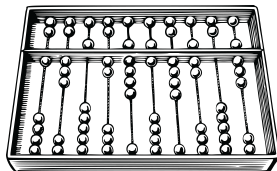
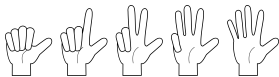


What really counts: the joy of enumeration

Lara Pudwell  Valparaiso
University
faculty.valpo.edu/lpudwell

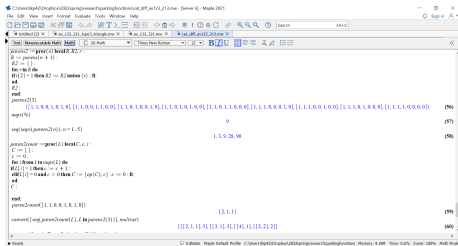
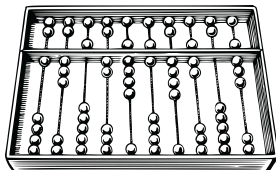
Professorial Lecture
April 14, 2022

What *is* counting (as research)?



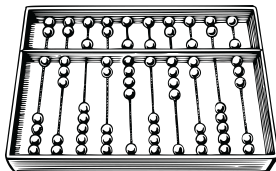


What *is* counting (as research)?



$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2 + 1)}{1 - 2x - x^2 - 2x^3}$$

What *is* counting (as research)?



```

C:\Users\k94\Desktop>cd prog\exercise2\part5\func02\src_01_12_21.exe - [Semir G] - Made 2017
File Edit View Insert Format Database Tools Windows Help
[Icons] [List] [Search] [History] [Filter] [Recent] [Address Bar] [Search] [Find]
C:\Users\k94\Desktop\prog\exercise2\part5\func02\src_01_12_21.exe
[Std] [View] [New Window] [Print] [Page Setup] [Page Properties] [Print Range] [Print Range]
jarnes27 ==> g++(C:\k94\Z_S27_
# -o jarnes27 -s 11 -
#2 - { }
for: skip: do
Rf(2) := 1 then #2 := #2 mod(n) := B
end
#2
end
jarnes23)
[1111111111][1100111000][1111111111][1111111000][1111110000][1111110000][1111001000][1111111000] (150)
next(N)
9
next N(jarnes23) := n + 1 - 5
1.59,26.96 (151)
jarnes2count ==> g++(Z:\k94\c.c:1:
C := { }
for: 3 then 1 then skip[2] do
Rf(2) := 1 then #2 := #2 + 1
end[2] := 0 then #2 := skip(C, #2) := 0 - B
end
C:
end
jarnes2count([1,1,1,1,1,1,1,1,1,1])
[1,1,1] (155)
convert([jarnes2count(Z, [ B(jarnes2) ])], wchstr)
[111,1,11,3][1,3,1,1,1][11,11,11,21,31] (160)

```

$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2 + 1)}{1 - 2x - x^2 - 2x^3}$$

Goal: Answer “How many?” in an efficient, strategic way.

What to count?

REFEREED PUBLICATIONS

(* denotes undergraduate student)

37. Jonathan Beagley and Lara Pudwell, Colorful tilings and **permutations**, *Journal of Integer Sequences* **24** (2021) 21.10.4.
 36. Lara Pudwell, From **permutation patterns** to the periodic table, *Notices of the American Mathematical Society* **67.7** (2020), 994–1001.
 35. Lara Pudwell and Rebecca Smith, Two-stack-sorting with pop stacks, *Australasian Journal of Combinatorics* **74.1** (2019), 179–195.
 34. Michael Bukata*, Ryan Kulwicki*, Nicholas Lewandowski*, Lara Pudwell, Jacob Roth*, and Teresa Wheeland*, Distributions of Statistics over **Pattern-Avoiding Permutations**, *Journal of Integer Sequences* **22** (2019) 19.2.6.
-
33. Monica Anderson*, Marika Diepenbroek*, Lara Pudwell, and Alex Stoll*, **Pattern avoidance** in reverse double lists, *Discrete Mathematics and Theoretical Computer Science* **19.2** (2018), #13.
 32. Lara Pudwell and Eric Rowland, **Avoiding** fractional powers over the natural numbers, *Electronic Journal of Combinatorics* **25.2** (2018), P2.27.
 31. Michael Dorff, Allison Henrich, and Lara Pudwell, Successfully Mentoring Undergraduates in Research: A How To Guide for Mathematicians, *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, **27.3** (2017), 320–336.
 30. Charles Cratty*, Samuel Erickson*, Frehiwet Negassi*, and Lara Pudwell, **Pattern avoidance** in double lists, *Involve, a Journal of Mathematics* **10.3** (2017), 379–398.

