What really counts: the joy of enumeration

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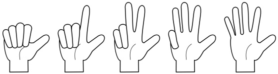
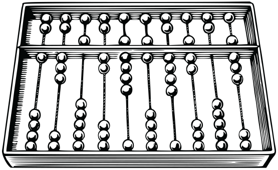
Professorial Lecture
April 14, 2022

What *is* counting (as research)?





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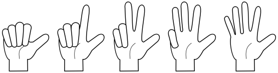

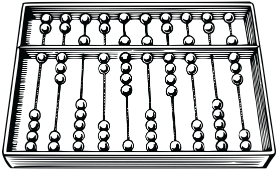

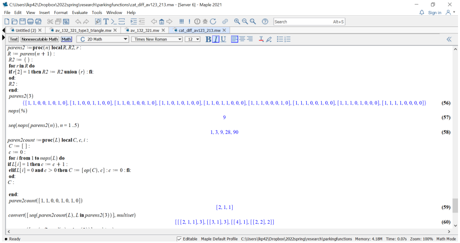

$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2+1)}{1-2x-x^2-2x^3}$$

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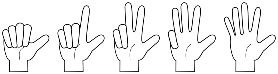

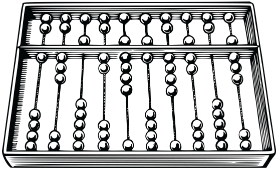

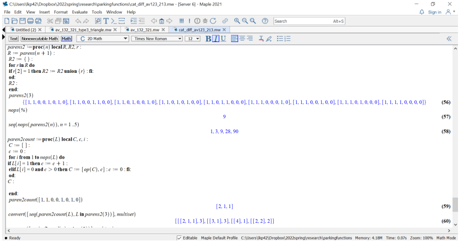

$$\sum_{\pi \in \mathcal{P}_2} x^{|\pi|} = \frac{x(x^2+1)}{1-2x-x^2-2x^3}$$

Goal: Answer "How many?" in an efficient, strategic way.

What to count?

REFEREED PUBLICATIONS

(* denotes undergraduate student)

37. Jonathan Beagley and Lara Pudwell, Colorful tilings and permutations, *Journal of Integer Sequences* **24** (2021) 21.10.4.
36. Lara Pudwell, From permutation patterns to the periodic table, *Notices of the American Mathematical Society* **67.7** (2020), 994–1001.
35. Lara Pudwell and Rebecca Smith, Two-stack-sorting with pop stacks, *Australasian Journal of Combinatorics* **74.1** (2019), 179–195.
34. Michael Bukata*, Ryan Kulwicki*, Nicholas Lewandowski*, Lara Pudwell, Jacob Roth*, and Teresa Wheeland*, Distributions of Statistics over Pattern-Avoiding Permutations, *Journal of Integer Sequences* **22** (2019) 19.2.6.

33. Monica Anderson*, Marika Diepenbroek*, Lara Pudwell, and Alex Stoll*, Pattern avoidance in reverse double lists, *Discrete Mathematics and Theoretical Computer Science* **19.2** (2018), #13.
32. Lara Pudwell and Eric Rowland, Avoiding fractional powers over the natural numbers, *Electronic Journal of Combinatorics* **25.2** (2018), P2.27.
31. Michael Dorff, Allison Henrich, and Lara Pudwell, Successfully Mentoring Undergraduates in Research: A How To Guide for Mathematicians, *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies*, **27.3** (2017), 320–336.
30. Charles Cratty*, Samuel Erickson*, Frehiwet Negassi*, and Lara Pudwell, Pattern avoidance in double lists, *Involve, a Journal of Mathematics* **10.3** (2017), 379–398.

A permutation of length n is an list of the numbers $1, 2, \ldots, n$ where order matters.

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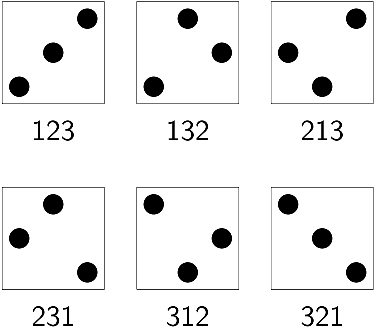
Fact

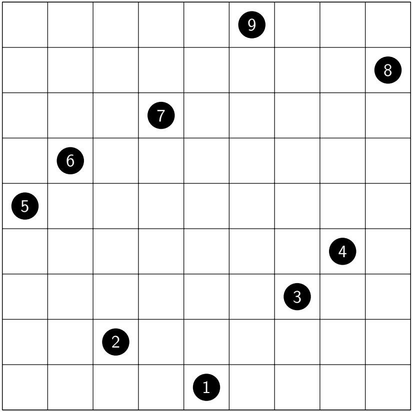
There are $n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$ permutations of length n.

