3)	2/2 (100%)
(2,1)	3/3 (100%)
(1,1,1)	2/2 (100%)
(4)	8/8 (100%)
(3,1)	12/12 (100%)
(2,2)	3/8 (37.5%)
(2,1,1)	4/12 (25%)
(1,2,1)	6/8 (75%)
(1,1,1,1)	2/7 (28.6%)
(5)	32/32 (100%)
(4,1)	32/32 (100%)
other length 5	80/415 (19.3%)

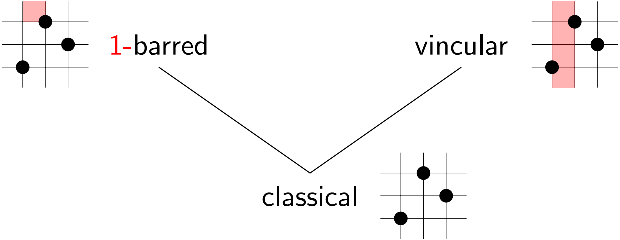
Interesting? Sequences

Q	OEIS No.	Description
$\{13{-}2, 213\}$	A105633	Number of Dyck paths of semilength $n+1$ avoiding UUDU
$\{1{-}2{-}3, 231\}$	A135307	Number of Dyck paths of semilength n avoiding UDDU
$\{12{-}3, 1{-}3{-}2, 312\}$	A005314	Number of compositions of n into parts congruent to $\{1,2\}$ mod 4
$\{1{-}2{-}3, 231, 3{-}1{-}2\}$	A089071	Number of liberties a big eye of size n gives in the game of Go

Where next?

classical 

Where next?



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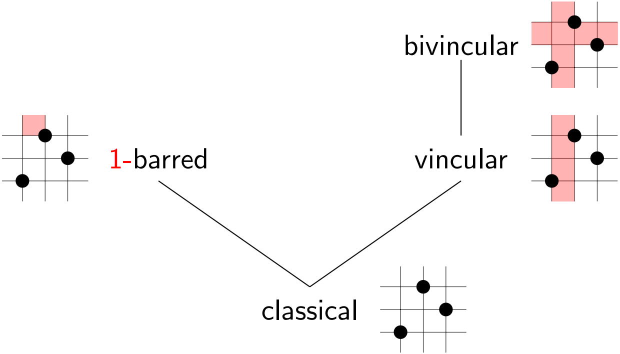
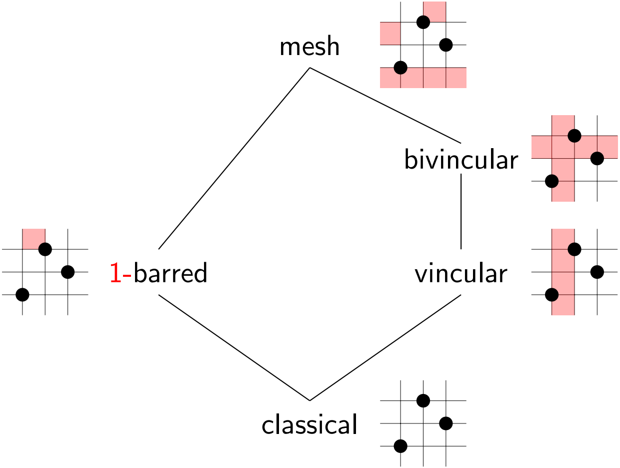


Where next?



Thank You!

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