MATH ENCOUNTERS

PATTERNS and PERMUTATIONS:
the HIDDEN and SURPRISING STRUCTURES that emerge from ORDERED LISTS

Speaker: Lara Pudwell

Special introduction by
Dr. Jennifer Quinn,
 professor of Mathematics,
 University of Washington Tacoma,
 and President, Mathematical
 Association of America

Wednesday, June 2, 2021
Afternoon Presentation
 4:00 PM ET (New York)
Evening Presentation
 7:00 PM ET (New York)

A "permutation" is a list where order matters. Despite this familiar definition, permutations offer a wealth of beautiful mathematics that can be applied across scientific disciplines. Starting simple with small ordered lists, mathematician Lara Pudwell demonstrates how smaller permutations can be embedded into larger permutations. The results are both unexpected and beautiful – not only do some familiar patterns suddenly appear but the underlying process yields a host of interesting counting problems with connections not just to mathematics but to computer science, chemistry, and more.

Register now at mathencounters.org to reserve your place!

MATH ENCOUNTERS
 a public perception intervention
 for an emerging world of mathematics

ELWELL FOUNDATION

MOXMAH
 MATHS ON THE MARCH

A **permutation** of length n is a list of the numbers $1, 2, \dots, n$ where order matters.

- Permutations of length 1? 1

A **permutation** of length n is a list of the numbers $1, 2, \dots, n$ where order matters.

- Permutations of length 1? 1
- Permutations of length 2? 12 21

A **permutation** of length n is an list of the numbers $1, 2, \dots, n$ where order matters.

- Permutations of length 1? 1
- Permutations of length 2? 12 21
- Permutations of length 3? 123 132
 213 231
 312 321

A **permutation** of length n is a list of the numbers $1, 2, \dots, n$ where order matters.

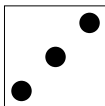
- Permutations of length 1? 1
- Permutations of length 2? 12 21
- Permutations of length 3? 123 132
 213 231
 312 321

Fact

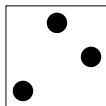
There are $n \cdot (n - 1) \cdot (n - 2) \cdots 1 = n!$ permutations of length n .

Permutations in picture form

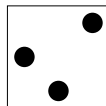
Visualize permutations by plotting the points in the plane.
(Heights correspond to digits in the permutation.)