Permutations in picture form

Visualize permutations by plotting the points in the plane.
(Heights correspond to digits in the permutation.)

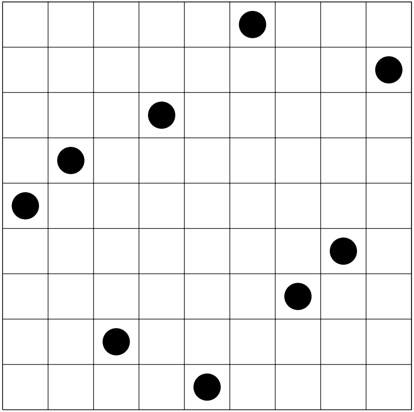


A bigger permutation picture



562719348

Patterns in permutations



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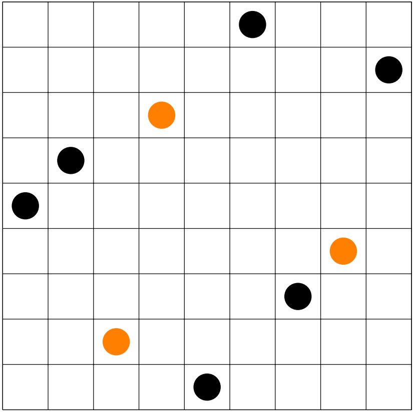

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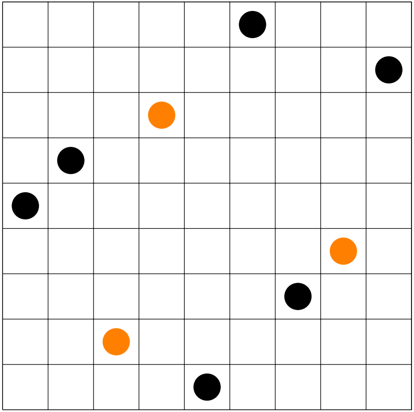
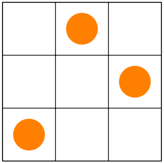
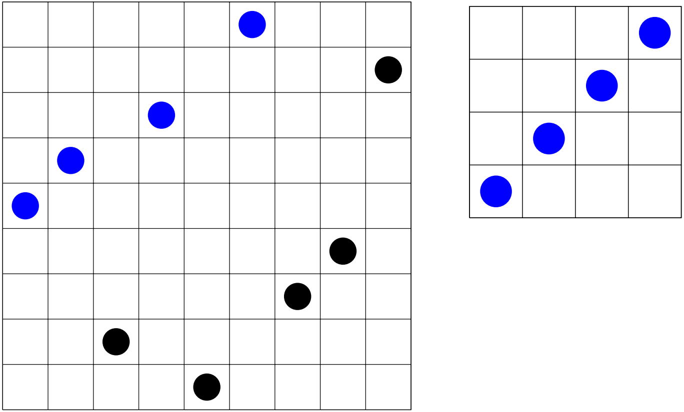




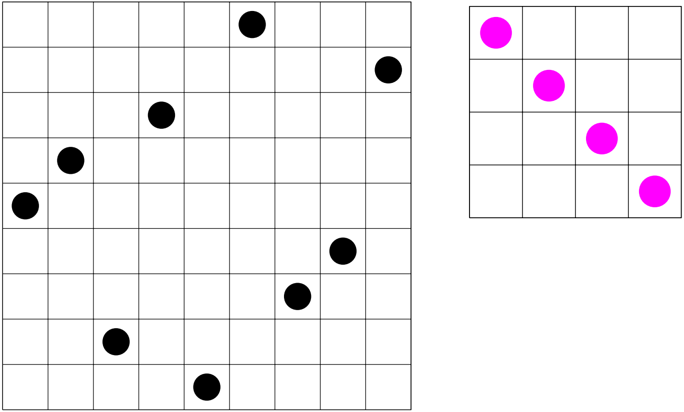
562719348 contains the pattern 132

Patterns in permutations



562719348 contains the pattern 1234

Patterns in permutations



562719348 avoids the pattern 4321

Big question

How many permutations of length n contain the permutation p?

