Patterns in Trees

Lara Pudwell
faculty.valpo.edu/lpudwell

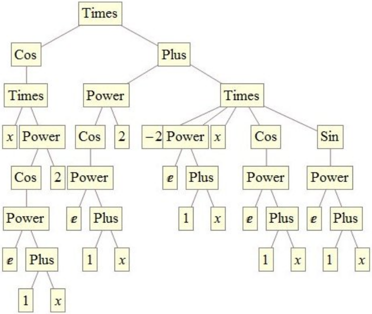
Valparaiso University

2018 Joint Mathematics Meetings
AMS Special Session on Open and Accessible Problems for Undergraduate Research
January 11, 2018

Why tree patterns?

compactly storing expressions in computer memory

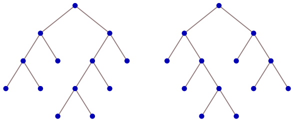
e.g. $\frac{d}{dx}\left(\sin(x\cos^2(e^{x+1}))\right) =$



Notation

Our trees are:

- rooted (root vertex at top, children below)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)



$\mathbb{T}_n$ is the set of n-leaf binary trees.

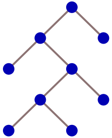
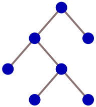
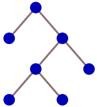
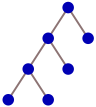
Question: How many trees in $\mathbb{T}_n$ avoid a given tree pattern?

Tree patterns

Contiguous tree pattern

Tree T contains tree t if and only if t is a contiguous rooted ordered subtree of T.

Example:

 contains  and  but avoids .

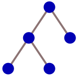
What is the number $a_t(n)$ of n-leaf binary trees avoiding t?

$t =$ 

$a_t(n) = 0$

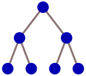
$t =$ 

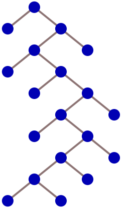
$$a_t(n) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$$

$t =$ 

$a_t(n) = 1$

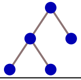
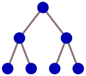
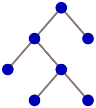
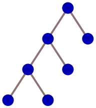
What is the number $a_t(n)$ of n-leaf binary trees avoiding t?

$t =$ 

"Typical" tree avoiding t: 

$$a_t(n) = \begin{cases} 1 & n = 1 \\ 2^{n-2} & n > 1 \end{cases}$$

Contiguous pattern enumeration data

t	$a_t(n)$
	0
	$\begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$
	1
 	2^{n-2}
	M_{n-1} (Motzkin numbers)

Contiguous tree pattern results

- 2009 (Rowland): contiguous pattern avoidance in binary trees
- 2010 (Gabriel, Peske, P., Tay): extended Rowland's results to ternary trees

Contiguous tree pattern results

- 2009 (Rowland): contiguous pattern avoidance in binary trees
  - Algorithm to determine $\sum_{n\geq 1} a_t(n)x^n$ for any binary tree pattern.
  - $\sum_{n\geq 1} a_t(n)x^n$ is always algebraic.

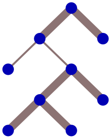
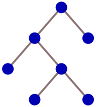
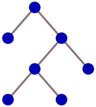
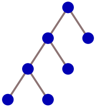
- 2010 (Gabriel, Peske, P., Tay): extended Rowland's results to ternary trees
  - Counting results from avoiding ternary tree patterns include Catalan numbers, little Schröder numbers, and other known combinatorial sequences.

Tree patterns

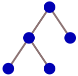
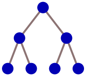
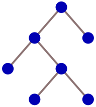
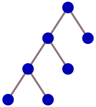
Noncontiguous tree pattern

Tree T contains tree t if and only if there exists a sequence of edge contractions of T (by pairs) that produces t.

Example:

 contains , , and .