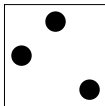
123



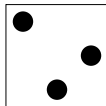
132



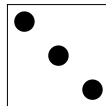
213



231

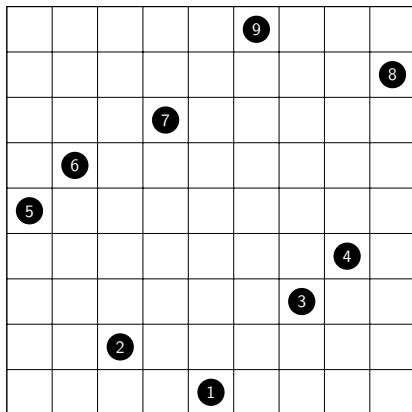


312



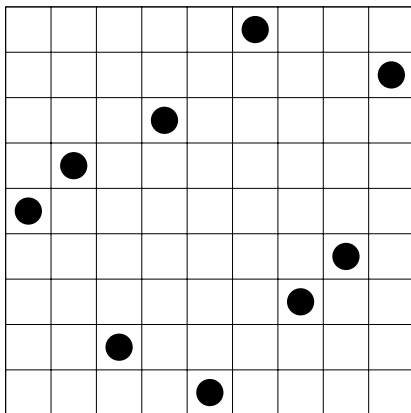
321

A bigger permutation picture

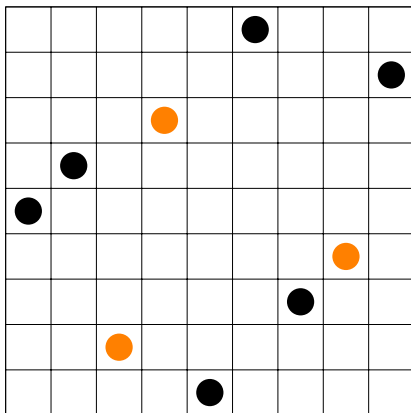


562719348

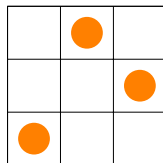
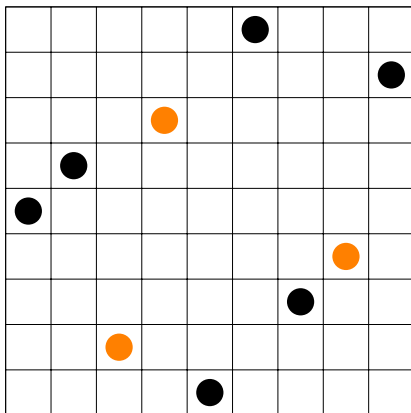
Patterns in permutations



Patterns in permutations

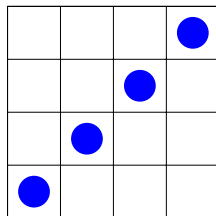
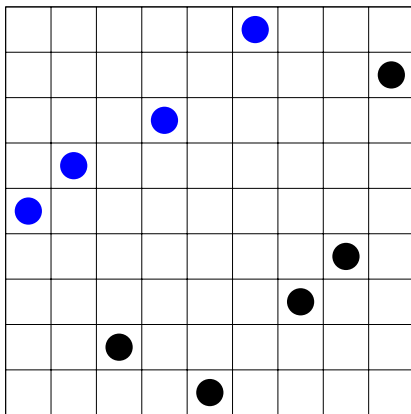


Patterns in permutations



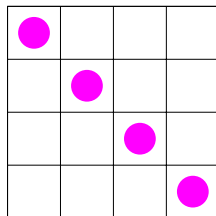
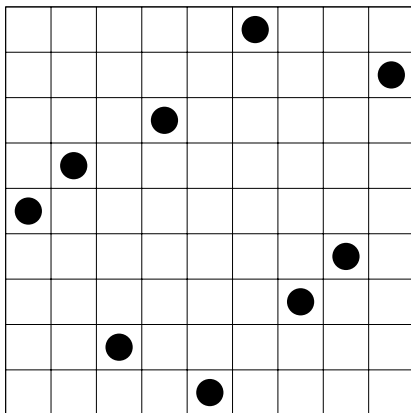
562719348 contains the pattern 132

Patterns in permutations



562719348 contains the pattern 1234

Patterns in permutations



562719348 avoids the pattern 4321

Big question

How many permutations of length n contain the permutation p ?

Or, alternatively...

Big question

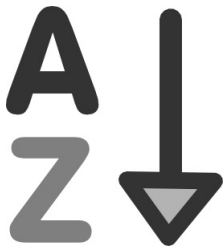
How many permutations of length n avoid the permutation p ?

(depends on what p is!)

Why *avoid* patterns?



Why *avoid* patterns?



Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

What's my ~~application~~ *motivation*?

- Knuth (1968): avoiding 231 is useful in computer science
- Mathematicians (1980s): what interesting things happen when we avoid other patterns?

What's my application *motivation*?

- Knuth (1968): avoiding 231 is useful in computer science
- Mathematicians (1980s): what interesting things happen when we avoid other patterns?
- Directions in past 30+ years:



What's my application *motivation*?

- Knuth (1968): avoiding 231 is useful in computer science
- Mathematicians (1980s): what interesting things happen when we avoid other patterns?
- Directions in past 30+ years:



Theme: studying new mathematical structure for its own sake

Bonus: spotting surprise connections

Isn't that peculiar?

----- Forwarded message -----

From: **David Harris**

Date: Thu, Jun 4, 2009 at 9:24 PM

Subject: a series of numbers

To: <Lara.Pudwell@valpo.edu>

Hi-

I happened across a web site of yours at Rutgers website via a google search. (<http://www.math.rutgers.edu/~lpudwell/maple/schemes/334out>).

