Patterns in Permutations

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Definition

A permutation of length n is an ordered list of the numbers $1, 2, \ldots, n$.
$\mathcal{S}_n$ is the set of all permutations of length n.

$$\mathcal{S}_1 = \{1\}$$

$$\mathcal{S}_2 = \{12, 21\}$$

$$\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$$

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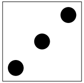
$$\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}$$

$$|\mathcal{S}_n| = n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$$

Note

Permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ is often visualized by plotting the points (i, π_i) in the Cartesian plane.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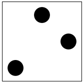
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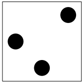
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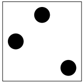
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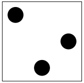
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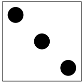
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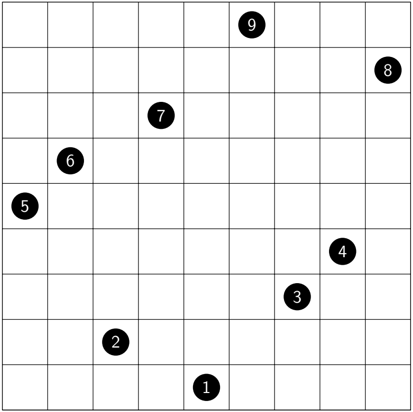
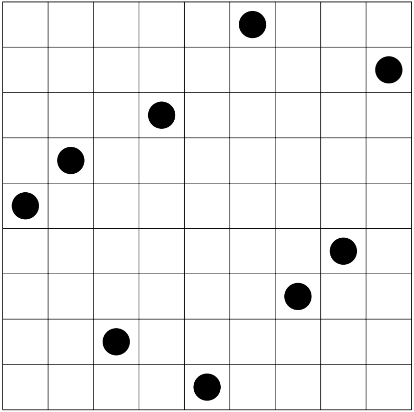
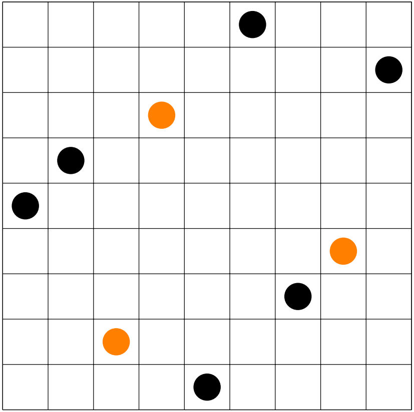
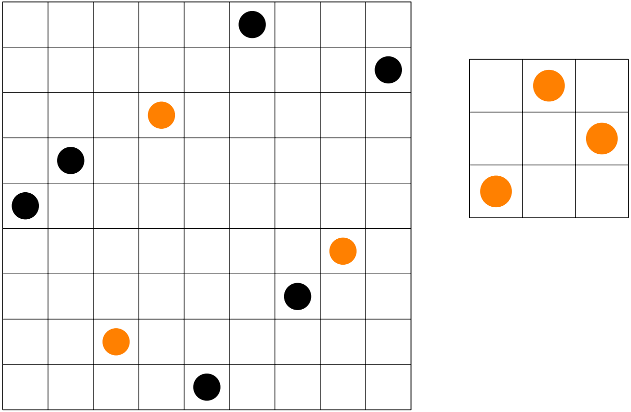


