Ascent sequences avoiding 0021

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0021-avoiders

Goal: Prove

Theorem

$$a_{0021}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

The plan:

- What is $a_{0021}(n)$?
- Why is $a_{0021}(n)$ interesting?
- Proof scribble:
  generating tree $\rightarrow$ recurrence $\rightarrow$ system of functional equations $\rightarrow$ experimental solution $\rightarrow$ plug in for catalytic variables

Ascent Sequences

Definition

An ascent sequence is a string $x_1 \cdots x_n$ of non-negative integers such that:

- $x_1 = 0$
- $x_n \leq 1 + \mathrm{asc}(x_1 \cdots x_{n-1})$ for $n \geq 2$

$\mathcal{A}_n$ is the set of ascent sequences of length n

$$\mathcal{A}_2 = \{00, 01\}$$
$$\mathcal{A}_3 = \{000, 001, 010, 011, 012\}$$

More examples: 01234, 01013

Non-example: 01024

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Theorem

$|\mathcal{A}_n|$ is the nth Fishburn number (OEIS A022493).

$$\sum_{n \geq 0} |\mathcal{A}_n| x^n = \sum_{n \geq 0} \prod_{i=1}^{n} (1 - (1-x)^i)$$

The pattern 0021

Pattern containment/avoidance

$a = a_1 \cdots a_n$ contains 0021 iff there exist $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ such that $a_{i_1} = a_{i_2} < a_{i_4} < a_{i_3}$.

$a_{0021}(n) = |\{a \in \mathcal{A}_n \mid a \text{ avoids } 0021\}|$

001010324 contains 0021.
001010345 avoids 0021.

Wilf Equivalence

Definition

Patterns σ and ρ are Wilf-equivalent if $a_\sigma(n) = a_\rho(n)$ for $n \geq 1$. In this case, write: $\sigma \sim \rho$.

Example: $00 \sim 01$.
$a_{00}(n) = 1$ (the strictly increasing sequence)
$a_{01}(n) = 1$ (the all zeros sequence).

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Example: $00 \sim 01$.
$a_{00}(n) = 1$ (the strictly increasing sequence)
$a_{01}(n) = 1$ (the all zeros sequence).

Known from Duncan/Steingrímsson: All possible Wilf equivalences of length at most 4 are:

$$00 \sim 01$$
$$10 \sim 001 \sim 010 \sim 011 \sim 012$$
$$102 \sim 0102 \sim 0112$$
$$101 \sim 021 \sim 0101 \sim 0012$$
$$0021 \sim 1012$$

(Duncan & Steingrímsson / Mansour & Shattuck / P.)

Avoiding 0021

Conjecture (Duncan & Steingrímsson)

$$a_{0021}(n) = a_{1012}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Theorem (Mansour & Shattuck)

$$a_{1012}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Avoiding 0021

Conjecture (Duncan & Steingrímsson)

$$a_{0021}(n) = a_{1012}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Theorem (Mansour & Shattuck)

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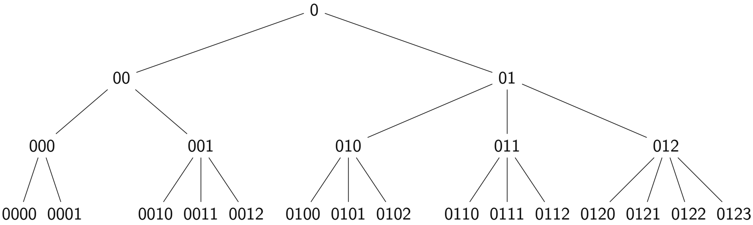
Theorem (P.)

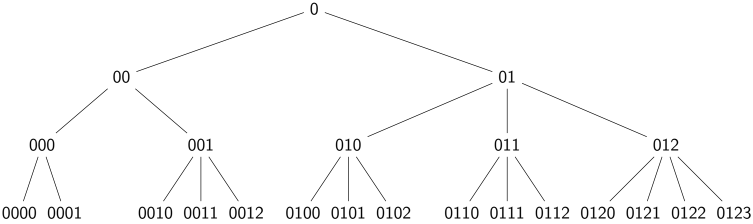
$$a_{0021}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Proof scribble:
generating tree $\to$ recurrence $\to$ system of functional equations $\to$ experimental solution $\to$ plug in for catalytic variables

Generating Tree



Generating Tree



- S_a is set of possible integers that may be appended to a to form a member of $\mathcal{A}_{0021}(n+1)$
- $S_a^i = \{x \in S_a \mid x > (\text{smallest repeated digit in } a)\}$
- $S_a^u = S_a \setminus S_a^i$

Example: $S_{0121235} = \{0, 1, 5, 6\}$, $S_a^u = \{0, 1\}$, and $S_a^i = \{5, 6\}$.