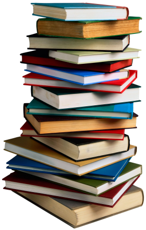
Or, alternatively...

Big question

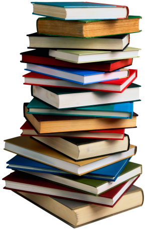
How many permutations of length n avoid the permutation p?

(depends on what p is!)

Why *avoid* patterns?



Why *avoid* patterns?



Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

What's my ~~application~~ *motivation*?

- Knuth (1968): avoiding 231 is useful in computer science
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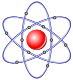


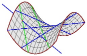

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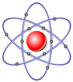
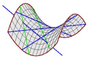


Theme: studying new mathematical structure for its own sake

Bonus: spotting surprise connections

Isn't that peculiar?

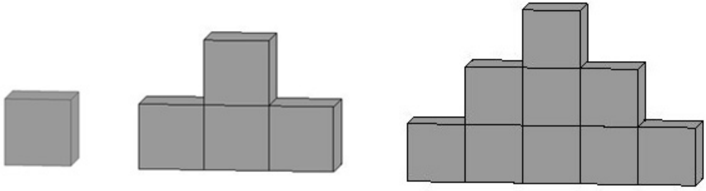
---------- Forwarded message ---------
From: **David Harris**
Date: Thu, Jun 4, 2009 at 9:24 PM
Subject: a series of numbers
To: <Lara.Pudwell@valpo.edu>

Hi-
I happened across a web site of yours at Rutgers website via a google search.(http://www.math.rutgers.edu/~lpudwell/maple/schemes/334out).

I got there because my Middle school aged daughter had a math project that generated the following series of numbers:
6 18 34 54 78 106 138 174
When I googled the series, only your site came up. We got the numbers from the following problem, which we're trying to find an equation to describe:

(This is a triangle series but with cubes).
We have a single cube(n=1) total surface area equals 6 sides.
If we add a cube on either side and one on top, forming a triangle surrounding the original (n=2) total surface area is 18 sides.

Pyramids of cubes



6 18 34

Theorem

The surface area of Harris's nth pyramid of blocks is...

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the number of lists with *two* 1s, *two* 2s, . . . , and *two* $(n+1)$s that avoid 132, 231, and 2134.

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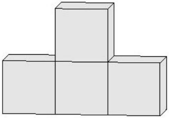
Example:

 has surface area 18.

18 pattern-avoiding lists:

112233, 121233, 122133, 211233, 212133, 221133
311223, 312123, 312213, 321123, 321213, 322113
331122, 331212, 331221, 332112, 332121, 332211

The next step is...

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A permutation is pop-stack sortable if and only if it avoids 231 and 312.

Theorem (West, 1990)

A permutation is sortable after two passes through a stack if and only if it avoids 2341 and $3\bar{5}241$.

Theorem (Pudwell and Smith, 2019)

A permutation is sortable after two passes through a pop-stack if and only if it avoids 2341, 3412, 3421, 4123, 4231, 4312, $4\bar{1}352$, and $413\bar{5}2$.

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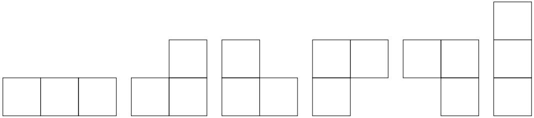

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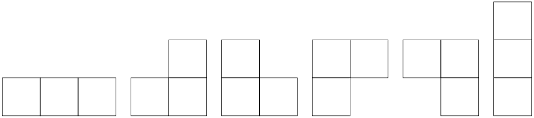

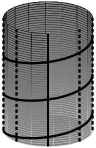

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that can be drawn on a "twisted cylinder" of width 3.

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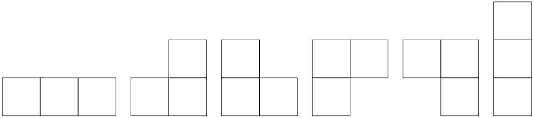

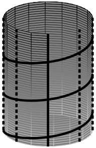

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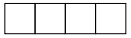
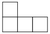
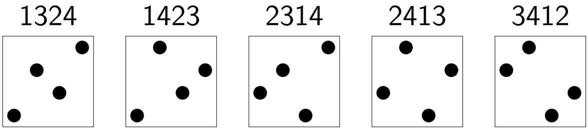
e.g.  and  are now the same

What if...?

