# Pattern-avoiding forests

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joint work with
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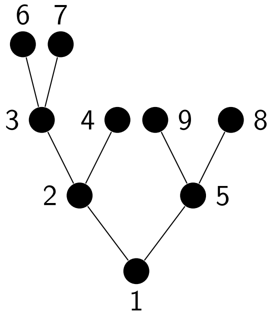
# Binary Heap

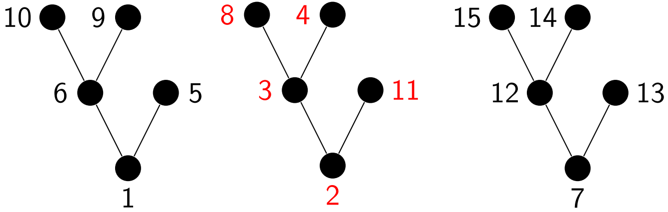


$h =$ 


$\pi_h = 125349867$

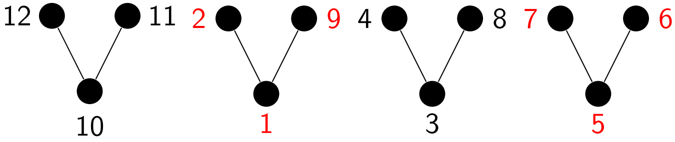
# Heap Forest

$f =$




$$\pi_f = 1\ 6\ 5\ 10\ 9\ \color{red}{2\ 3\ 11\ 8\ 4}\ 7\ 12\ 13\ 15\ 14$$

# Binary Shrub Forest

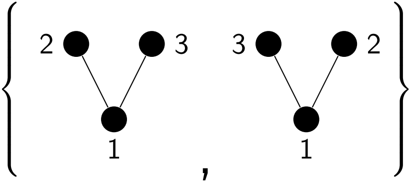
$f =$ 


$\pi_f = 10\ 12\ 11\ 1\ 2\ 9\ 3\ 4\ 8\ 5\ 7\ 6$

# Notation

- $\mathcal{F}_n$, the set of all binary shrub forests of $n$ heaps.

  e.g. $\mathcal{F}_1 =$




- $\mathcal{S}_n^* = \{\pi \in \mathcal{S}_{3n} \mid \pi = \pi_f \text{ for some } f \in \mathcal{F}_n\}$.
  e.g. $\mathcal{S}_1^* = \{123, 132\}$

Note:

$$|\mathcal{S}_n^*| = \frac{(3n)!}{3^n}.$$

# More Notation



- $\mathcal{S}_n^*(\rho) = \{\pi \in \mathcal{S}_n^* \mid \pi \text{ avoids } \rho\}$.

**Goal**

Determine $|\mathcal{S}_n^*(\rho)|$ for $\rho \in \mathcal{S}_3$.

Note:

$$\mathcal{S}_n^*(\rho) \subseteq \mathcal{S}_{3n}(\rho).$$

So, for $\rho \in \mathcal{S}_3$,

$$|\mathcal{S}_n^*(\rho)| < C_{3n}.$$

# Overview

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| $\rho$ | $\lvert\mathcal{S}_n^*(\rho)\rvert$, $n \geq 1$ | | | | | OEIS |
|---|---|---|---|---|---|---|
| 123 | 1, | 3, | 12, | 55, | 273, . . . | A001764 |
| 132 | 1, | 4, | 22, | 140, | 969, . . . | A002293 |
| 213 | 2, | 14, | 134, | 1482, | 17818, . . . | A144097 |
| 312 | 2, | 14, | 134, | 1482, | 17818, . . . | A144097 |
| 231 | 2, | 23, | 377, | 7229, | 151491, . . . | A060941 |
| 321 | 2, | 37, | 866, | 23285, | 679606, . . . | new |