Idea: Replace a with (a_n, S_a^u, S_a^i).

Generating Tree

- 0 (0,{0,1},{})
  - 00 (0,{0},{1})
    - 000 (0,{0},{1})
    - 001 (1,{0},{1,2})
  - 01 (1,{0,1,2},{})
    - 010 (0,{0},{1,2})
    - 011 (1,{0,1},{2})
    - 012 (2,{0,1,2,3},{})

Generating Tree

- 0 (0,{0,1},{})
  - 00 (0,{0},{1})
    - 000 (0,{0},{1})
    - 001 (1,{0},{1,2})
  - 01 (1,{0,1,2},{})
    - 010 (0,{0},{1,2})
    - 011 (1,{0,1},{2})
    - 012 (2,{0,1,2,3},{})

Idea: Replace (a_n, S_a^u, S_a^i) with $(p, |S_a^u|, |S_a^i|)$ where a_n is the $(p-1)$st smallest digit in (a_n, S_a^u, S_a^i).

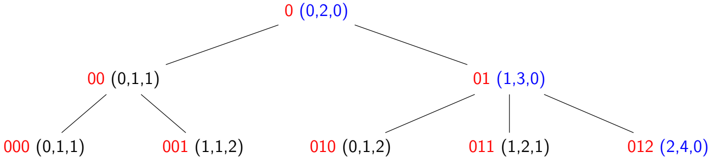
Example: $S_{0121235} = \{0, 1, 5, 6\}$, $S_a^u = \{0, 1\}$, and $S_a^i = \{5, 6\}$, so

$(a_n, S_a^u, S_a^i) = (5, \{0, 1\}, \{5, 6\})$, and $(p, |S_a^u|, |S_a^i|) = (2, 2, 2)$.

Generating Tree

- 0 (0,2,0)
  - 00 (0,1,1)
    - 000 (0,1,1)
    - 001 (1,1,2)
  - 01 (1,3,0)
    - 010 (0,1,2)
    - 011 (1,2,1)
    - 012 (2,4,0)

Generating Tree



- **root:** $(0, 2, 0)$
- **rules:**
  $(q-2, q, 0) \to (q-1, q+1, 0), (i, i+1, q-1-i)_{i=0}^{q-2}$

Generating Tree

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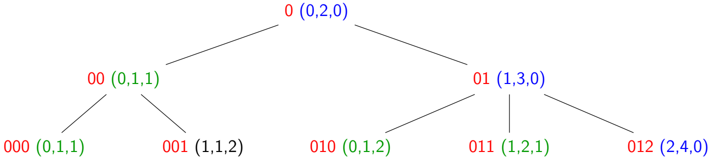
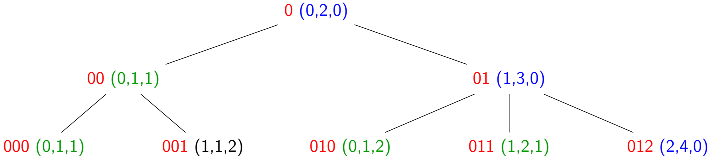


- **root:** $(0, 2, 0)$
- **rules:**

  $$(q-2, q, 0) \to (q-1, q+1, 0), (i, i+1, q-1-i)_{i=0}^{q-2}$$

  $$(q-1, q, r) \to$$
  $$(q-1, q, r), (i, i+1, q+r-1-i)_{i=1}^{q-2}, (q, q, i)_{i=2}^{r+1}$$

Generating Tree



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  $$(q, q, r) \to$$
  $$(q, q, r), (i, i+1, q+r-1-i)_{i=0}^{q-1}, (q, q, i)_{i=2}^{r}$$

Counting Nodes

- **root:** $(0, 2, 0)$
- **rules:**
  $(q-2, q, 0) \to (q-1, q+1, 0), (i, i+1, q-1-i)_{i=0}^{q-2}$

  $(q-1, q, r) \to$
  $(q-1, q, r), (i, i+1, q+r-1-i)_{i=1}^{q-2}, (q, q, i)_{i=2}^{r+1}$

  $(q, q, r) \to$
  $(q, q, r), (i, i+1, q+r-1-i)_{i=0}^{q-1}, (q, q, i)_{i=2}^{r}$

Note: One blue node per level of tree.