Noncontiguous pattern enumeration data

Pattern t	Number of n-leaf trees avoiding t
	0
	$\begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$
	1
  	2^{n-2}

The Main Theorem

Notation

- Let $\mathrm{av}_t(n)$ be the number trees in $\mathbb{T}_n$ that avoid t noncontiguously.
- Let $g_t(x) = \sum_{n=1}^{\infty} \mathrm{av}_t(n) x^n$.

Theorem (Dairyko, P., Tyner, & Wynn, 2011)

Fix $k \in \mathbb{Z}^+$. Let $t, s \in \mathbb{T}_k$. Then $g_t(x) = g_s(x)$.

Generating functions

k	$g_t(x),\ t \in \mathbb{T}_k$	OEIS number
1	0	trivial
2	x	trivial
3	$\frac{x}{1-x}$	trivial
4	$\frac{x-x^2}{1-2x}$	A000079
5	$\frac{x-2x^2}{1-3x+x^2}$	A001519
6	$\frac{x-3x^2+x^3}{1-4x+3x^2}$	A007051
7	$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$	A080937
8	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$	A024175

Coefficient sightings...

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

Coefficient sightings...

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

Coefficient sightings...

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

$$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$$

An explicit formula

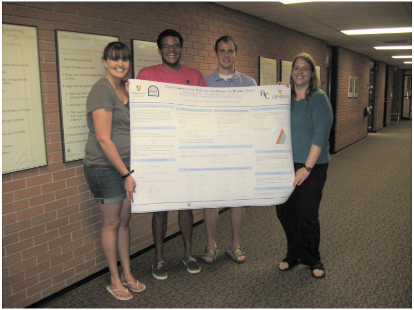
Theorem (Dairyko, P., Tyner, & Wynn, 2011)

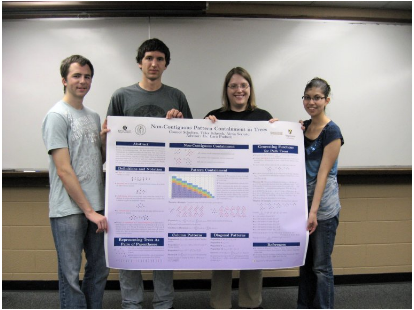
Let $k \in \mathbb{Z}^+$ and let $t \in \mathbb{T}_k$. Then

$$g_t(x) = \frac{\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot \binom{k-(i+2)}{i} \cdot x^{i+1}}{\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot \binom{k-(i+1)}{i} \cdot x^i}.$$

Noncontiguous tree pattern results

- 2011 (Dairyko, P., Tyner, Wynn): **noncontiguous** pattern avoidance in binary trees
- 2012 (P., Serrato, Scholten, Schrock): noncontiguous pattern **containment** in *m*-ary trees





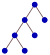
Contiguous Containment

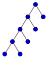
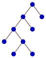
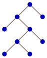
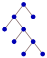
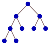
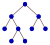
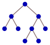
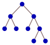
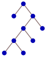
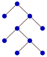
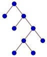
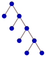
Theorem (Flajolet & Steyaert, 1983)

The number of copies of a k-leaf tree in the set of all n-leaf trees is independent of the tree pattern and is $\binom{2n-k}{n-k}$.

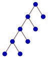
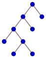
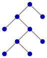
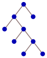
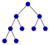
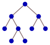
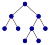
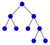
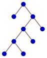
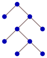
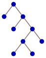
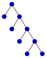
Example: Any 4-leaf tree is contained in the set of 5-leaf trees $\binom{2\cdot 5-4}{5-4} = \binom{6}{1} = 6$ times.