I got there because my Middle school aged daughter had a math project that generated the following series of numbers:

6 18 34 54 78 106 138 174

When I googled the series, only your site came up. We got the numbers from the following problem, which we're trying to find an equation to describe:

(This is a triangle series but with cubes).

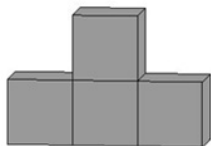
We have a single cube($n=1$) total surface area equals 6 sides.

If we add a cube on either side and one on top, forming a triangle surrounding the original ($n=2$) total surface area is 18 sides.

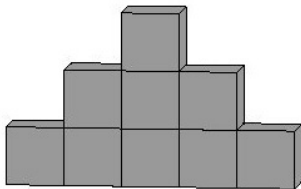
Pyramids of cubes



6



18



34

Theorem

The surface area of Harris's n th pyramid of blocks is...

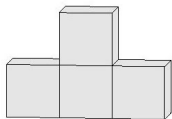
Theorem

The surface area of Harris's n th pyramid of blocks is...
the number of lists with *two* 1s, *two* 2s, ..., and *two* $(n + 1)$ s
that avoid 132, 231, and 2134.

Theorem

The surface area of Harris's n th pyramid of blocks is...
the number of lists with *two* 1s, *two* 2s, ..., and *two* $(n + 1)$ s
that avoid 132, 231, and 2134.

Example:



has surface area 18.

18 pattern-avoiding lists:

112233, 121233, 122133, 211233, 212133, 221133
311223, 312123, 312213, 321123, 321213, 322113
331122, 331212, 331221, 332112, 332121, 332211

The next step is...

Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

The next step is...

Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

Theorem (Avis and Newborn, 1981)

A permutation is pop-stack sortable if and only if it avoids 231 and 312.

The next step is...

Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

Theorem (Avis and Newborn, 1981)

A permutation is pop-stack sortable if and only if it avoids 231 and 312.

Theorem (West, 1990)

A permutation is sortable after two passes through a stack if and only if it avoids 2341 and $3\bar{5}241$.

Theorem (Pudwell and Smith, 2019)

A permutation is sortable after two passes through a pop-stack if and only if it avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

Theorem (Pudwell and Smith, 2019)

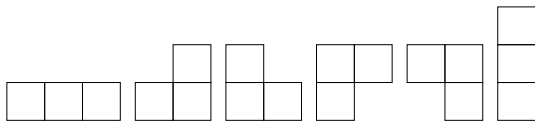
A permutation is sortable after two passes through a pop-stack if and only if it avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

The number of such permutations is...

Theorem (Pudwell and Smith, 2019)

A permutation is sortable after two passes through a pop-stack if and only if it avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

The number of such permutations is...

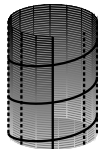
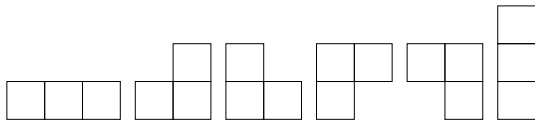


the number of n -square polyominoes...

Theorem (Pudwell and Smith, 2019)

A permutation is sortable after two passes through a pop-stack if and only if it avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

The number of such permutations is...

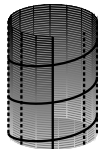
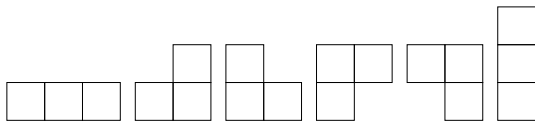


the number of n -square polyominoes...
that can be drawn on a “twisted cylinder” of width 3.



Theorem (Pudwell and Smith, 2019)

A permutation is sortable after two passes through a pop-stack if and only if it avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

The number of such permutations is...



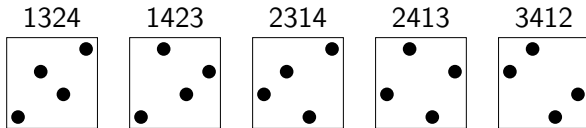
the number of n -square polyominoes...
that can be drawn on a “twisted cylinder” of width 3.

e.g.  and  are now the same

What if...?