Need to look at green and black nodes more closely.

Tree rules to functional equations

Define:

- $g0_{n,q,r}$ is number of (q, q, r) nodes at level n.
- $B(x, y, z) = \sum_{n\geq 3}\sum_{q\geq 1}\sum_{r\geq 2} g0_{n,q,r}x^q y^r z^n$.
- $g1_{n,q,r}$ is number of $(q-1, q, r)$ nodes at level n.
- $G(x, y, z) = \sum_{n\geq 2}\sum_{q\geq 1}\sum_{r\geq 1} g1_{n,q,r}x^q y^r z^n$.

From the rules, we obtain:

$$\begin{aligned}
B(x,y,z) =& \frac{z(1-2y)}{1-y}B(x,y,z) + \frac{zy^2}{1-y}B(x,1,z) + \frac{xy^2z^2}{(1-z)(1-yz)(1-xz)} \\
&- \frac{zy^2}{1-y}\left(G(x,y,z) - \frac{xyz^2}{(1-xz)(1-yz)}\right) \\
&+ \frac{zy^2}{1-y}\left(G(x,1,z) - \frac{xz^2}{(1-xz)(1-z)}\right)
\end{aligned}$$

$$\begin{aligned}
G(x,y,z) =& \frac{xyz^2}{(1-xz)(1-yz)} + \frac{zx}{x-y}G(x,y,z) \\
&- \frac{zx}{x-y}G(y,y,z) + \frac{zx}{x-y}B(x,y,z) - \frac{zx}{x-y}B(y,y,z)
\end{aligned}$$

Black node data

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$A0_n$ is an $(n-2)\times(n-2)$ array with $g0_{n,q,r}$ in row q, column $r-1$.

$$A0_3 = \begin{bmatrix} 1 \end{bmatrix} \quad A0_4 = \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \quad A0_5 = \begin{bmatrix} 14 & 6 & 1 \\ 4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A0_6 = \begin{bmatrix} 50 & 27 & 8 & 1 \\ 14 & 6 & 1 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A0_7 = \begin{bmatrix} 187 & 113 & 44 & 10 & 1 \\ 50 & 27 & 8 & 1 & 0 \\ 14 & 6 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A0_8 = \begin{bmatrix} 730 & 468 & 212 & 65 & 12 & 1 \\ 187 & 113 & 44 & 10 & 1 & 0 \\ 50 & 27 & 8 & 1 & 0 & 0 \\ 14 & 6 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$A0_8 = \begin{bmatrix} 730 & 468 & 212 & 65 & 12 & 1 \\ 187 & 113 & 44 & 10 & 1 & 0 \\ 50 & 27 & 8 & 1 & 0 & 0 \\ 14 & 6 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Let $f(z) = \frac{1-z-\sqrt{1-6z+5z^2}}{2z}$,
- Let $g(z) = \frac{16z^2(z-1)}{\left(1-z+\sqrt{1-6z+5z^2}\right)^3\left(-1+3z+\sqrt{1-6z+5z^2}\right)}$.
- Experimentally predict:
  Column i has generating function $\frac{f(z)-1}{1-z}g(z)^{i-1}$.
  $B(x,y,z) = \frac{2xy^2z^3}{(1-xz)((1-(y+1)z)\sqrt{5z^2-6z+1}+(1-(y+3)z)(1-z))}$

Green node data

$A1_n$ is an $(n-1)\times(n-1)$ array with $g1_{n,q,r}$ in row q column r.

