Avoiding the Pattern $\overline{31}542$

Lara Pudwell
Valparaiso University

Permutation Patterns 2009
July 16, 2009

Bar Notation

A *barred permutation pattern* is a permutation where each number may or may not have a bar over it.
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A barred pattern p encodes two permutation patterns,

1. The smaller pattern p_s formed by the unbarred letters of p.
(in this case, 542 forms a 321 pattern.)
2. The larger pattern p_ℓ formed by all letters of p.
(in this case, 31542.)

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We say that permutation π avoids the barred pattern p iff every copy of p_s in π is part of a copy of p_ℓ in π.

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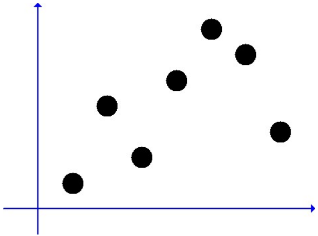

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Barred Patterns

Gaussian Polynomials and Partitions

The sequence A047970

The pattern $\overline{31}542$

Conclusion

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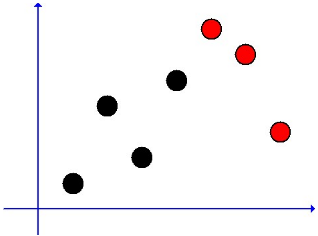

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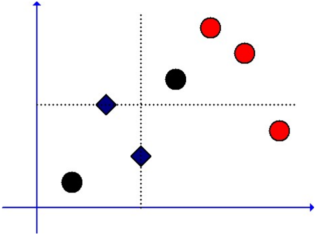

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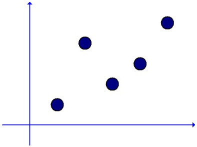

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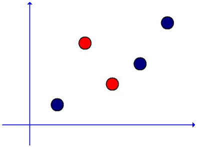

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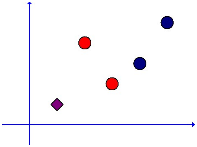

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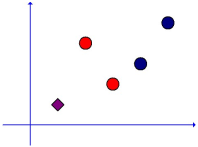

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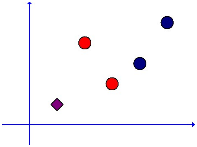

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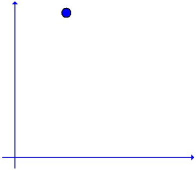

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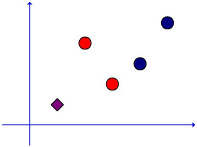

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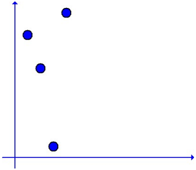

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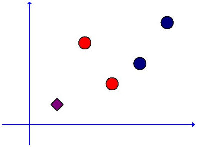

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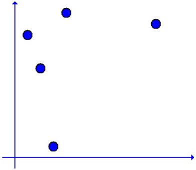

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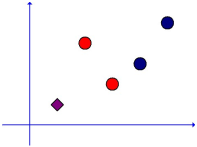

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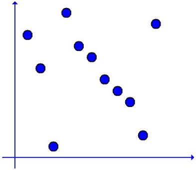

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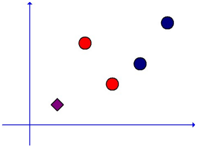

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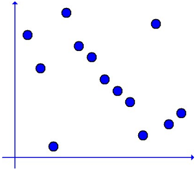

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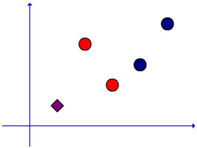

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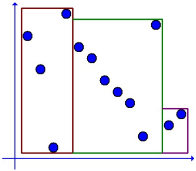

Barred Pattern Sightings

- West, 1990: A permutation is 2-stack sortable if and only if it avoids 2341 and $3\bar{5}241$.

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- Burstein and Lankham, 2006: A permutation is a reverse patience word if and only if it avoids $3{-}\bar{1}{-}42$.