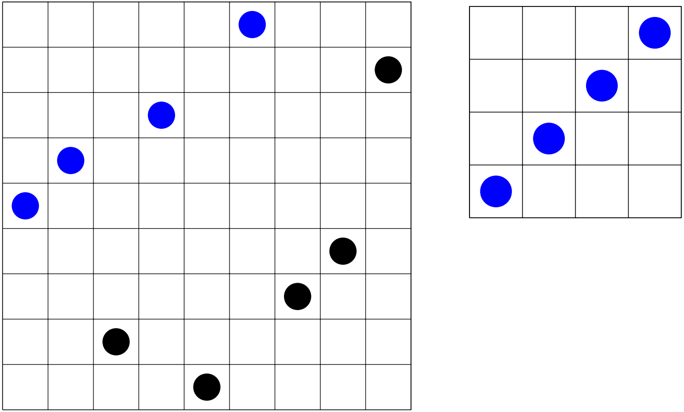
$$\pi = 562719348$$



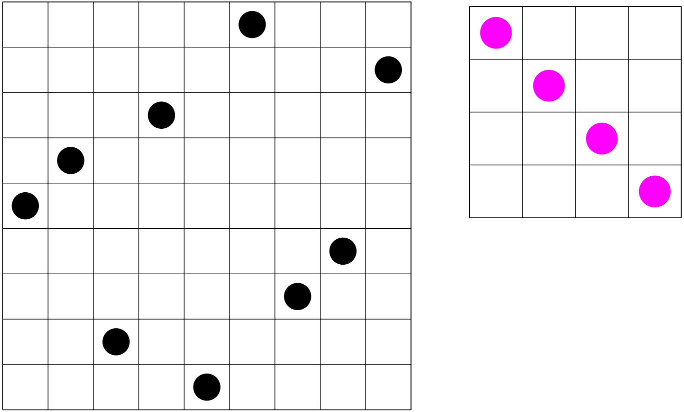




562719348 contains the pattern 132



562719348 contains the pattern 1234



562719348 avoids the pattern 4321

Big question

How many permutations of length n contain the permutation π?

Or, alternatively...

Big question

How many permutations of length n avoid the permutation π?

(depends on what π is!)

Question

How many permutations of length n avoid the permutation ?

Length 1?

Question

How many permutations of length n avoid the permutation ?

Length 1? (1)

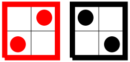
Length 2?

Question

How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



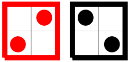
Length 3?

Question

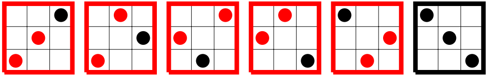
How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



Length 3? (1)

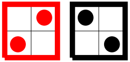


Question

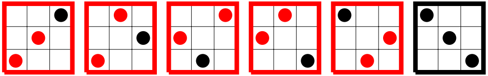
How many permutations of length *n* avoid the permutation ?

Length 1? (1)

Length 2? (1)



Length 3? (1)



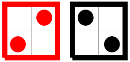
The decreasing permutation is the only permutation of length *n* that avoids 12.

Question

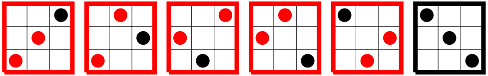
How many permutations of length n avoid the permutation ?

Length 1? (1)

Length 2? (1)



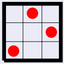
Length 3? (1)



The decreasing permutation is the only permutation of length n that avoids 12.

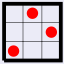
Similar: the increasing permutation is the only permutation of length n that avoids 21.

Question

How many permutations of length n avoid the permutation ?

Length 1?

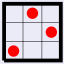
Question

How many permutations of length n avoid the permutation ?

Length 1? (1)

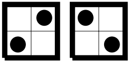
Length 2?

Question

How many permutations of length *n* avoid the permutation ?

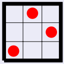
Length 1? (1)

Length 2? (2)



Length 3?

Question

How many permutations of length *n* avoid the permutation ?

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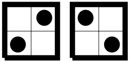
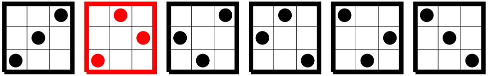
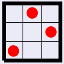
Length 2? (2)



Length 3? (5)

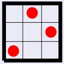


Question

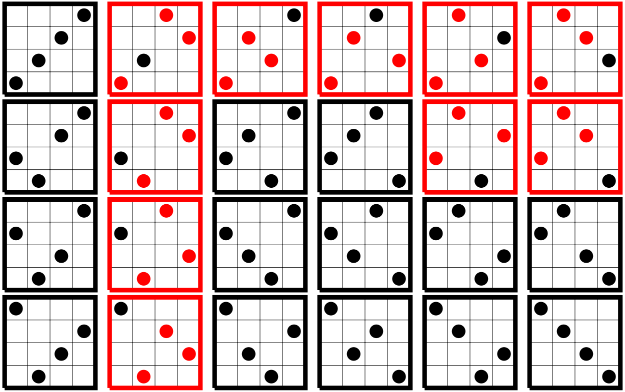
How many permutations of length n avoid the permutation ?

Length 4?

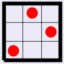
Question

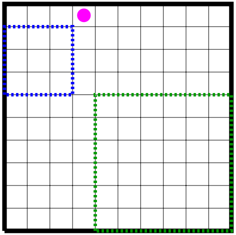
How many permutations of length n avoid the permutation ?