A permutation is alternating if its adjacent pairs of digits alternate between increasing and decreasing pairs.

Examples:



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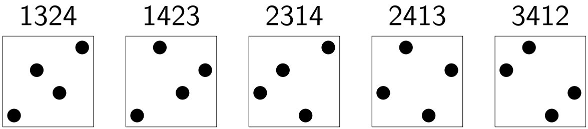
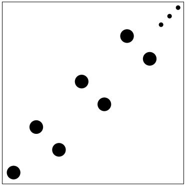


Which of these examples has the most 123 patterns?

Many 123s

The alternating permutation with the most 123s possible looks like this:



Question: How many 123 patterns does it have?

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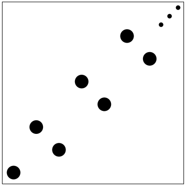

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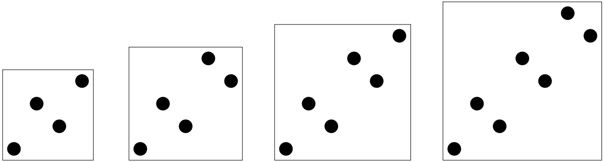

2 copies 4 copies 12 copies 20 copies

Many 123s

$$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$$

Many 123s

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, . . .

A099956	Atomic numbers of the alkaline earth metals. 9
	4, 12, 20, 38, 56, 88 (list; graph; refs; listen; history; text; internal format)
OFFSET	1,1
LINKS	Table of n, a(n) for n=1..6.
EXAMPLE	12 is the atomic number of magnesium.
CROSSREFS	Cf. A099955, alkali metals; A101648, metalloids; A101647, nonmetals (except halogens and noble gases); A097478, halogens; A018227, noble gases; A101649, poor metals. Sequence in context: A057317 A008068 A008183 * A301066 A008092 A316299 Adjacent sequences: A099953 A099954 A099955 * A099957 A099958 A099959
KEYWORD	nonn,fini,full
AUTHOR	Parthasarathy Nambi, Nov 12 2004
STATUS	approved

Online Encyclopedia of Integer Sequences (oeis.org)

Many 123s

$$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$$

A168380 Row sums of A168281. +20 14

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, 1140, 1340, 1540, 1782, 2024, 2312, 2600, 2938, 3276, 3668, 4060, 4510, 4960, 5472, 5984, 6562, 7140, 7788, 8436, 9158, 9880, 10680, 11480, 12362, 13244, 14212, 15180, 16238, 17296, 18448, 19600, 20850, 22100 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS The atomic numbers of the augmented alkaline earth group in Charles Janet's spiral periodic table are 0 and the first eight terms of this sequence (see Stewart reference). - Alonso del Arte, May 13 2011

LINKS Vincenzo Librandi, Table of n, a(n) for n = 1..10000
Stewart, Philip, Charles Janet: unrecognized genius of the Periodic System. Foundations of Chemistry (2010), p. 9.
Index entries for linear recurrences with constant coefficients, signature (2,1,-4,1,2,-1).

