Pattern Avoidance on k-ary Heaps

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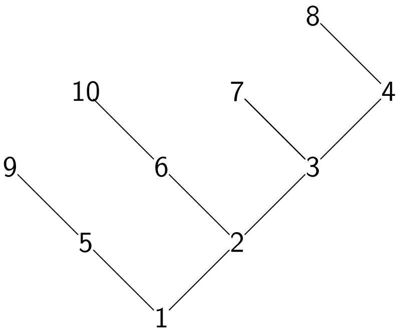
University of Wisconsin - Eau Claire, Valparaiso University

AMS Section meeting - Georgetown University - March 8, 2015

Motivation

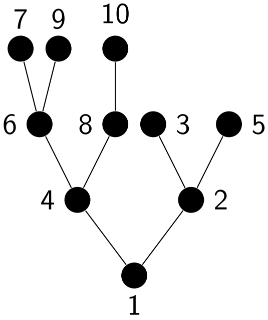
Sophia Yakoubov, PP2013, Pattern Avoidance on Combs



Something like combs, but not combs

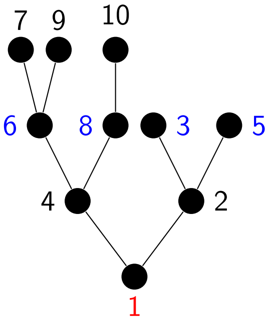
Definition

A *heap* is a complete k-ary tree labeled with $\{1, \ldots, n\}$ such that every child has a larger label than its parent.



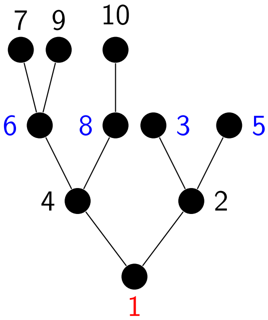
Where's the pattern?

$h =$



$$\pi_h = 1\ 4\ 2\ 6\ 8\ 3\ 5\ 7\ 9\ 10$$

Where's the pattern?

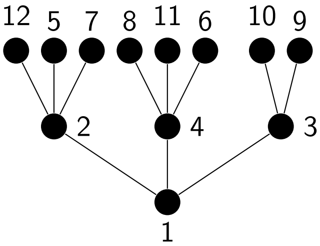
$h =$ 

$\pi_h = 1\ 4\ 2\ 6\ 8\ 3\ 5\ 7\ 9\ 10$

h avoids 321.

k-ary Heaps

$h =$



$$\pi_h = 1\ 2\ 4\ 3\ 12\ 5\ 7\ 8\ 11\ 6\ 10\ 9$$

h avoids 231.

Notation

$\mathcal{H}_n^k(P)$ is the set of k-ary heaps on n nodes avoiding P.

Goal

Determine $\left|\mathcal{H}_n^k(P)\right|$.

Start with $k = 2$, $P \subset \mathcal{S}_3$.

Crunch the numbers, cross your fingers

P	$\{\|\mathcal{H}_n^2(P)\|\}_{n\geq 1}$	OEIS#
$\emptyset$	$1, 1, 2, 3, 8, 20, 80, 210, 896, \ldots$	
$\{123\}$	$1, 1, 1, 0, 0, 0, 0, 0, \ldots$	
$\{132\}$	$1, 1, 1, 1, 1, 1, 1, 1, \ldots$	
$\{213\}$	$1, 1, 2, 2, 5, 5, 14, 14, 42, \ldots$	
$\{231\}$ $\{312\}$	$1, 1, 2, 3, 7, 14, 37, 80, 222, \ldots$	
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$\{213, 231\}$ $\{213, 312\}$	$1, 1, 2, 2, 4, 4, 8, 8, 16, \ldots$	
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$\{231, 312\}$ $\{231, 321\}$ $\{312, 321\}$	$1, 1, 2, 3, 6, 11, 22, 42, 84, \ldots$	

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$\emptyset$	$1, 1, 2, 3, 8, 20, 80, 210, 896, \ldots$	A056971
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$\{213, 231\}$ $\{213, 312\}$	$1, 1, 2, 2, 4, 4, 8, 8, 16, \ldots$	A016116
$\{213, 321\}$	$1, 1, 2, 2, 4, 4, 7, 7, 11, \ldots$	A000124($\lceil \frac{n}{2} \rceil$)
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The friendly cases

All heaps: $\left|\mathcal{H}_n^2\right| = \binom{n-1}{n_\ell} \left|\mathcal{H}_{n_\ell}^2\right| \left|\mathcal{H}_{n-1-n_\ell}^2\right|$
(n_ℓ = number of vertices left of root.)

