Pattern avoidance in double lists

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joint work with
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Introduction
Length 4 Patterns
1342
2143
1423, 1432, 1243
1234
2413
1324
Summary

Variations on Pattern Avoidance

- in words (Burstein 1998, and others)
- in centrosymmetric permutations (Egge 2007; Barnabei, Bonetti, Silimbani 2010)
- in centrosymmetric words (Ferrari 2011)
- in circular permutations (Callan 2002, Vella 2003)
  - Here, for example, $1324 = 3241 = 2413 = 4132$.

Our study: pattern avoidance in words with a special kind of symmetry/repetition.

Definitions/Notation

- $\mathcal{S}_n$ is the set of permutations of length n.

$$\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

- $\mathcal{D}_n = \{\pi\pi \mid \pi \in \mathcal{S}_n\}$.

$$\mathcal{D}_3 = \{123123, 132132, 213213, 231231, 312312, 321321\}.$$

- $\mathcal{D}_n(\rho) = \{\sigma \mid \sigma \in \mathcal{D}_n \text{ and } \sigma \text{ avoids } \rho\}$.

$$\mathcal{D}_3(12) = \emptyset.$$
$$\mathcal{D}_3(123) = \{321321\}.$$

Goal: Characterize $\mathcal{D}_n(\rho)$/compute $|\mathcal{D}_n(\rho)|$ where $\rho \in \mathcal{S}_4$.

Warmup

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- $\mathcal{D}_n(1) = \emptyset$ for $n \geq 1$.
- $\mathcal{D}_n(12) = \mathcal{D}_n(21) = \emptyset$ for $n \geq 2$.

Warmup

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- $\mathcal{D}_n(1) = \emptyset$ for $n \geq 1$.
- $\mathcal{D}_n(12) = \mathcal{D}_n(21) = \emptyset$ for $n \geq 2$.
- $|\mathcal{D}_n(\rho)| = |\mathcal{D}_n(\rho^r)| = |\mathcal{D}_n(\rho^c)|$
  but $|\mathcal{D}_n(\rho)| \neq |\mathcal{D}_n(\rho^{-1})|$ in general.

Warmup

- $\mathcal{D}_n(1) = \emptyset$ for $n \geq 1$.
- $\mathcal{D}_n(12) = \mathcal{D}_n(21) = \emptyset$ for $n \geq 2$.
- $|\mathcal{D}_n(\rho)| = |\mathcal{D}_n(\rho^r)| = |\mathcal{D}_n(\rho^c)|$ **but** $|\mathcal{D}_n(\rho)| \neq |\mathcal{D}_n(\rho^{-1})|$ in general.
- $\mathcal{D}_n(123) = \{n \cdots 1 n \cdots 1\}$ for $n \geq 3$.
- $$\mathcal{D}_n(132) = \begin{cases} \{11\} & n = 1 \\ \{1212, 2121\} & n = 2 \\ \{231231\} & n = 3 \\ \emptyset & n \geq 4 \end{cases}.$$

Length 4 Trivial Wilf Classes

Pattern ρ	$\{\lvert\mathcal{D}_n(\rho)\rvert\}_{n=1}^{10}$
1342, 2431, 3124, 4213	1, 2, 6, 12, 15, 15, 15, 15, 15, 15
2143, 3412	1, 2, 6, 12, 13, 14, 16, 18, 20, 22
1423, 2314, 3241, 4132	1, 2, 6, 12, 17, 23, 27, 30, 33, 36
1432, 2341, 3214, 4123	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
1243, 2134, 3421, 4312	1, 2, 6, 12, 19, 25, 34, 44, 55, 67
2413, 3142	1, 2, 6, 12, 18, 29, 47, 76, 123, 199
1324, 4231	1, 2, 6, 12, 21, 38, 69, 126, 232, 427
1234, 4321	1, 2, 6, 12, 27, 58, 121, 248, 503, 1014

Contrast: For large n, $\lvert\mathcal{S}_n(1342)\rvert < \lvert\mathcal{S}_n(1234)\rvert < \lvert\mathcal{S}_n(1324)\rvert$.

$\mathcal{D}_n(1342)$

$(1, 2, 6, 12, 15, 15, 15, 15, 15, 15, \ldots)$

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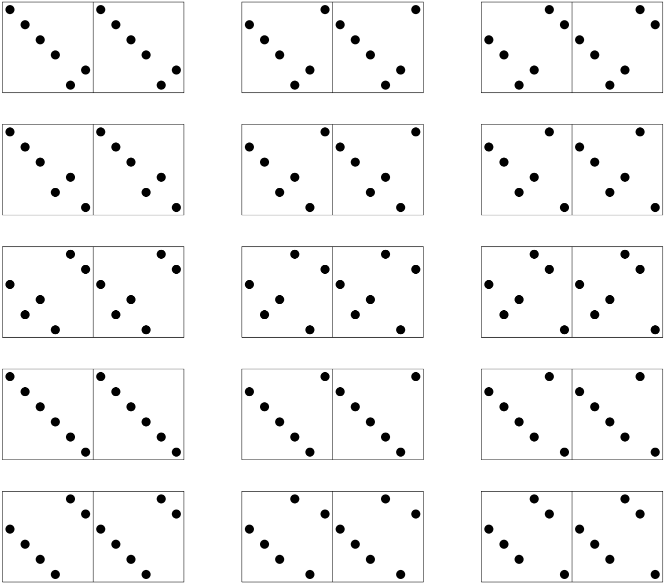
$\mathcal{D}_n(1342)$