FORMULA
```
a(n) = 2*A005993(n-1).
a(n) = (n+1)*(3 + 2*n^2 + 4*n - 3*(-1)^n)/12.
a(n+1) - a(n) = A093907(n) = A137583(n+1).
a(2n+1) = A035597(n+1)          a(2n)=A002492(n).
a(n) = A099956(n-1), 2<=n<=7.
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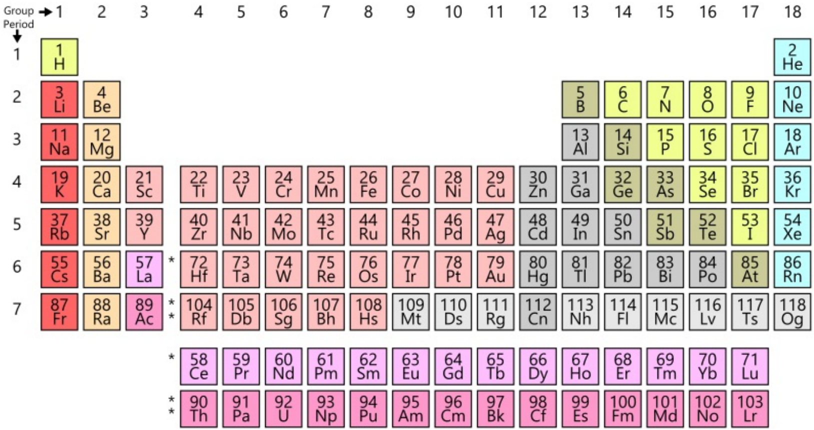
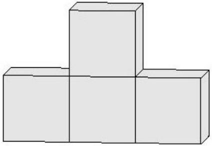
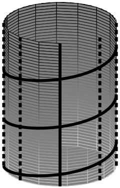
Online Encyclopedia of Integer Sequences (oeis.org)

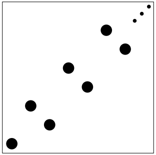
Alkaline Earth Metals (Group 2)

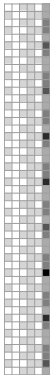


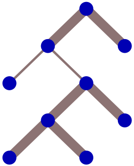
...and more!

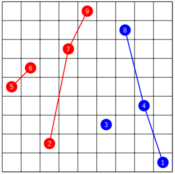


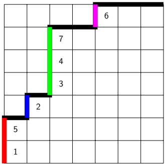












My (personal) motivation

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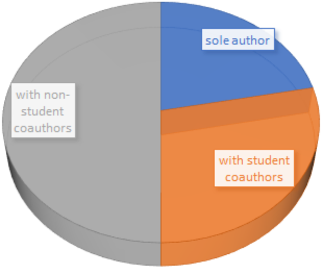
- Community

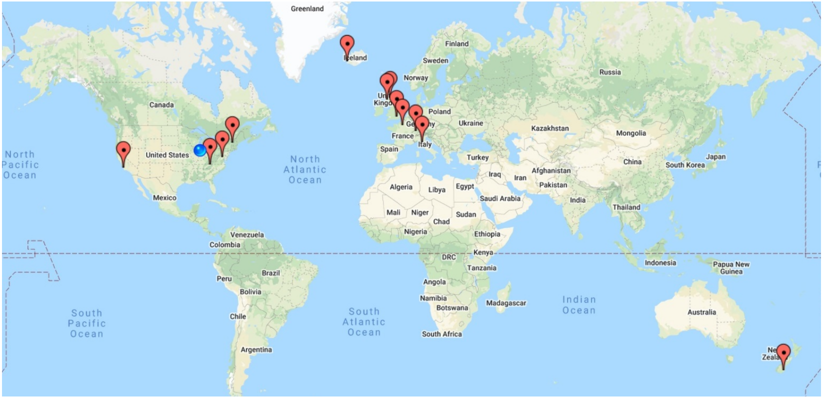
  (Sharing the (pursuit of) beauty and truth with others)



Community: with other researchers

Permutation Patterns 2019, Zurich, Switzerland

Community: with other researchers



Permutation Patterns Locations

Community: with other researchers

Permutation Patterns 2020

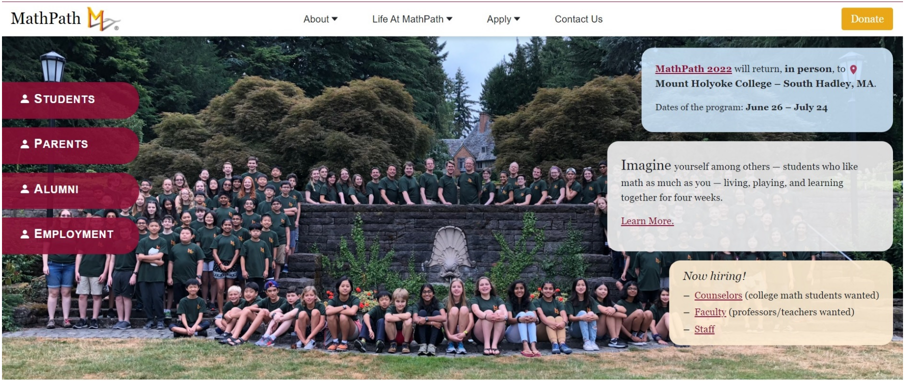
Community: with other researchers

Community: with students

Community: with students

Community: with students

Community: with young thinkers



MathPath (mathpath.org)
a national residential summer camp for 11-14 year olds
showing high interest in mathematics.

Math in Action?

$$0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1$$
$$0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{-25a+28b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1\,0^{11a-12b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{2a-2b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1$$
$$0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1$$
$$0^{11a-12b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1$$
$$0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1$$
$$0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1$$
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$$0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1$$
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$$0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1$$
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$$0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2$$
$$0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{a-b-1}\,2\,0^{a-b-1}\,1\,0^{2a-2b-1}\,1$$