$$A1_2 = \begin{bmatrix} 1 \end{bmatrix} \quad A1_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad A1_4 = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A1_5 = \begin{bmatrix} 1 & 8 & 5 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad A1_6 = \begin{bmatrix} 1 & 23 & 19 & 7 & 1 \\ 1 & 8 & 5 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A1_7 = \begin{bmatrix} 1 & 74 & 69 & 34 & 9 & 1 \\ 1 & 23 & 19 & 7 & 1 & 0 \\ 1 & 8 & 5 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad A1_8 = \begin{bmatrix} 1 & 262 & 256 & 147 & 53 & 11 & 1 \\ 1 & 74 & 69 & 34 & 9 & 1 & 0 \\ 1 & 23 & 19 & 7 & 1 & 0 & 0 \\ 1 & 8 & 5 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Green node data

$A1_n$ is an $(n-1) \times (n-1)$ array with $g1_{n,q,r}$ in row q column r.

$$A1_8 = \begin{bmatrix} 1 & 262 & 256 & 147 & 53 & 11 & 1 \\ 1 & 74 & 69 & 34 & 9 & 1 & 0 \\ 1 & 23 & 19 & 7 & 1 & 0 & 0 \\ 1 & 8 & 5 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A091698 Matrix inverse of triangle A063967. 4

1, -1, 1, 1, -3, 1, -1, 8, -5, 1, 1, -23, 19, -7, 1, -1, 74, -69, 34, -9, 1, 1, -262, 256, -147, 53, -11, 1, -1, 993, -986, 615, -265, 76, -13, 1, 1, -3943, 3935, -2571, 1235, -431, 103, -15, 1, -1, 16178, -16169, 10862, -5591, 2216, -653, 134, -17, 1, 1 (list; table; graph; refs; listen; history; text; internal format)

OFFSET 0,5

COMMENTS Riordan array (1/(1+x), (sqrt(1+6x+5x^2)-x-1)/(2(1+x))). The absolute value array is (1/(1-x),xc(x)/(1-xc(x))) where c(x) is the g.f. of A000108. It factorizes as (1/(1-x),x/(1-x))(1,xc(x)). - Paul Barry, Jun 10 2005

LINKS Table of n, a(n) for n=0..55.

EXAMPLE 1; -1,1; 1,-3,1; -1,8,-5,1; 1,-23,19,-7,1; ...
Contribution from Paul Barry, Apr 15 2010: (Start)
Triangle begins
1,
-1, 1,
1, -3, 1,
-1, 8, -5, 1,
1, -23, 19, -7, 1,
-1, 74, -69, 34, -9, 1,
1, -262, 256, -147, 53, -11, 1,
-1, 993, -986, 615, -265, 76, -13, 1,
1, -3943, 3935, -2571, 1235, -431, 103, -15, 1

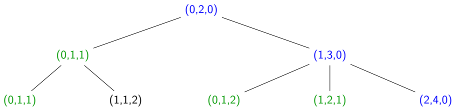
$$G(x,y,z) = \frac{2xyz^2}{(1-xz)(y\sqrt{5z^2-6z+1}+yz-2z-y+2)}$$

Recap

Goal

$$a_{0021}(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} C_k$$

Technique: Count blue, black, and green nodes in the following generating tree:



governed by:

- **root:** $(0, 2, 0)$
- **rules:**
  $(q-2, q, 0) \rightarrow (q-1, q+1, 0), (i, i+1, q-1-i)_{i=0}^{q-2}$

  $(q-1, q, r) \rightarrow (q-1, q, r), (i, i+1, q+r-1-i)_{i=1}^{q-2}, (q, q, i)_{i=2}^{r+1}$

  $(q, q, r) \rightarrow (q, q, r), (i, i+1, q+r-1-i)_{i=0}^{q-1}, (q, q, i)_{i=2}^{r}$

Final Enumeration

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- blue nodes on level n

  Generating function: $\frac{z}{1-z}$

- black nodes of type (q, q, r) on level n

  Generating function:
  $$B(x,y,z) = \frac{2xy^2z^3}{(1-xz)((1-(y+1)z)\sqrt{5z^2-6z+1}+(1-(y+3)z)(1-z))}$$

- green nodes of type $(q-1, q, r)$ on level n

  Generating function:
  $$G(x,y,z) = \frac{2xyz^2}{(1-xz)(y\sqrt{5z^2-6z+1}+yz-2z-y+2)}$$

- total nodes at level n:

  $$\frac{z}{1-z} + B(1,1,z) + G(1,1,z) = \frac{1-z-\sqrt{5z^2-6z+1}}{2(1-z)}.$$

Thanks for listening!