Upper bounds:

| | Terms | | | | OEIS |
|---|---|---|---|---|---|
| $C_{3n}$ | 5, | 132, | 4862, | 208012, . . . | A187357 |
| $\frac{(3n)!}{3^n}$ | 2, | 80, | 13440, | 5913600, . . . | A210277 |

# Avoiding 123

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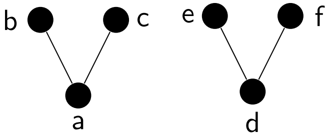
**Theorem**

$$|\mathcal{S}_n^*(123)| = \frac{\binom{3n}{n}}{2n+1}. \qquad (A001764)$$

e.g. $|\mathcal{S}_2^*(123)| = \frac{\binom{6}{2}}{2 \cdot 2 + 1} = \frac{15}{5} = 3.$




Know: $a > d, \quad b > c > e > f$

# Avoiding 123



**Theorem**

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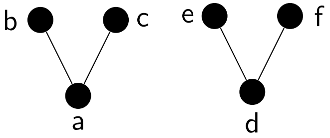



Know: $a > d, \quad b > c > e > f$

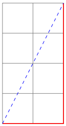
$\mathcal{S}_2^*(123) = \{\underline{265}\ \underline{143},\ \underline{365}\ \underline{142},\ \underline{465}\ \underline{132}\}$

# Avoiding 123

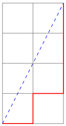
**Theorem**

$$|\mathcal{S}_n^*(123)| = \frac{\binom{3n}{n}}{2n+1}. \qquad \text{(A001764)}$$

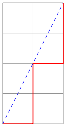
- Choose roots, then put all other vertices in decreasing order.
- In bijection with NE paths from $(0,0)$ to $(n,2n)$ weakly below $y=2x$.



EENNNN
265143



ENENNN
365142



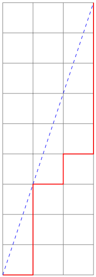
ENNENN
465132

# Avoiding 132

**Theorem**

$$|\mathcal{S}_n^*(132)| = \frac{\binom{4n}{n}}{3n+1}. \qquad \text{(A002293)}$$

- In bijection with NE paths from $(0,0)$ to $(n,3n)$ weakly below $y=3x$.



$\rightarrow 034 \rightarrow 145 \rightarrow 541 \rightarrow 567\ 489\ 123$

# Avoiding 132

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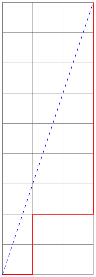
**Theorem**

$$|\mathcal{S}_n^*(132)| = \frac{\binom{4n}{n}}{3n+1}. \qquad \text{(A002293)}$$

- In bijection with NE paths from $(0,0)$ to $(n,3n)$ weakly below $y=3x$.



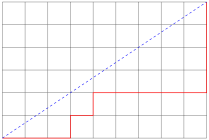
$\rightarrow 022 \rightarrow 133 \rightarrow 331 \rightarrow 345\ 678\ 129$

# Avoiding 231

**Theorem**

$|\mathcal{S}_n^*(231)| = \text{A060941(n)}$.

- In bijection with NE paths from $(0,0)$ to $(3n,2n)$ weakly below $y=\frac{2}{3}x$.



$\rightarrow E^3N\ EN\ E^5N\ N\ N\ N$

Idea:

- Read $E^kN$ blocks from right to left.
- $k$ determines new digit to prepend to front of permutation.
- After every even index block prepend an extra 1.

# Avoiding 231

**Theorem**

$|\mathcal{S}_n^*(231)| =$ A060941(n).

- In bijection with NE paths from $(0,0)$ to $(3n, 2n)$ weakly below $y = \frac{2}{3}x$.

$$E^3N\ EN\ E^5N\ N\ N\ N$$

$$\pi = 1$$

# Avoiding 231



**Theorem**

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- In bijection with NE paths from $(0,0)$ to $(3n, 2n)$ weakly below $y = \frac{2}{3}x$.

$$E^3N\ EN\ E^5N\ N\ N\ N$$

1

$k = 0$, prepend 1.