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$$0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{-24a+27b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1\,0^{-7a+8b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{-25a+28b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1\,0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{a-b-1}\,1\,0^{-26a+29b-1}\,1\,0^{28a-31b-1}\,1\,0^{-25a+28b-1}\,1\,0^{10a-11b-1}\,1$$
$$0^{a-b-1}\,1\,0^{9a-10b-1}\,1\,0^{-7a+8b-1}\,1\,0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1$$
$$0^{2a-2b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1$$
$$0^{10a-11b-1}\,1\,0^{-8a+9b-1}\,1\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{-25a+28b-1}\,1\,0^{27a-30b-1}\,1$$
$$0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{2a-2b-1}\,2\,0^{a-b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{2a-2b-1}\,2$$
$$0^{a-b-1}\,1\,0^{-25a+28b-1}\,1\,0^{3a-3b-1}\,1\,0^{10a-11b-1}\,1\,0^{10a-11b-1}\,1\,0^{3a-3b-1}\,1\,0^{-25a+28b-1}\,1$$

- $D(x,y) = \sum_{n\geq 1}\sum_{i=1}^{n} d_{n,i}x^i y^n$,
- $C_2(y) = \sum_{n\geq 2} c_{n,2}y^n$.

$c_{n,i} = c_{n,i-1} - d_{n,i-1}$ for $3 \leq i \leq n$ implies that

$$(1-x)C(x,y) + xD(x,y) = \frac{xy}{1-y} + x^2C_2(y).$$

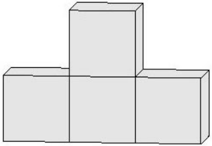
$d_{n,i} = d_{n-1,i} + c_{n-1,i-1}$ for $2 \leq i \leq n$ implies that

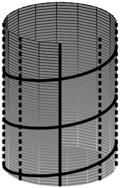
$$(1-y)D(x,y) - xyC(x,y) = xy.$$

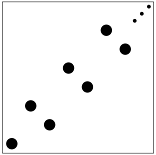
We also know that $c_{n,n} = 1$ for $n \geq 1$, which implies that

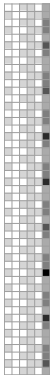
$$\begin{aligned} C\left(\frac{1}{y}, yz\right)\bigg|_{y=0} &= \left(\sum_{n\geq 1}\sum_{i=1}^{n} c_{n,i}\left(\frac{1}{y}\right)^i (yz)^n\right)\bigg|_{y=0} \\ &= \left(\sum_{n\geq 1}\sum_{i=1}^{n} c_{n,i}y^{n-i}z^n\right)\bigg|_{y=0} \\ &= \sum_{n\geq 1}\sum_{i=1}^{n} c_{n,i}0^{n-i}z^n \\ &= \sum_{n\geq 1} c_{n,n}z^n \\ &= \frac{z}{1-z}. \end{aligned}$$

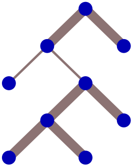
Beautiful Structure

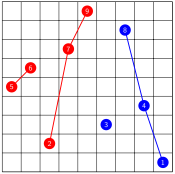


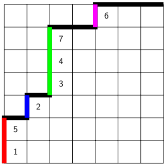




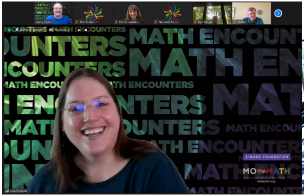




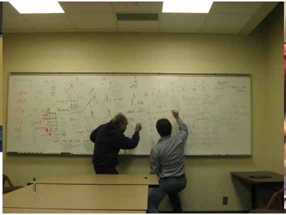




Beauty + Truth + Community = Joy






For more technical details...

- Lara Pudwell, Stacking blocks and counting permutations, *Mathematics Magazine* **83** (2010), 297–302.
- Lara Pudwell and Rebecca Smith, Two-stack-sorting with pop stacks, *Australasian Journal of Combinatorics* **74.1** (2019), 179–195.
- Lara Pudwell, From permutation patterns to the periodic table, *Notices of the American Mathematical Society* **67.7** (2020), 994–1001.
- Lara Pudwell, The hidden and surprising structure of ordered lists, *Math Horizons* **29.3** (February 2022), 5–7.

Thanks for listening!

slides at `faculty.valpo.edu/lpudwell`