Length 4? (14)



Question

How many permutations of length n avoid the permutation ? (C_n)



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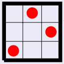
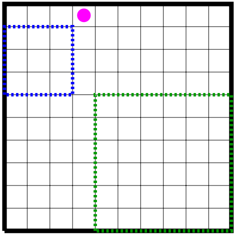
How many permutations of length n avoid the permutation ? (C_n)



Answer: $C_0 = 1$, $C_1 = 1$, and for larger n:

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0 = \sum_{i=1}^{n} C_{i-1} C_{n-i}$$

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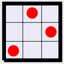
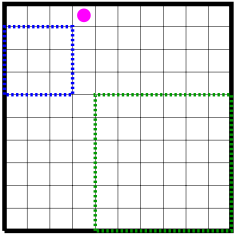
How many permutations of length n avoid the permutation ? (C_n)



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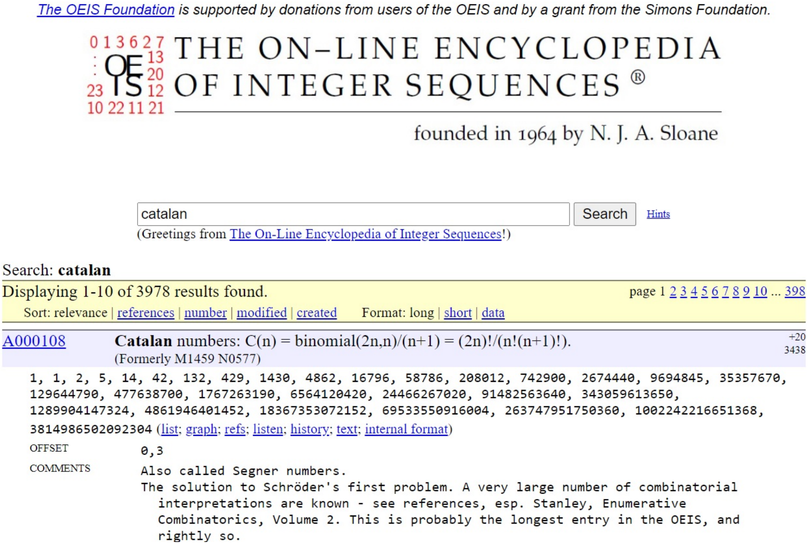
$1, 1, 2, 5, 14, 42, 132, \ldots$ (Catalan numbers!)



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catalan Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **catalan**

Displaying 1-10 of 3978 results found. page 1 2 3 4 5 6 7 8 9 10 ... 398

Sort: relevance | references | number | modified | created Format: long | short | data

A000108 **Catalan** numbers: C(n) = binomial(2n,n)/(n+1) = (2n)!/(n!(n+1)!). +20
(Formerly M1459 N0577) 3438

```
1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670,
129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650,
1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368,
3814986502092304
```

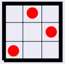
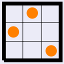
(list; graph; refs; listen; history; text; internal format)

OFFSET `0,3`

COMMENTS

```
Also called Segner numbers.
The solution to Schröder's first problem. A very large number of combinatorial
  interpretations are known - see references, esp. Stanley, Enumerative
  Combinatorics, Volume 2. This is probably the longest entry in the OEIS, and
  rightly so.
```

Question

How many permutations of length n avoid the permutations  and ?

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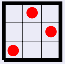
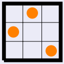
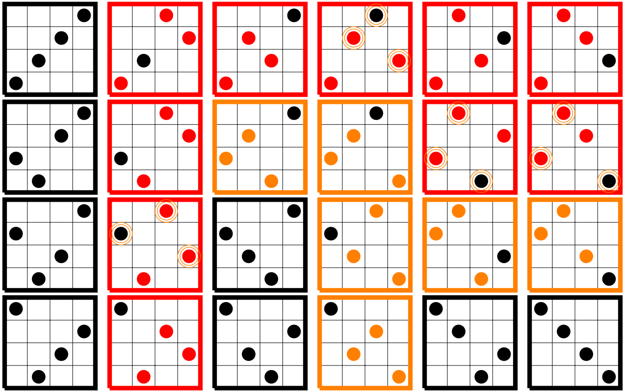
How many permutations of length n avoid the permutations  and ?