A permutation is **alternating** if its adjacent pairs of digits alternate between increasing and decreasing pairs.

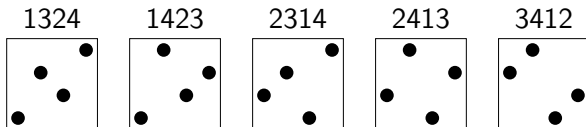
Examples:



What if...?

A permutation is **alternating** if its adjacent pairs of digits alternate between increasing and decreasing pairs.

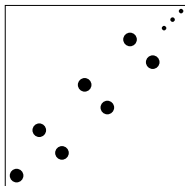
Examples:



Which of these examples has the most 123 patterns?

Many 123s

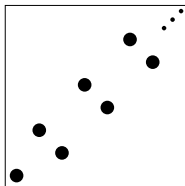
The alternating permutation with the most 123s possible looks like this:



Question: How many 123 patterns does it have?

Many 123s

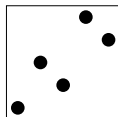
The alternating permutation with the most 123s possible looks like this:



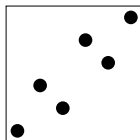
Question: How many 123 patterns does it have?



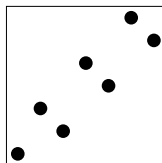
2 copies



4 copies



12 copies



20 copies

Many 123s

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, ...

Many 123s

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, ...

A099956	Atomic numbers of the alkaline earth metals.	9
	4, 12, 20, 38, 56, 88 (list ; graph ; refs ; listen ; history ; text ; internal format)	
OFFSET	1,1	
LINKS	Table of n, a(n) for n=1..6.	
EXAMPLE	12 is the atomic number of magnesium.	
CROSSREFS	Cf. A099955 , alkali metals; A101648 , metalloids; A101647 , nonmetals (except halogens and noble gases); A097478 , halogens; A018227 , noble gases; A101649 , poor metals. Sequence in context: A057317 A008068 A008183 * A301066 A008092 A316299 Adjacent sequences: A099953 A099954 A099955 * A099957 A099958 A099959	
KEYWORD	nonn,fini,full	
AUTHOR	Parthasarathy Nambi , Nov 12 2004	
STATUS	approved	

Online Encyclopedia of Integer Sequences (oeis.org)

Many 123s

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, ...

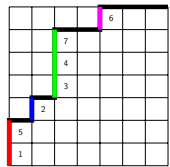
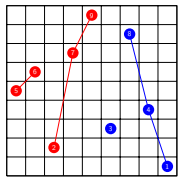
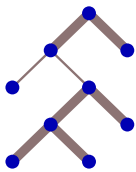
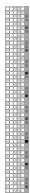
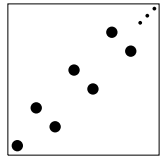
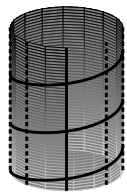
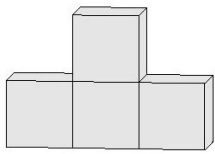
A168380	Row sums of A168281 .	+20 14
	2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, 1140, 1340, 1540, 1782, 2024, 2312, 2600, 2938, 3276, 3668, 4060, 4510, 4960, 5472, 5984, 6562, 7140, 7788, 8436, 9158, 9880, 10680, 11480, 12362, 13244, 14212, 15180, 16238, 17296, 18448, 19600, 20850, 22100 (list ; graph ; refs ; listen ; history ; text ; internal format)	
OFFSET	1,1	
COMMENTS	The atomic numbers of the augmented alkaline earth group in Charles Janet's spiral periodic table are 0 and the first eight terms of this sequence (see Stewart reference). - Alonso del Arte , May 13 2011	
LINKS	Vincenzo Librandi, Table of n, a(n) for n = 1..10000 Stewart, Philip, Charles Janet: unrecognized genius of the Periodic System . Foundations of Chemistry (2010), p. 9. Index entries for linear recurrences with constant coefficients , signature (2,1,-4,1,2,-1).	
FORMULA	$a(n) = 2 \cdot \text{A005993}(n-1)$. $a(n) = (n+1) \cdot (3 + 2 \cdot n^2 + 4 \cdot n - 3 \cdot (-1)^n) / 12$. $a(n+1) - a(n) = \text{A093907}(n) = \text{A137583}(n+1)$. $a(2n+1) = \text{A035597}(n+1)$ $a(2n) = \text{A002492}(n)$. $a(n) = \text{A099956}(n-1)$, $2 \leq n \leq 7$.	