Contiguous Containment Example

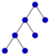
6 copies of  in 5-leaf trees

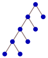
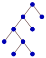
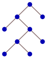
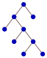
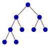
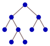
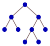
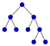
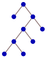
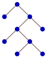
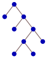
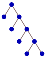
 2	 1	 1	 0	 0	 1	 0
 0	 0	 1	 0	 0	 0	 0

6 copies of  in 5-leaf trees

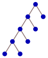
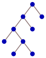
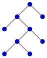
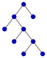
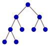
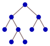
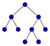
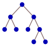
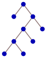
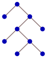
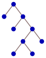
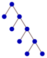
 0	 0	 1	 0	 0	 1	 1
 1	 1	 0	 0	 1	 0	 0

Noncontiguous Containment Example

10 copies of  in 5-leaf trees

						
4	2	1	1	0	1	0
						
0	0	1	0	0	0	0

10 copies of  in 5-leaf trees

						
0	0	1	0	0	2	2
						
2	2	0	0	1	0	0

Noncontiguous Containment

Theorem (P., Scholten, Schrock, Serrato, 2012)

If $\mathrm{occ}_k(n)$ is the number of noncontiguous occurrences of $t \in \mathbb{T}_k$ in $\mathbb{T}_n$, then $\mathrm{occ}_k(n)$ is independent of t and

$$\sum_{n\geq 1}\sum_{k\geq 1} \mathrm{occ}_k(n) x^n y^k = \frac{\sqrt{1-4x}(1-\sqrt{1-4x})y}{(y+2)\sqrt{1-4x}-y}.$$

Expansion:

$$\frac{\sqrt{1-4x}(1-\sqrt{1-4x})y}{(y+2)\sqrt{1-4x}-y} = xy + x^2(y+y^2) + x^3(2y+4y^2+y^3)+$$

$$x^4(5y+15y^2+7y^3+y^4) + x^5(14y+56y^2+37y^3+\mathbf{10}y^4+y^5)+\cdots$$

Recap

- Rowland: How many *binary* trees *avoid* a given *contiguous* tree pattern?
- REU 2010: How many *ternary* trees *avoid* a given *contiguous* tree pattern?
- REU 2011: How many *binary* trees *avoid* a given *noncontiguous* tree pattern?
- REU 2012: How many *m-ary* trees *contain* a given *noncontiguous* tree pattern?

Where next?

- non-ordered trees
- non-rooted trees
- non-full trees
- semi-contiguous tree patterns
(some parts must be contiguous; other parts not)

For more details...

- M. Dairyko, L. Pudwell, S. Tyner, and C. Wynn, Non-contiguous pattern avoidance in binary trees, *Electron. J. Combin.* **19** (3) (2012), P22.
- P. Flajolet and J. M. Steyaert, Patterns and pattern-matching in trees: an analysis. *Info. Control* **58** (1983), 19–58.
- N. Gabriel, K. Peske, L. Pudwell, and S. Tay, Pattern avoidance in ternary trees, *J. Integer Seq.* **15** (2012), 12.1.5.
- L. Pudwell, C. Scholten, T. Schrock, and A. Serrato, Non-contiguous pattern containment in binary trees, *ISRN Combinatorics* vol. 2014, Article ID 316535, 8 pages, 2014.
- E. S. Rowland, Pattern avoidance in binary trees, *J. Combin. Theory, Ser. A* **117** (2010), 741–758.

- slides at **faculty.valpo.edu/lpudwell**

For more details...

- M. Dairyko, L. Pudwell, S. Tyner, and C. Wynn, Non-contiguous pattern avoidance in binary trees, *Electron. J. Combin.* **19** (3) (2012), P22.
- P. Flajolet and J. M. Steyaert, Patterns and pattern-matching in trees: an analysis. *Info. Control* **58** (1983), 19–58.
- N. Gabriel, K. Peske, L. Pudwell, and S. Tay, Pattern avoidance in ternary trees, *J. Integer Seq.* **15** (2012), 12.1.5.
- L. Pudwell, C. Scholten, T. Schrock, and A. Serrato, Non-contiguous pattern containment in binary trees, *ISRN Combinatorics* vol. 2014, Article ID 316535, 8 pages, 2014.
- E. S. Rowland, Pattern avoidance in binary trees, *J. Combin. Theory, Ser. A* **117** (2010), 741–758.

- slides at **faculty.valpo.edu/lpudwell**

Thanks for listening!