Observations

Based on computation:

- Conjecture: If q is a barred pattern of length k with $k-2$ bars then either $S_n(q) = 1$ or $S_n(q) = (n-(k-2))!$.
- Conjecture: $S_n(\overline{31}542)$ gives the number of ordered factorizations over the Gaussian polynomials. (OEIS A047970)
- Conjecture: $S_n(\bar{1}43\bar{5}2)$ has generating function $\prod_{n\geq 0} \frac{1}{(1-\frac{x}{(1-x)^n})^{(1/2)^{n+1}}}$ (OEIS A122993).
- There are at least 19 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.

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A Little Help from OEIS

Search: **id:A047970**

Displaying 1-1 of 1 results found.

Format: long | short | internal | text   Sort: relevance | references | number   Highlight: on | off

A047970   Sums along antidiagonals of A047969.   +0 19

1, 2, 5, 14, 43, 144, 523, 2048, 8597, 38486, 182905, 919146, 4866871, 27068420, 157693007, 959873708, 6091057009, 40213034874, 275699950381, 1959625294310, 14418124498211, 109655727901592, 860946822538675, 6969830450679864 (list; graph; listen)

OFFSET	0,2
COMMENT	Number of ordered factorizations over the Gaussian polynomials. Apparently, also the number of permutations in S_n avoiding {bar 3}{bar 1}542 (i.e. every occurrence of 542 is contained in an occurrence of a 31542). - Lara Pudwell (Lara.Pudwell(AT)valpo.edu), Apr 25 2008
REFERENCES	See Andrews, Partitions, (1976) page 242 for table of Gaussian polynomials.
EXAMPLE	a(3)=1+5+7+1=14.
CROSSREFS	Partial sums are in A026898, A003101. Adjacent sequences: A047967 A047968 A047969 this_sequence A047971 A047972 A047973 Sequence in context: A137549 A014327 A137550 this_sequence A137551 A160701 A148333
KEYWORD	nonn
AUTHOR	Alford Arnold (Alford1940(AT)aol.com)

Gaussian Polynomials

Recall that the *q-analogue* of an integer n is the polynomial

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

From this, we can define the *q-factorial* as

$$[n]_q! = [n]_q \cdot [n-1]_q \cdots [1]_q.$$

and the *q-binomial* as

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q![m-n]_q!}.$$

These q-binomial coefficients turn out to be polynomials in the variable q, and are called *Gaussian Polynomials*.

Gaussian Polynomials and Partitions

The coefficient of q^k in $\begin{bmatrix} m \\ n \end{bmatrix}_q$ gives the number of partitions of k that fit inside an $n \times (m-n)$ box.
For example,

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4.$$

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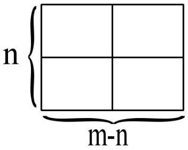

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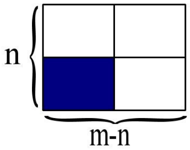

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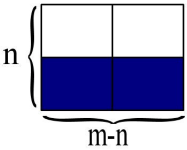

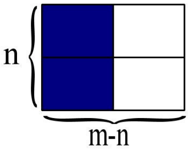

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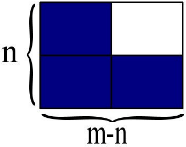

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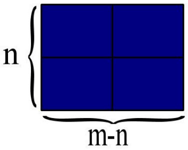

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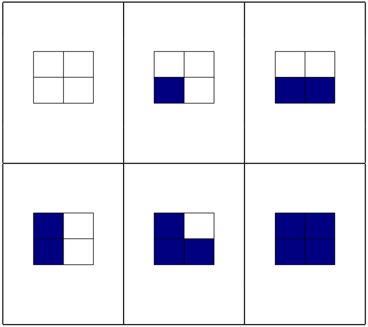
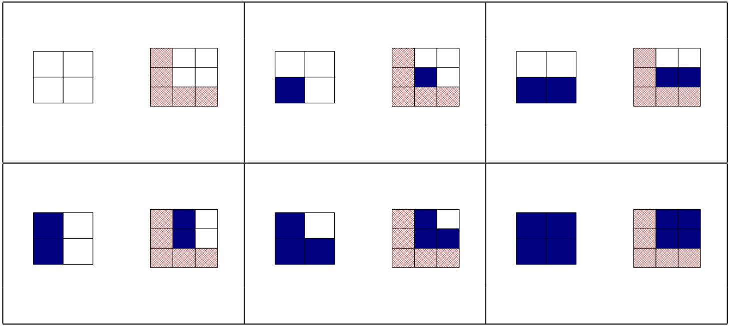


But what if we wanted to use the full dimensions of the box?