1.2

even block, prepend 1.

1.2.3

# Avoiding 231

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$|\mathcal{S}_n^*(231)| = \text{A060941(n)}$.

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$$E^3N\ EN\ E^5N\ N\ N\ N$$

1.2.3

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1.2.3.4

# Avoiding 231

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**Theorem**

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$$E^3N\ EN\ E^5N\ N\ N\ N$$

1.2.3.4

$k = 0$, prepend 1.

1.2.3.4.5

even block, prepend 1.

1.2.3.4.5.6

# Avoiding 231



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$$E^3N\ EN\ E^5N\ N\ N\ N$$

1.2.3.4.5.6

$k = 5$, prepend max(5)+1=6 and merge first 5 parts.

612345.7

# Avoiding 231

**Theorem**

$|\mathcal{S}_n^*(231)| = $ A060941(n).

- In bijection with NE paths from $(0, 0)$ to $(3n, 2n)$ weakly below $y = \frac{2}{3}x$.

$$E^3N\ EN\ E^5N\ N\ N\ N$$

612345.7

$k = 1$, prepend max(6,1,2,3,4,5)+1=7.

7612345.8

even block, prepend 1.

1.8723456.9

# Avoiding 231

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$$E^3N\ EN\ E^5N\ N\ N\ N$$

1.8723456.9

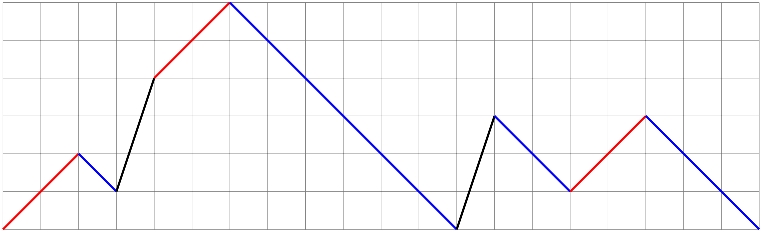
$$E^3N\ EN\ E^5N\ N\ N\ N \to \underline{1\ 8\ 7}\ \underline{2\ 3\ 4}\ \underline{5\ 6\ 9}$$

# Avoiding 213 or 312

**Theorem**

$|\mathcal{S}_n^*(213)| = |\mathcal{S}_n^*(312)| = A144097(n).$

- In bijection with paths from $(0,0)$ to $(4n,0)$ using the steps $(1,3)$, $(2,2)$ and $(1,-1)$.



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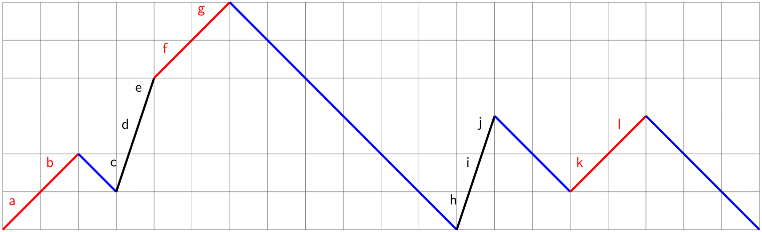

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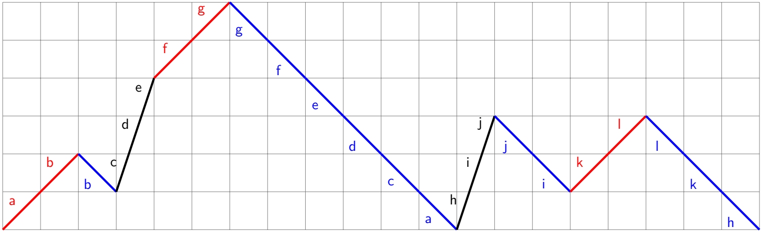