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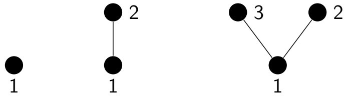
123-avoiders:



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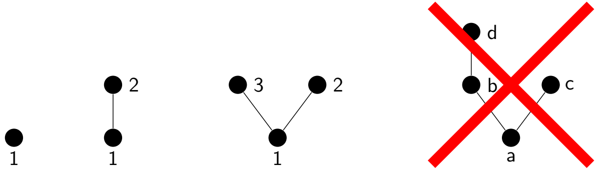

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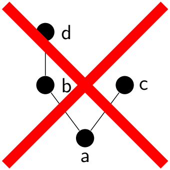
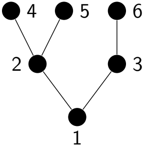


132-avoiders:



Onward...

P	$\{\lvert\mathcal{H}_n^2(P)\rvert\}_{n\geq 1}$	OEIS#
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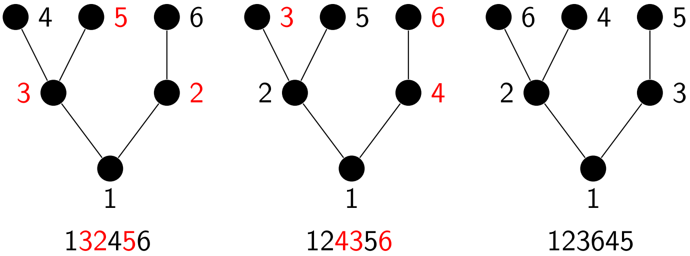
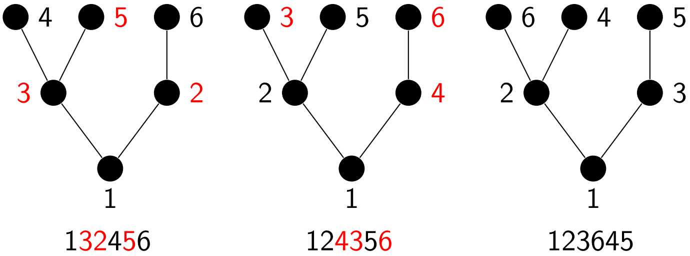
Heaps Avoiding 213



Heaps Avoiding 213



$$\left|\mathcal{H}_n^2(213)\right| = C_{\lceil \frac{n}{2} \rceil}$$

Progress...

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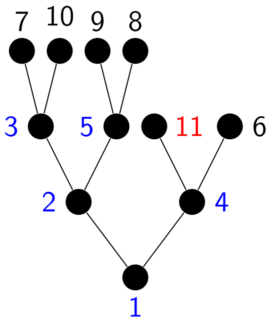
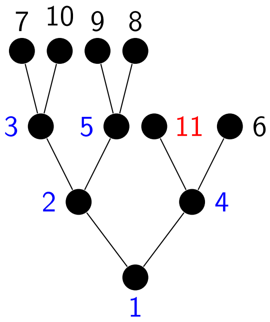
Heaps Avoiding 231



- *n* appears on a leaf.
- All labels before *n* are less than all labels after *n*.
- Labels before *n* are a heap avoiding 231.
- Labels after *n* are a permutation avoiding 231.

Heaps Avoiding 231



- n appears on a leaf.
- All labels before n are less than all labels after n.
- Labels before n are a heap avoiding 231.
- Labels after n are a permutation avoiding 231.

$$\left|\mathcal{H}_n^2(231)\right| = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} C_i \cdot \left|\mathcal{H}_{n-i-1}^2(231)\right|$$

Heaps Avoiding 321

n	$\lvert\mathcal{H}_n^2(321)\rvert$	n	$\lvert\mathcal{H}_n^2(321)\rvert$	n	$\lvert\mathcal{H}_n^2(321)\rvert$
1	1	11	2686	21	395303480
2	1	12	8033	22	1379160685
3	2	13	25470	23	4859274472
4	3	14	80480	24	17195407935
5	7	15	263977	25	61310096228
6	16	16	862865	26	219520467207
7	45	17	2891344	27	790749207801
8	111	18	9706757	28	2859542098634
9	318	19	33178076	29	10391610220375
10	881	20	113784968	30	37897965144166
				31	138779392289785

Heaps Avoiding 321

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For $n \geq 9$, $$2^{n-1} < \left\lvert\mathcal{H}_n^2(321)\right\rvert < 4^n.$$

Progress...