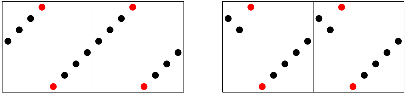
$(1, 2, 6, 12, 15, 15, 15, 15, 15, 15, \ldots)$



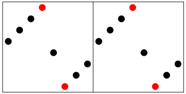
$\mathcal{D}_n(2143)$

$(1, 2, 6, 12, 13, 14, 16, 18, 20, 22, \ldots)$

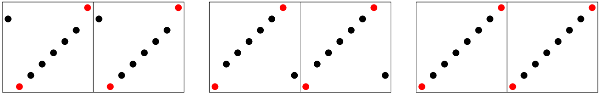
Case 1: $(\ldots, n, 1, \ldots, n, 1, \ldots)$ $(n + 1$ lists$)$



Case 2: $(\ldots, n, a, 1, \ldots, n, a, 1, \ldots)$ $(n - 2$ lists$)$



Case 3: $(\ldots, 1, \ldots, n, \ldots, 1, \ldots, n, \ldots)$ $(3$ lists$)$



$$(n + 1) + (n - 2) + 3 = 2n + 2.$$

$\mathcal{D}_n(\rho),\ \rho \in \{1423, 1432, 1243\}$

(More case analysis...)

For $\rho = 1423$, $(1,\ 2,\ 6,\ 12,\ 17,\ 23,\ 27,\ 30,\ 33,\ 36, \ldots)$

For $\rho = 1432$, $(1,\ 2,\ 6,\ 12,\ 17,\ 23,\ 31,\ 40,\ 50,\ 61, \ldots)$

For $\rho = 1243$, $(1,\ 2,\ 6,\ 12,\ 19,\ 25,\ 34,\ 44,\ 55,\ 67, \ldots)$

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(More case analysis...)

For $\rho = 1423$, $(1, 2, 6, 12, 17, 23, 27, 30, 33, 36, \ldots)$

$|\mathcal{D}_n(1423)| = 3n + 6$ $(n \geq 7)$

For $\rho = 1432$, $(1, 2, 6, 12, 17, 23, 31, 40, 50, 61, \ldots)$

For $\rho = 1243$, $(1, 2, 6, 12, 19, 25, 34, 44, 55, 67, \ldots)$

$\mathcal{D}_n(\rho)$, $\rho \in \{1423, 1432, 1243\}$

(More case analysis...)

For $\rho = 1423$, $(1, 2, 6, 12, 17, 23, 27, 30, 33, 36, \ldots)$

$|\mathcal{D}_n(1423)| = 3n + 6 \ (n \geq 7)$

For $\rho = 1432$, $(1, 2, 6, 12, 17, 23, 31, 40, 50, 61, \ldots)$

$|\mathcal{D}_n(1432)| = |\mathcal{D}_{n-1}(1432)| + (n+1) \ (n \geq 7)$

$|\mathcal{D}_n(1432)| = \frac{1}{2}n^2 + \frac{3}{2}n - 4 \ (n \geq 6)$

For $\rho = 1243$, $(1, 2, 6, 12, 19, 25, 34, 44, 55, 67, \ldots)$

$\mathcal{D}_n(\rho)$, $\rho \in \{1423, 1432, 1243\}$

(More case analysis...)

For $\rho = 1423$, $(1, 2, 6, 12, 17, 23, 27, 30, 33, 36, \ldots)$

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$|\mathcal{D}_n(1432)| = |\mathcal{D}_{n-1}(1432)| + (n+1)$ $(n \geq 7)$

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For $\rho = 1243$, $(1, 2, 6, 12, 19, 25, 34, 44, 55, 67, \ldots)$

$|\mathcal{D}_n(1243)| = |\mathcal{D}_{n-1}(1243)| + (n+2)$ $(n \geq 7)$

$|\mathcal{D}_n(1243)| = \frac{1}{2}n^2 + \frac{5}{2}n - 8$ $(n \geq 6)$

$\mathcal{D}_n(1234)$

$(1, 2, 6, 12, 27, 58, 121, 248, 503, 1014, \ldots)$

For $n \geq 4$, in OEIS, "Number of different permutations of a deck of n cards that can be produced by a single shuffle".

1. Begin with ordered deck $n \cdots 1$.
2. Cut.
3. Each card either comes from upper or lower partial deck.

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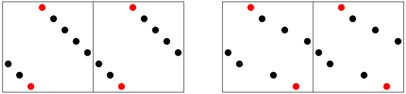
Picture of 1234-avoiding double list:



$\mathcal{D}_n(1234)$

(1, 2, 6, 12, 27, 58, 121, 248, 503, 1014,. . .)