Online Encyclopedia of Integer Sequences (oeis.org)

Alkaline Earth Metals (Group 2)

Group	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
Period 1	1 H																		2 He
Period 2	3 Li	4 Be											5 B	6 C	7 N	8 O	9 F	10 Ne	
Period 3	11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 Ar	
Period 4	19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr	
Period 5	37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe	
Period 6	55 Cs	56 Ba	57 La *	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn	
Period 7	87 Fr	88 Ra	89 Ac *	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og	
				* 58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu		
				* 90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr		

...and more!



My (personal) motivation

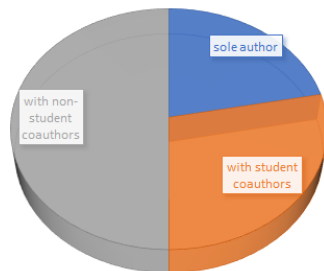
- (Mathematical) Beauty
(Delight in surprise connections)

My (personal) motivation

- (Mathematical) Beauty
(Delight in surprise connections)
- (Mathematical) Truth
(Rising to the challenge of writing rigorous proofs)

My (personal) motivation

- (Mathematical) Beauty
(Delight in surprise connections)
- (Mathematical) Truth
(Rising to the challenge of writing rigorous proofs)
- Community
(Sharing the (pursuit of) beauty and truth with others)

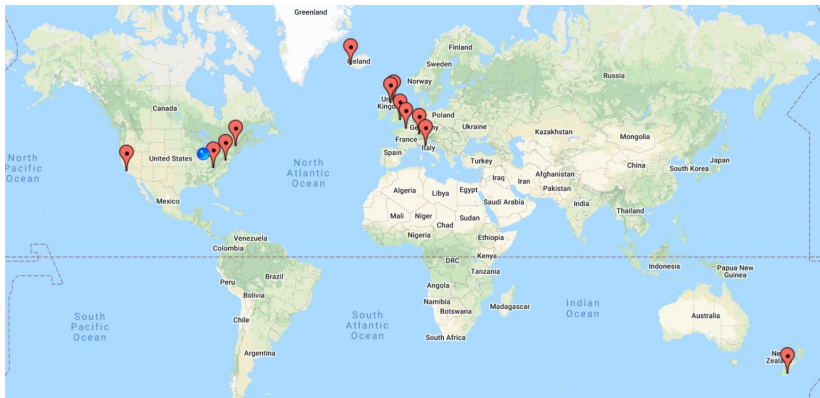


Community: with other researchers



Permutation Patterns 2019, Zurich, Switzerland

Community: with other researchers

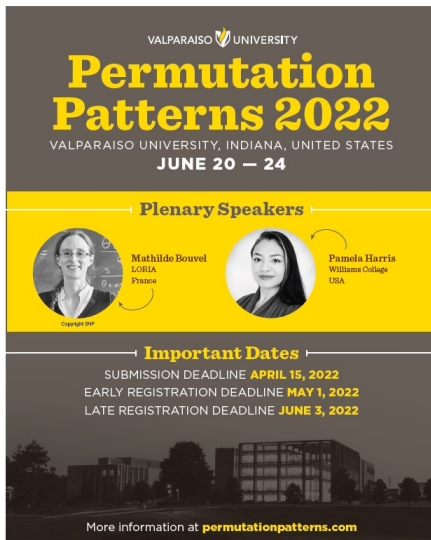


Permutation Patterns Locations

Community: with other researchers



Community: with other researchers





VALPARAISO UNIVERSITY

Permutation Patterns 2022

VALPARAISO UNIVERSITY, INDIANA, UNITED STATES
JUNE 20 – 24

Plenary Speakers

 **Mathilde Bouvel**
LORIA
France

 **Pamela Harris**
Williams College
USA

Important Dates

SUBMISSION DEADLINE **APRIL 15, 2022**
EARLY REGISTRATION DEADLINE **MAY 1, 2022**
LATE REGISTRATION DEADLINE **JUNE 3, 2022**

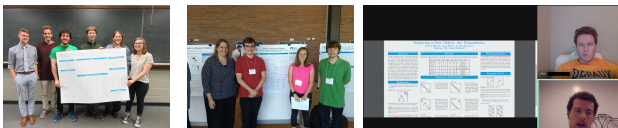
More information at permutationpatterns.com

The poster features a dark grey background with a yellow band for the plenary speakers. At the bottom, there is a dark image of a building and navigation icons.