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Heaps Avoiding $\{231, 312\}$

Narayana-Zidek-Capell Numbers:
$1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, \ldots$

(Count types of compositions, trees, etc.)

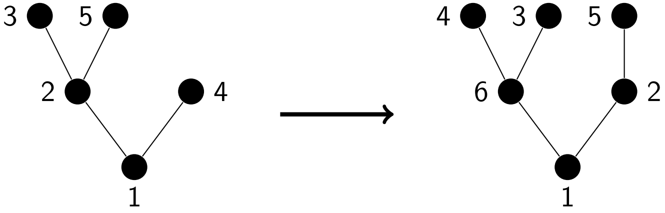
Given by:

- $a_1 = a_2 = 1$
- $a_{n+1} = \begin{cases} 2a_n & n \text{ even} \\ 2a_n - a_{\frac{n-1}{2}} & n \text{ odd} \end{cases}$

Heaps Avoiding $\{231, 312\}$

Narayana-Zidek-Capell Numbers:
$1, 1, 2, 3, 6, 11, 22, 42, 84, 165, 330, \ldots$

Insert $n + 1$, but maintain a heap.



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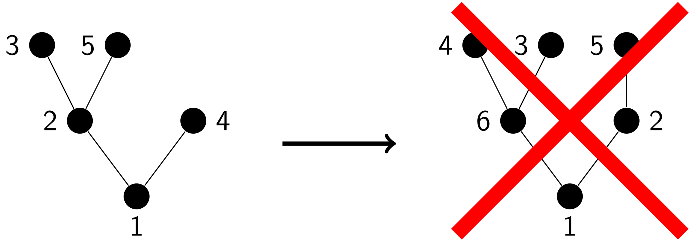


Heaps Avoiding $\{231, 312\}$

Observation

Before insertion, n is a leaf.
After insertion, $n+1$ is a leaf.

Lemma

In order to avoid 231 and 312, $n+1$ must be inserted immediately before n or at the end.

Heaps Avoiding $\{231, 312\}$: $n+1$ must be right before n, or at end

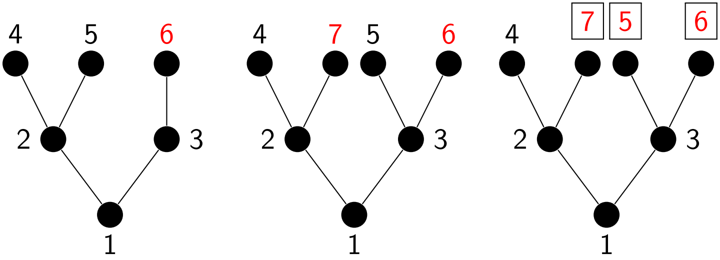
Proof of Lemma:

- $n+1$ at last leaf: OK
- $n+1$ right before n: OK

Heaps Avoiding $\{231, 312\}$: $n+1$ must be right before n, or at end

Proof of Lemma:

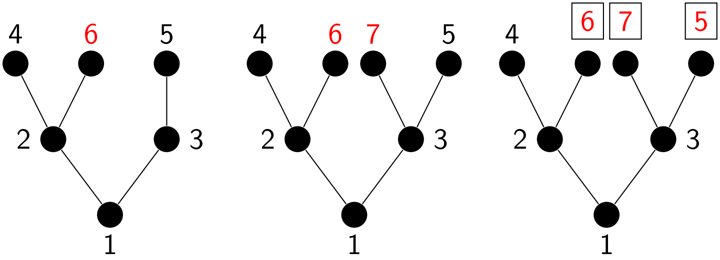
- If $n+1$ is inserted further before n, we create a 312.



Heaps Avoiding $\{231, 312\}$: $n+1$ must be right before n, or at end

Proof of Lemma:

- If $n+1$ is inserted after n, but not at the end, we create a 231.



Heaps Avoiding $\{231, 312\}$

Easy Case (n is even.):
The new leaf is the sibling of a current leaf.
Internal nodes stay internal, leaves stay leaves.

We can put $n+1$ at the (new) last leaf, or we can insert it right before n.

$$\left|\mathcal{H}_{n+1}^{2}(\{231, 312\})\right| = 2\left|\mathcal{H}_{n}^{2}(\{231, 312\})\right|.$$

Heaps Avoiding $\{231, 312\}$

Second Case (n is odd.):
The new leaf is child of a former leaf.