Gaussian Polynomials and Partitions

By adding a new bottom row and new leftmost column to each partition, the coefficient of q^k in $\begin{bmatrix} m \\ n \end{bmatrix}_q$ gives the number of partitions of $k + m + 1$ into $m - n + 1$ parts with largest part $n + 1$. We will call these *modified partitions*.



"Ordered Factorizations over the Gaussian Polynomials"

To generate A047970, Arnold considered the modified partitions corresponding to the Gaussian polynomials in the nth row of Pascal's triangle:

$$\begin{bmatrix} n-1 \\ 0 \end{bmatrix}_q, \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q, \dots, \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}_q.$$

There are 2^{n-1} such partitions.

How many *compositions* can be obtained from these partitions?

Arnold's construction for $n = 2$

Polynomials	Partitions	Compositions
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = 1$		
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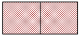
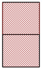
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = 1$		
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = 1$		

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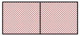
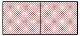
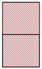
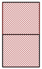
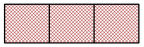
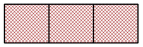
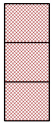
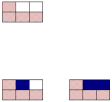
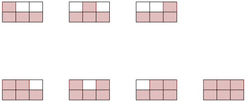
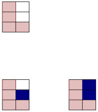
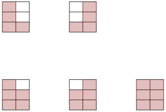
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_q = 1$		
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = 1$		
	2	2

Arnold's construction for $n = 3$

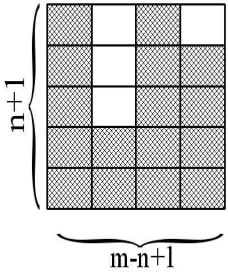
Polynomials	Partitions	Compositions
$\begin{bmatrix} 2 \\ 0 \end{bmatrix}_q = 1$		
$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1 + q$		
		
$\begin{bmatrix} 2 \\ 2 \end{bmatrix}_q = 1$		
	4	5

Arnold's construction for $n = 4$

Polynomials	Partitions	Compositions
$\begin{bmatrix} 3 \\ 0 \end{bmatrix}_q = 1$		
$\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q = 1 + q + q^2$		
$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q = 1 + q + q^2$		
$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_q = 1$		
	8	14

A formula for A047970

There is an easy bijection from the compositions corresponding to $\begin{bmatrix} m \\ n \end{bmatrix}_q$ in Arnold's construction and set partitions of $\{1, \ldots, m-n+1\}$ into $n+1$ ordered blocks.



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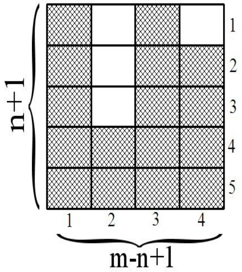

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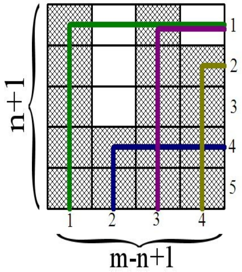


This composition corresponds to the set partition $\{1,3\}, \{4\}, \{\}, \{2\}, \{\}$.

A formula for A047970

The compositions corresponding to the Gaussian polynomial $\begin{bmatrix} m \\ n \end{bmatrix}_q$ are in bijection with the ways to partition $m-n+1$ elements into $n+1$ ordered blocks, where the first block must be non-empty.

There are $(n+1)^{m-n+1} - n^{m-n+1}$ such set partitions.