Community: with students



Community: with students



Community: with students



Explore Your Future in Mathematics!

The **Valparaiso Experience in Research by Undergraduate Mathematicians (VERUM)** program seeks exceptional rising sophomore and junior students looking for a research experience in mathematical sciences. It's the perfect opportunity to determine whether graduate studies in the mathematical sciences should be part of your future plans. First generation college students, minority students, and women are particularly encouraged to apply.

Each participant will be provided with residence hall accommodations on campus, \$4,050 stipend, travel reimbursement to Valparaiso University for the summer, and partial travel reimbursement to the Joint Mathematics Meeting in January 2013.

Most projects are in combinatorics and mathematical biology, with additional projects selected from other areas of mathematics, statistics, and computer science. A complete list of current and past projects can be found on the program website.

Program Dates: May 30 – July 31, 2012

Program Highlights

- Learn from expert mathematicians.
- Participate in two undergraduate research conferences.
- Participate in the Joint Mathematics Meeting.
- Travel to area graduate schools.
- Take fun trips to Chicago and the Indiana Dunes Lakeshore.



Application Deadline: February 27, 2012

Applicant Requirement

- Must be a citizen or permanent resident of the United States or its possessions.
- Must be a full-time undergraduate student in 2012-2013.
- Must have completed linear algebra, or another proof-based course.



Valparaiso
University

Mathematics and
Computer Science

valpo.edu/mcs/verum



Community: with young thinkers

STUDENTS

PARENTS

ALUMNI

EMPLOYMENT

MathPath 2022 will return, **in person**, to **Mount Holyoke College – South Hadley, MA**.

Dates of the program: **June 26 – July 24**

Imagine yourself among others – students who like math as much as you – living, playing, and learning together for four weeks.

[Learn More](#)

Now hiring!

- [Counselors](#) (college math students wanted)
- [Faculty](#) (professors/teachers wanted)
- [Staff](#)

MathPath (mathpath.org)
a national residential summer camp for 11-14 year olds
showing high interest in mathematics.

Math in Action?

$$\begin{aligned}
 & 0^{2a-25a+28b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} \\
 & 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} \\
 & 0^{25a+28b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} \\
 & 0^{25a+28b-1} 1_0^{a-b-1} 1_0^{a-b-1} 2_0^{a-b-1} 1_0^{2a-2b-1} 1_0^{10a-11b-1} 1_0^{11a-12b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} \\
 & 0^{2a-2b-1} 1_0^{-24a+27b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} \\
 & 0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{27a-30b-1} 1_0^{-24a+27b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{10a-11b-1} \\
 & 0^{11a-12b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} \\
 & 0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{28a-31b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{-7a+8b-1} 1_0^{10a-11b-1} 1_0^{-7a+8b-1} 1_0^{10a-11b-1} \\
 & 0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{-8a+9b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{10a-11b-1} \\
 & 0^{a-b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} \\
 & 0^{a-b-1} 1_0^{9a-10b-1} 1_0^{-7a+8b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{9a-10b-1} 1_0^{a-b-1} 1_0^{9a-10b-1} 1_0^{a-b-1} 1_0^{9a-10b-1} \\
 & 0^{-7a+8b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} 1_0^{a-b-1} \\
 & 0^{-25a+28b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{-25a+28b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} \\
 & 0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} \\
 & 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} \\
 & 0^{3a-3b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} \\
 & 0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{a-b-1} 2_0^{a-b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} 1_0^{2a-2b-1} \\
 & 0^{11a-12b-1} 1_0^{-25a+28b-1} 1_0^{2a-2b-1} 1_0^{11a-12b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} \\
 & 0^{-25a+28b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{11a-12b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} \\
 & 0^{25a+28b-1} 1_0^{27a-30b-1} 1_0^{-24a+27b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} \\
 & 0^{2a-2b-1} 1_0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{-7a+8b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} \\
 & 0^{-25a+28b-1} 1_0^{28a-31b-1} 1_0^{-25a+28b-1} 1_0^{27a-30b-1} 1_0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} \\
 & 0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} \\
 & 0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{a-b-1} 1_0^{-26a+29b-1} 1_0^{28a-31b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} \\
 & 0^{a-b-1} 1_0^{9a-10b-1} 1_0^{-7a+8b-1} 1_0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} \\
 & 0^{2a-2b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} \\
 & 0^{10a-11b-1} 1_0^{-8a+9b-1} 1_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{-25a+28b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} 1_0^{27a-30b-1} \\
 & 0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} 1_0^{2a-2b-1} 2_0^{a-b-1} 1_0^{10a-11b-1} \\
 & 0^{a-b-1} 1_0^{-25a+28b-1} 1_0^{3a-3b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{3a-3b-1} 1_0^{-25a+28b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1} 1_0^{10a-11b-1}
 \end{aligned}$$