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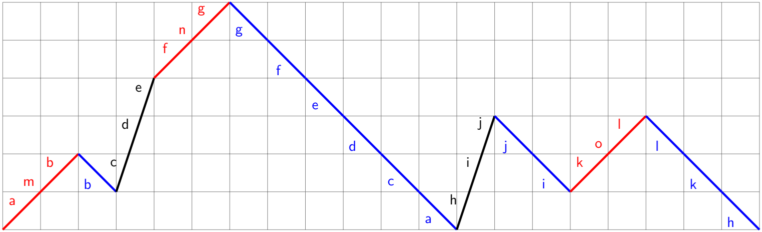

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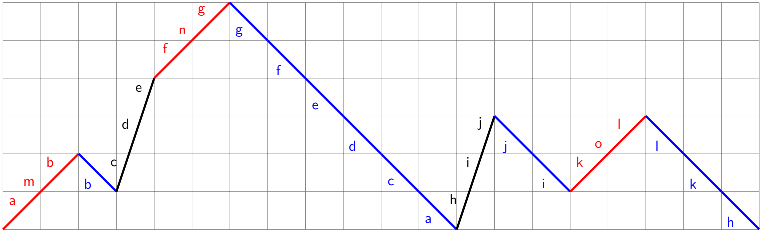


left to right word: amb cde fng hij kol

right to left word: hkl oij acd efg nbm

permutation: 7 15 14 8 9 10 11 13 12 1 5 6 2 4 3

# Avoiding 321

**Open**

$|\mathcal{S}_n^*(321)|$.

- Track $d(\pi)$, number of digits larger than the highest '1' in a 21 pattern.

  e.g. $d(123) = 3$, $d(132) = 1$, $d(145236) = 3$

# Avoiding 321

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## Open

$|\mathcal{S}_n^*(321)|$.

- Track $d(\pi)$, number of digits larger than the highest '1' in a 21 pattern.
  e.g. $d(123) = 3$, $d(132) = 1$, $d(145236) = 3$
- Leads to generating tree:
  Root: (0)
  Rules: $(0) \to (1)(3)$
  $(1) \to (1)^3(2)^2(3)^1(4)^1$
  $(i) \to (1)^{\binom{i+2}{2}}(2)^{\binom{i+2}{2}-1}(3)^{\binom{i+1}{2}}(4)^{\binom{i}{2}} \cdots (i+2)^{\binom{2}{2}}(i+3)^1$

# Avoiding 321

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## Open

$|\mathcal{S}_n^*(321)|$.

- Track $d(\pi)$, number of digits larger than the highest '1' in a 21 pattern.
  e.g. $d(123) = 3$, $d(132) = 1$, $d(145236) = 3$
- Production matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 3 & 2 & 1 & 1 & 0 & 0 & \dots \\ 0 & 6 & 5 & 3 & 1 & 1 & 0 & \dots \\ 0 & 10 & 9 & 6 & 3 & 1 & 1 & \dots \\ 0 & 15 & 14 & 10 & 6 & 3 & 1 & \dots \\ 0 & 21 & 20 & 15 & 10 & 6 & 3 & \dots \\ 0 & 28 & 27 & 21 & 15 & 10 & 6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

# Avoiding 321

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**Open**

$|\mathcal{S}_n^*(321)|$.

- Used generating tree to compute $|\mathcal{S}_n^*(321)|$ for $n \leq 250$.
- Naively know
  $7^n < |\mathcal{S}_n^*(321)| < |\mathcal{S}_{3n}(321)| = C_{3n} < 4^{3n} = 64^n$.
- Numerically predict $|\mathcal{S}_n^*(321)| \sim c^n$ where $39 < c < 40$.

# Summary & Future Work





- $\mathcal{S}_n^*(\rho)$ has nice connections to lattice paths for $\rho \in \{123, 132, 213, 231, 312\}$.
- $|\mathcal{S}_n^*(321)|$ is open.

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  - Other heap forests
  - Other pattern sets

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# Thanks for listening!

(slides at `faculty.valpo.edu/lpudwell`)