Inserting $n + 1$ at the last leaf: Still OK!

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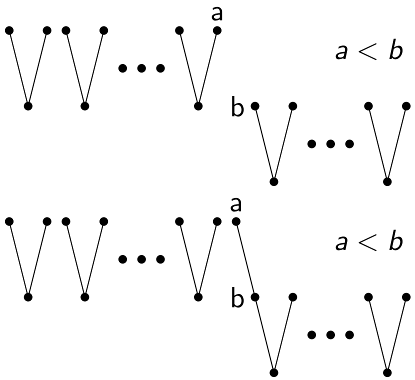
The new leaf is child of a former leaf.

Inserting $n+1$ at the last leaf: Still OK!

Inserting $n+1$ before n:
The last leaf a becomes a child of the first leaf b. What if $a < b$?

Heaps Avoiding $\{231, 312\}$

Inserting $n+1$ anywhere except the first or last leaf:



$a < b$

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Heaps Avoiding $\{231, 312\}$

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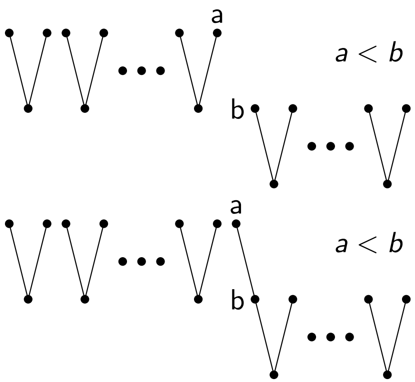
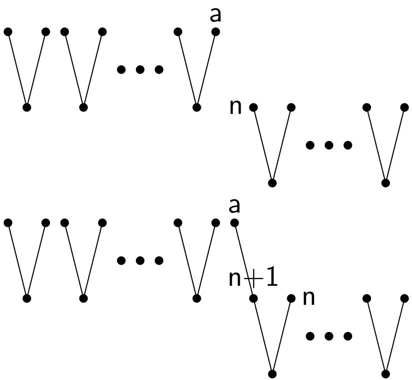


bna was already a copy of 231.

Heaps Avoiding $\{231, 312\}$

n was first leaf, insert $n+1$ as new first leaf:



Not a heap! Don't count this.

Heaps Avoiding $\{231, 312\}$

How many members of $\mathcal{H}_n^2(\{231, 312\})$ have n on the first leaf?

- Other leaves are the largest elements in decreasing order.
- The heap obtained by removing all leaves:
  - avoids 231 and 312.
  - has $\frac{n-1}{2}$ nodes.
- There are $\left|\mathcal{H}^2_{\frac{n-1}{2}}(\{231, 312\})\right|$ such heaps.

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$$\left|\mathcal{H}^2_{n+1}(\{231, 312\})\right| = 2\left|\mathcal{H}^2_n(\{231, 312\})\right| - \left|\mathcal{H}^2_{\frac{n-1}{2}}(\{231, 312\})\right|$$

when n is odd.

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What about k-ary heaps?

Patterns P	$\left\lvert\mathcal{H}_n^k(P)\right\rvert$ (where $\ell = \left\lceil \frac{(k-1)n-(k-2)}{k} \right\rceil$)
$\{213\}$	C_ℓ
$\{231\}$ $\{312\}$	$\begin{cases} 1 & n = 1 \\ \sum_{i=0}^{\ell-1} C_i \cdot \left\lvert\mathcal{H}_{n-i-1}^k(231)\right\rvert & n \geq 2 \end{cases}$
$\{321\}$	OPEN
$\{213, 231\}$ $\{213, 312\}$	$2^{\ell-1}$
$\{213, 321\}$	$\binom{\ell}{2} + 1$
$\{231, 312\}$ $\{231, 321\}$ $\{312, 321\}$	$\begin{cases} 1 & n \leq 2 \\ 2a_{n-1} & k \nmid n-2 \\ 2a_{n-1} - a_{\frac{n-2}{k}} & k \mid n-2. \end{cases}$

Ongoing work/ Thank you!

Ongoing work:

- Trees that aren't heaps (Unary-binary, binary, some k-ary)
- How many permutations avoiding σ can be realized as trees?
- Forests of heaps (Stay tuned for the next talk!)

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Thank you to:

- UWEC Department of Mathematics
- UWEC Office of Research and Sponsored Programs

paper to appear in *Australasian Journal of Combinatorics*
preprint and slides at `faculty.valpo.edu/lpudwell`