A formula for A047970

Now if we consider $\begin{bmatrix} m \\ i \end{bmatrix}_q$ for $0 \leq i \leq m$, there are

$$\sum_{i=0}^{m} (i+1)^{m-i+1} - i^{m-i+1}$$

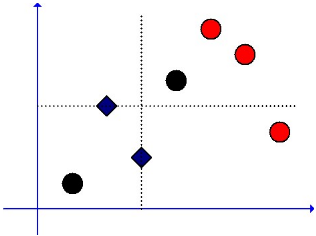
compositions corresponding to the mth row of Pascal's triangle.

Or, with suitable change of variables

$$\sum_{k=0}^{n} (n-k+1)^k - (n-k)^k.$$

Back to Permutation Patterns

Recall, we are interested in $S_n(\overline{31}542)$.



Back to Permutation Patterns

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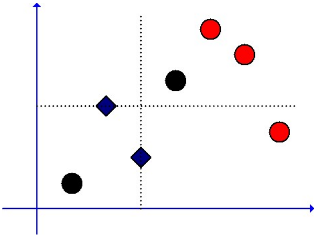


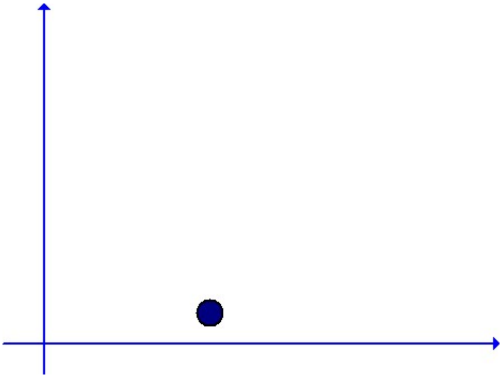
Using brute force computation, for $n \leq 15$ we have:

$$\left|S_n(\overline{31}542)\right| = \sum_{k=0}^{n} (n-k+1)^k - (n-k)^k.$$

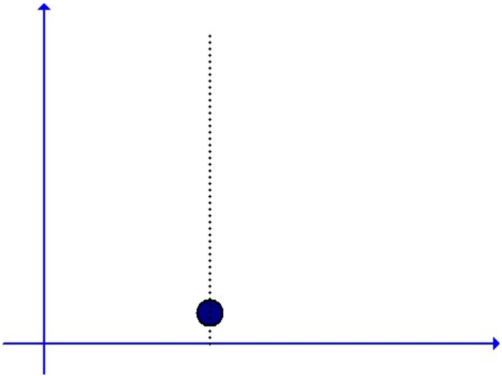
This seems to match Arnold's sequence.

Does the pattern continue?