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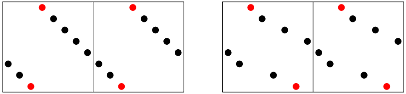
Picture of 1234-avoiding double list:



There are 2^n strings on $\{U, L\}^n$, but the $(n+1)$ decks of the form $U \cdots UL \cdots L$ are all equivalent to the original deck.

$$|\mathcal{D}_n(1234)| = 2^n - n \text{ for } n \geq 4.$$

$\mathcal{D}_n(2413)$

$(1, 2, 6, 12, 18, 29, 47, 76, 123, 199, \ldots)$

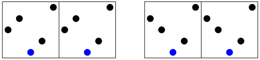
Key observation:

1 must appear in position 1, $n-2$, $n-1$, or n.

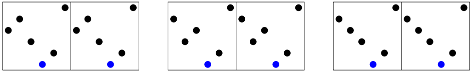
- e.g. If 1 is in position $n-2$:
  must begin with $n-1$ or $(n-2)(n-1)$.

$\mathcal{D}_n(2413)$ (1 in position $n-2$)

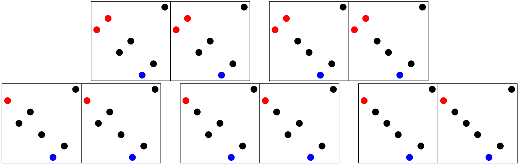
In $\mathcal{D}_5(2413)$...



In $\mathcal{D}_6(2413)$...



In $\mathcal{D}_7(2413)$...



$\mathcal{D}_n(2413)$

$(1, 2, 6, 12, 18, 29, 47, 76, 123, 199, \ldots)$

Key observation: 1 must appear in position 1, $n-2$, $n-1$, or n.

- e.g. If 1 is in position $n-2$: must begin with $n-1$ or $(n-2)(n-1)$.
- similar recursions for other positions of 1.

$\mathcal{D}_n(2413)$

(1, 2, 6, 12, 18, 29, 47, 76, 123, 199, ...)

Key observation: 1 must appear in position 1, $n-2$, $n-1$, or n.

- e.g. If 1 is in position $n-2$: must begin with $n-1$ or $(n-2)(n-1)$.
- similar recursions for other positions of 1.

For $n \geq 7$,

$$|\mathcal{D}_n(2413)| = |\mathcal{D}_{n-1}(2413)| + |\mathcal{D}_{n-2}(2413)|.$$

i.e. $|\mathcal{D}_n(2413)| = L_{n+2}$ $(n \geq 5)$.

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$\mathcal{D}_n(1324)$

$(1, 2, 6, 12, 21, 38, 69, 126, 232, 427, \ldots)$

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$\mathcal{D}_n(1324)$

$(1, 2, 6, 12, 21, 38, 69, 126, 232, 427, 785, 1444, 2656, \ldots)$

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$\mathcal{D}_n(1324)$

$(1, 2, 6, 12, 21, 38, 69, 126, 232, 427, 785, 1444, 2656, \ldots)$

$427 = 232 + 126 + 69$

$785 = 427 + 232 + 126$

$1444 = 785 + 427 + 232$

$2656 = 1444 + 785 + 427$

$\mathcal{D}_n(1324)$

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$(1, 2, 6, 12, 21, 38, 69, 126, 232, 427, 785, 1444, 2656, \ldots)$

427=232+126+69
785=427+232+126
1444=785+427+232
2656=1444+785+427

For $n \geq 10$,
$|\mathcal{D}_n(1324)| = |\mathcal{D}_{n-1}(1324)| + |\mathcal{D}_{n-2}(1324)| + |\mathcal{D}_{n-3}(1324)|$.

Summary

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Pattern ρ	$\lvert\mathcal{D}_n(\rho)\rvert$	
1342, 2431, 3124, 4213	15	$(n \geq 5)$
2143, 3412	$2n+2$	$(n \geq 6)$
1423, 2314, 3241, 4132	$3n+6$	$(n \geq 7)$
1432, 2341, 3214, 4123	$\frac{1}{2}n^2 + \frac{3}{2}n - 4$	$(n \geq 6)$
1243, 2134, 3421, 4312	$\frac{1}{2}n^2 + \frac{5}{2}n - 8$	$(n \geq 6)$
2413, 3142	L_{n+2}	$(n \geq 5)$
1324, 4231	$\lvert\mathcal{D}_{n-1}(\rho)\rvert + \lvert\mathcal{D}_{n-2}(\rho)\rvert + \lvert\mathcal{D}_{n-3}(\rho)\rvert$	$(n \geq 10)$
1234, 4321	$2^n - n$	$(n \geq 4)$

Future Work

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- Convert case-bash proofs to bijective proofs.
- Avoid longer patterns/patterns with repeated letters/sets of patterns.
- Consider words on $\{1, 1, 2, 2, \ldots, n, n\}$ with other structure.

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- Convert case-bash proofs to bijective proofs.
- Avoid longer patterns/patterns with repeated letters/sets of patterns.
- Consider words on $\{1, 1, 2, 2, \ldots, n, n\}$ with other structure.

Thanks for listening!