$$\bullet D(x, y) = \sum_{n \geq 1} \sum_{i=1}^n d_{n,i} x^i y^n,$$

$$\bullet C_2(y) = \sum_{n \geq 2} c_{n,2} y^n.$$

$c_{n,i} = c_{n,i-1} - d_{n,i-1}$ for $3 \leq i \leq n$ implies that

$$(1-x)C(x, y) + xD(x, y) = \frac{xy}{1-y} + x^2 C_2(y).$$

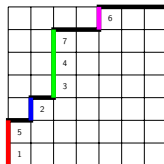
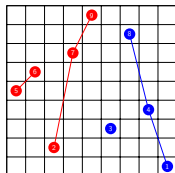
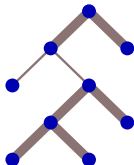
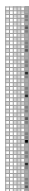
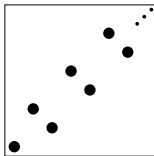
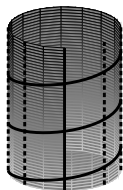
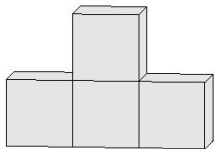
$d_{n,i} = d_{n-1,i} + c_{n-1,i-1}$ for $2 \leq i \leq n$ implies that

$$(1-y)D(x, y) - xyC(x, y) = xy.$$

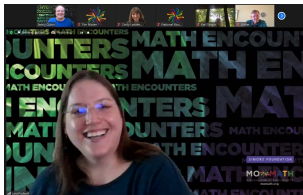
We also know that $c_{n,n} = 1$ for $n \geq 1$, which implies that

$$\begin{aligned}
 C\left(\frac{1}{y}, yz\right) \Big|_{y=0} &= \left(\sum_{n \geq 1} \sum_{i=1}^n c_{n,i} \left(\frac{1}{y}\right)^i (yz)^n \right) \Big|_{y=0} \\
 &= \left(\sum_{n \geq 1} \sum_{i=1}^n c_{n,i} y^{n-i} z^n \right) \Big|_{y=0} \\
 &= \sum_{n \geq 1} \sum_{i=1}^n c_{n,i} 0^{n-i} z^n \\
 &= \sum_{n \geq 1} c_{n,n} z^n \\
 &= \frac{z}{1-z}.
 \end{aligned}$$

Beautiful Structure



Beauty + Truth + Community = Joy



For more technical details...

- Lara Pudwell, Stacking blocks and counting permutations, *Mathematics Magazine* **83** (2010), 297–302.
- Lara Pudwell and Rebecca Smith, Two-stack-sorting with pop stacks, *Australasian Journal of Combinatorics* **74.1** (2019), 179–195.
- Lara Pudwell, From permutation patterns to the periodic table, *Notices of the American Mathematical Society* **67.7** (2020), 994–1001.
- Lara Pudwell, The hidden and surprising structure of ordered lists, *Math Horizons* **29.3** (February 2022), 5–7.

Thanks for listening!

slides at faculty.valpo.edu/lpudwell