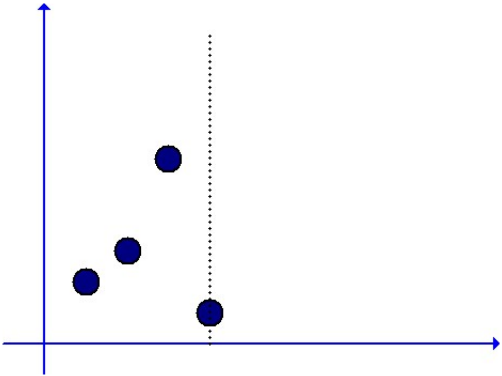
$S_n(\overline{31}542)$ Observation 1



$S_n(\overline{3}1542)$ Observation 1

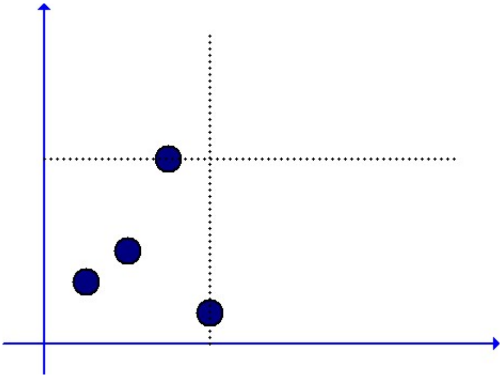


$S_n(\bar{3}1542)$ Observation 1



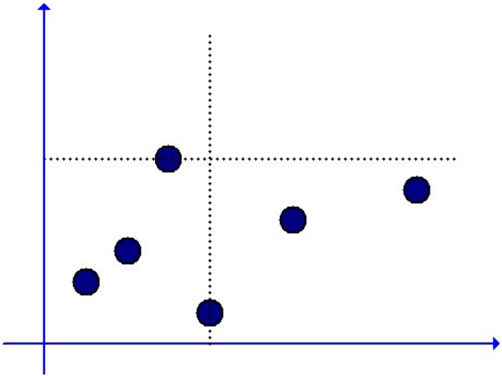
- All numbers before the 1 must be in increasing order.

$S_n(\overline{31}542)$ Observation 2



- All numbers before the 1 must be in increasing order.

$S_n(\bar{3}1542)$ Observation 2

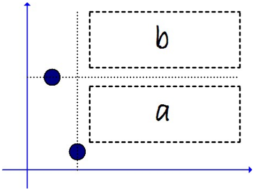


- All numbers before the 1 must be in increasing order.
- All numbers in the lower right quadrant must be in increasing order.

Subset Definitions

This structure allows us to define several useful subsets of $S_n(\overline{31}542)$.

$$G\begin{pmatrix} b \\ a \end{pmatrix}$$



Subset Definitions

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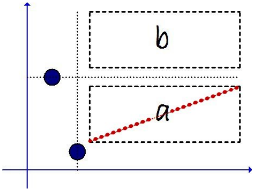
$$G\binom{b}{a}$$



Subset Definitions

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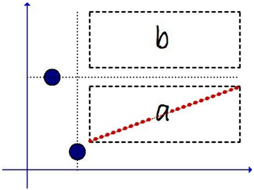
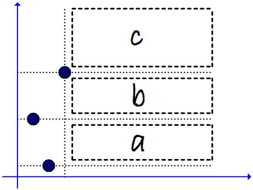
$$G\begin{pmatrix} b \\ a \end{pmatrix} \qquad\qquad F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$$





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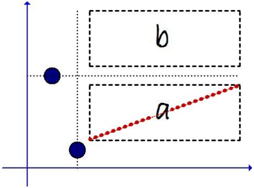
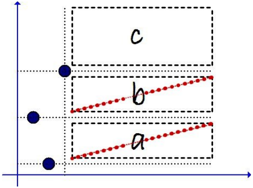
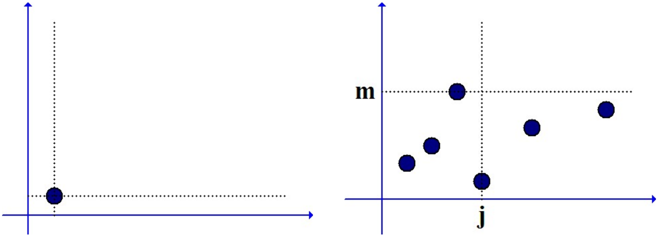




$$\left|G\begin{pmatrix} b \\ a \end{pmatrix}\right| = \left|F\begin{pmatrix} b \\ 0 \\ a \end{pmatrix}\right|$$

Recurrence for $S_n(\overline{31}542)$

For $\pi \in S_n(\overline{31}542)$, one of the following must happen



$j, m > 1$

Recurrence for $S_n(\overline{3}1542)$

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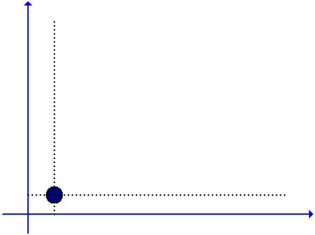
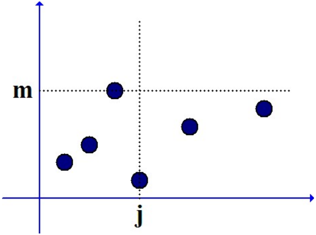




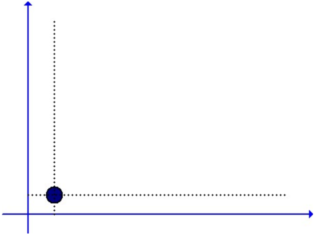
$j, m > 1$

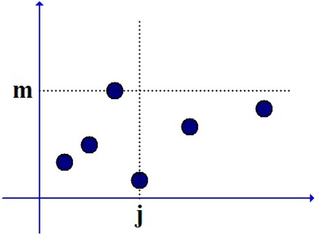
$$\left|S_n(\overline{3}1542)\right| =$$

$$\left|S_{n-1}(\overline{3}1542)\right|$$

Recurrence for $S_n(\bar{3}\bar{1}542)$

For $\pi \in S_n(\bar{3}\bar{1}542)$, one of the following must happen





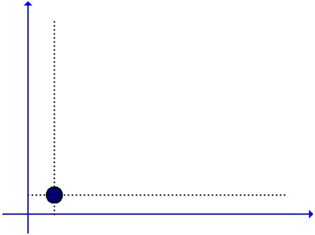
$j, m > 1$

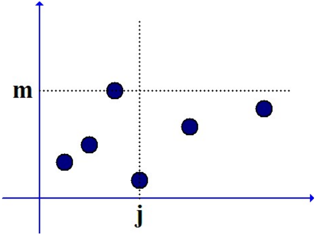
$$\left|S_n(\bar{3}\bar{1}542)\right| =$$

$$\left|S_{n-1}(\bar{3}\bar{1}542)\right| + \sum_{j=2}^{n}\sum_{m=j}^{n}\binom{m-2}{j-2}\cdot\left|G\left(\begin{array}{c} n-m \\ m-j \end{array}\right)\right|.$$

Recurrence for $S_n(\overline{31}542)$

For $\pi \in S_n(\overline{31}542)$, one of the following must happen





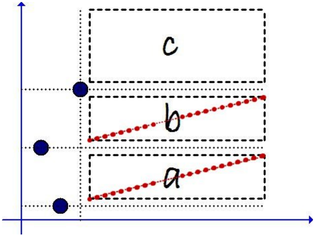
$j, m > 1$

$$\left|S_n(\overline{31}542)\right| =$$

$$\left|S_{n-1}(\overline{31}542)\right| + \sum_{j=2}^{n} \sum_{m=j}^{n} \binom{m-2}{j-2} \cdot \left| F \begin{pmatrix} n-m \\ 0 \\ m-j \end{pmatrix} \right| .$$

Recurrence for $F\begin{pmatrix}c\\b\\a\end{pmatrix}$

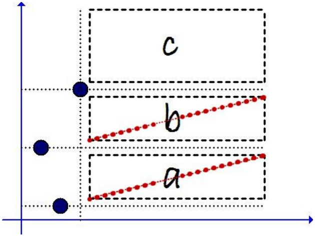
For $\pi \in F\begin{pmatrix}c\\b\\a\end{pmatrix}$



$$\left|F\begin{pmatrix}0\\b\\a\end{pmatrix}\right| = \binom{b+a}{a}, \left|F\begin{pmatrix}c\\0\\0\end{pmatrix}\right| = \left|S_c(\overline{31}542)\right|.$$

Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

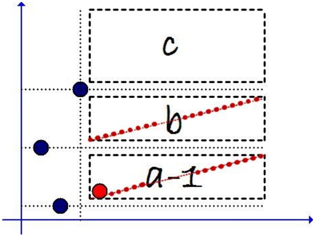
For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left| F\begin{pmatrix} c \\ b \\ a \end{pmatrix} \right| =$$

Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

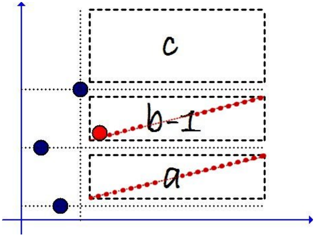
For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| = \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a-1}$$

Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

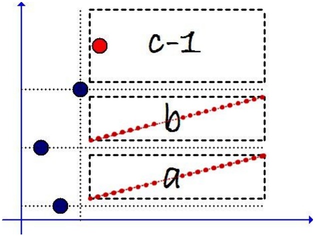
For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| = \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a-1} + \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a}$$

Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| = \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a-1} +$$

$$\left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a} + \sum_{\ell=1}^{c}\left|F\begin{pmatrix} c-\ell \\ b+\ell-1 \\ a \end{pmatrix}\right|.$$

Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

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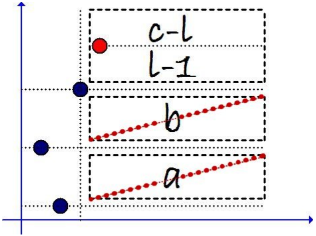

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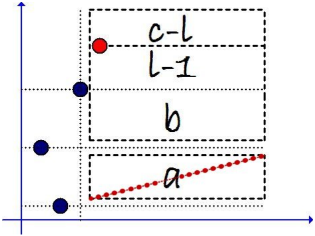
For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| = \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a-1} +$$

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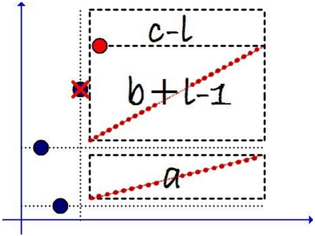
Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left| F\begin{pmatrix} c \\ b \\ a \end{pmatrix} \right| = \left| F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix} \right| \cdot \binom{a+b-1}{a-1} +$$

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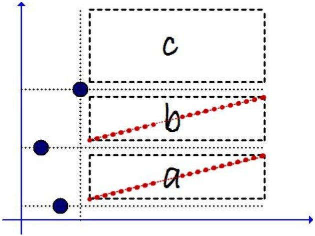
Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

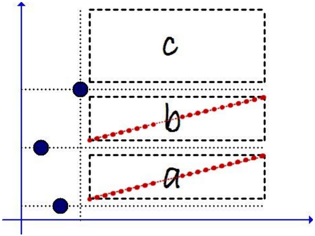
For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| = \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a-1} + \left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right| \cdot \binom{a+b-1}{a} + \sum_{\ell=1}^{c} \left|F\begin{pmatrix} c-\ell \\ b+\ell-1 \\ a \end{pmatrix}\right|.$$

Recurrence for $F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$

For $\pi \in F\begin{pmatrix} c \\ b \\ a \end{pmatrix}$



$$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| =$$

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Recurrence

Together:

$\left|S_n(\overline{31}542)\right| =$

$$\left|S_{n-1}(\overline{31}542)\right| + \sum_{j=2}^{n}\sum_{m=j}^{n}\binom{m-2}{j-2}\cdot\left|F\begin{pmatrix} n-m \\ 0 \\ m-j \end{pmatrix}\right|.$$

$\left|F\begin{pmatrix} c \\ b \\ a \end{pmatrix}\right| =$

$$\left|F\begin{pmatrix} c \\ 0 \\ b+a-1 \end{pmatrix}\right|\cdot\binom{a+b}{a} + \sum_{\ell=1}^{c}\left|F\begin{pmatrix} c-\ell \\ b+\ell-1 \\ a \end{pmatrix}\right|.$$

Note that $\left|S_n(\overline{31}542)\right|$ satisfies a recurrence with binomial coefficients rather than constant coefficients.

Conclusion

Both Arnold's composition sequence and the counting sequence for $S_n(\overline{3}\overline{1}542)$ are identical for at least 386 terms. Coincidence?

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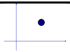
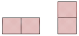
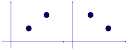
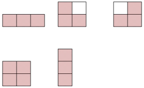
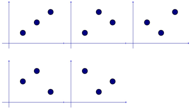
Open Problems:

- Prove that
$$\left|S_n(\overline{31}542)\right| = \sum_{k=0}^{n} \left((n+1-k)^k - (n-k)^k\right).$$
- Find a bijection between the elements of $S_n(\overline{31}542)$ and the compositions of Arnold's construction.

Compositions and Permutations

n	Compositions	Permutations
1		
2		
3		
4