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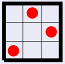
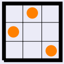
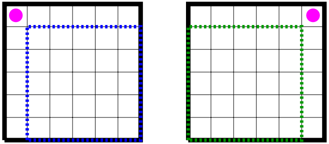
How many permutations of length n avoid the permutations  and ? (T_n)

or



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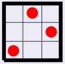
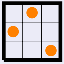
How many permutations of length n avoid the permutations  and ? (T_n)

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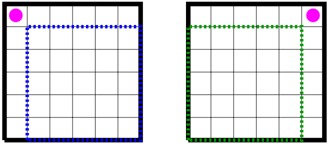
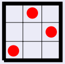
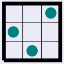
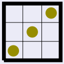


Answer: $T_1 = 1$ and $T_n = T_{n-1} + T_{n-1} = 2T_{n-1}$, so...

$$T_n = 2^{n-1}.$$

Question

How many permutations of length n avoid the permutations  and  and ?

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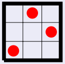
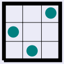
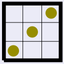
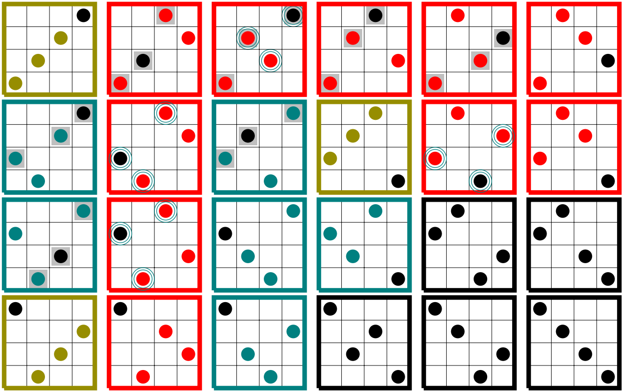
How many permutations of length n avoid the permutations  and  and ?



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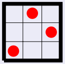
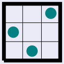
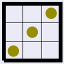
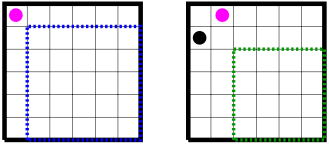
 and  and ? (F_n)

or



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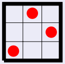
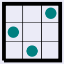
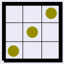
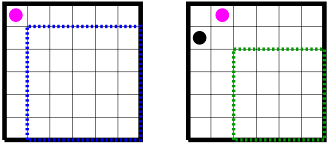
How many permutations of length n avoid the permutations  and  and ? (F_n)

or



Answer: $F_0 = F_1 = 1$ and for larger n,

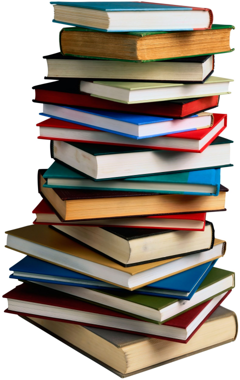
$$F_n = F_{n-1} + F_{n-2}.$$

$1, 1, 2, 3, 5, 8, 13, \ldots$ (Fibonacci numbers!)

How many permutations of length n avoid the pattern(s)...

- 12? 1
- 132? (Catalan)
- 132 and 231? 2^{n-1}
- 132 and 213 and 123? (Fibonacci)
- 1234? $1, 1, 2, 6, 23, 103, 513, 2761, \ldots$ (Gessel, 1990)
- 1342? $1, 1, 2, 6, 23, 103, 512, 2740, \ldots$ (Bóna, 1997)
- 1324? $1, 1, 2, 6, 23, 103, 513, 2762, \ldots$ (open question!)

Why *avoid* patterns?



Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Stack Operations

- *push* – first element of input to top of stack
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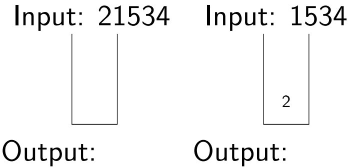


Input: 21534

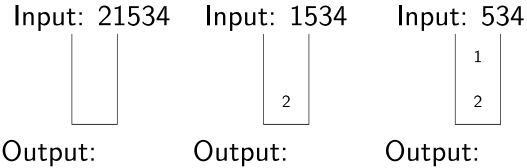
Input: 1534

Output:

Output:

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

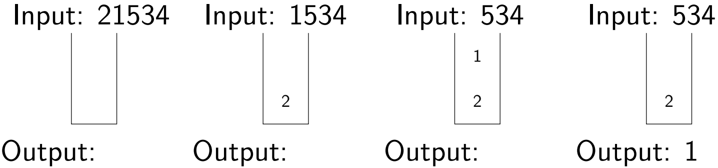


Input: 21534 Input: 1534 Input: 534

Output: Output: Output:

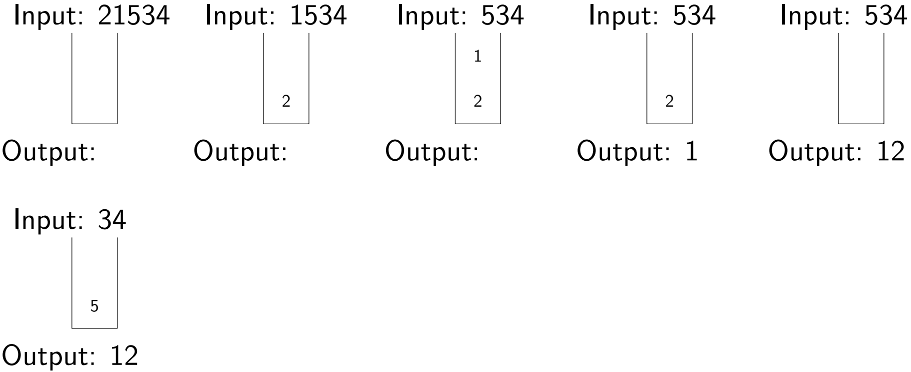
Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output



Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output



Input: 21534

Output:

Input: 1534

Output:

Input: 534

Output:

Input: 534

Output: 1

Input: 534

Output: 12

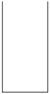
Input: 34

Output: 12

Stack Operations

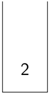
- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

Input: 21534



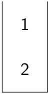
Output:

Input: 1534



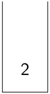
Output:

Input: 534



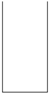
Output:

Input: 534



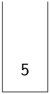
Output: 1

Input: 534



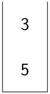
Output: 12

Input: 34



Output: 12

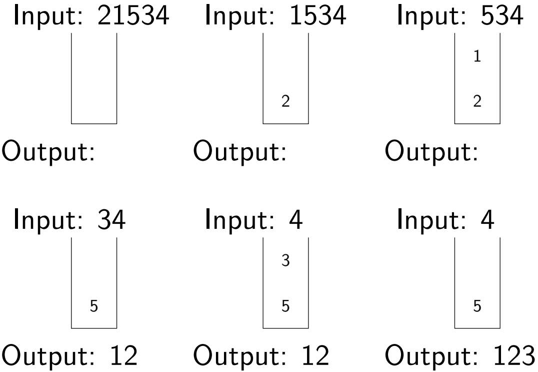
Input: 4



Output: 12

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output



Input: 21534

Output:

Input: 1534

Output:

Input: 534

Output:

Input: 534

Output: 1

Input: 534

Output: 12

Input: 34

Output: 12

Input: 4

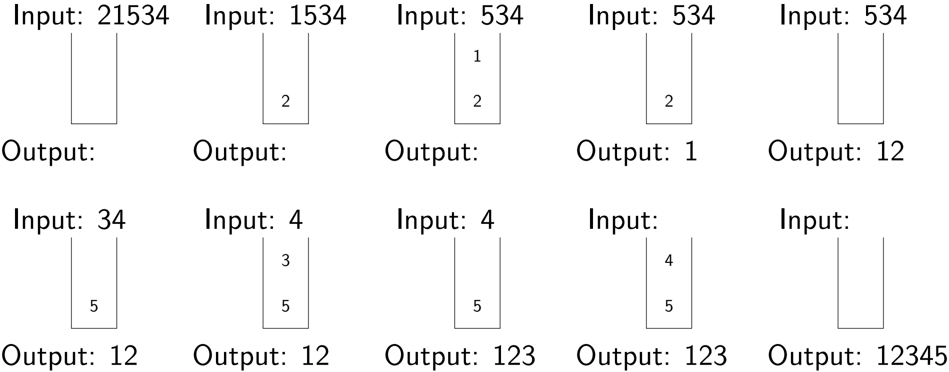
Output: 12

Input: 4

Output: 123

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output



Input: 21534
Output:

Input: 1534

Output:

Input: 534

Output:

Input: 534

Output: 1

Input: 534
Output: 12

Input: 34

Output: 12

Input: 4

Output: 12

Input: 4

Output: 123

Input:

Output: 123

Input:
Output: 12345

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

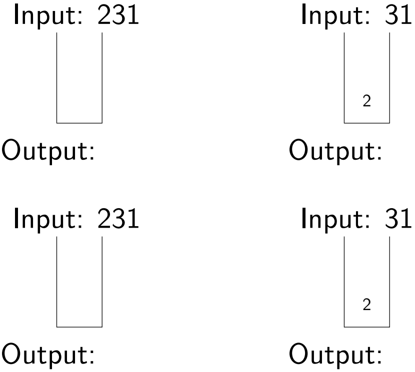
21534 can be sorted after one pass through a stack.

Can you find a permutation that *can't* be sorted after one pass through a stack?

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

What about 231?



Input: 231

Output:

Input: 31

2

Output:

Input: 231

Output:

Input: 31

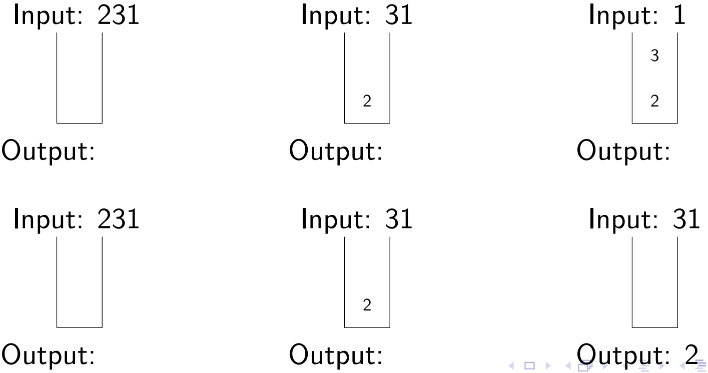
2

Output:

Stack Operations

- *push* – first element of input to top of stack
- *pop* – top element of stack to end of output

What about 231?



Input: 231

Output:

Input: 31

Output:

Input: 1

Output:

Input: 231

Output:

Input: 31

Output:

Input: 31

Output: 2

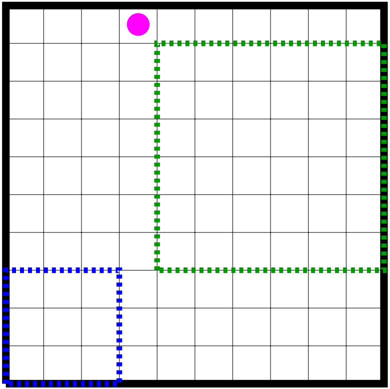
Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

Theorem (Knuth, 1968)

A permutation is stack sortable if and only if it avoids 231.

Proof sketch: (by induction)



Output? 🟦🟩🟪

Pattern Containment

How many permutations of $\mathcal{S}_3$ contain 12?

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How many times?

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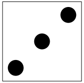
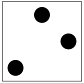
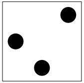


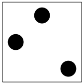
3 copies



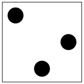
2 copies



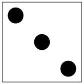
2 copies



1 copy



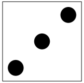
1 copy



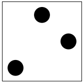
0 copies

Pattern Containment

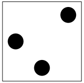
How many permutations of $\mathcal{S}_3$ contain 12?
How many times?



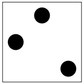
3 copies



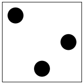
2 copies



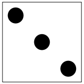
2 copies



1 copy



1 copy



0 copies

The maximum number of copies of 12 in a member of $\mathcal{S}_3$ is 3.

Alternating Permutations

A permutation $\pi = \pi_1 \cdots \pi_n$ is *alternating* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 \cdots$.

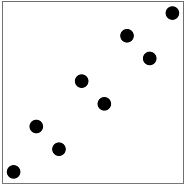
Examples:

1324 1423 2314 2413 3412

Interesting(?) Counting Question

What is the largest possible number of copies of 123 in π if π is alternating?

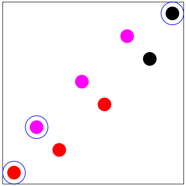
Packing 123

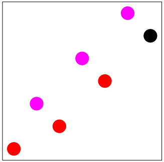


$1\ 32\ 54\ 76\cdots$ is the alternating permutation of length *n* with the most copies of 123.

Packing 123

n even vs. n odd





$\frac{n}{2} - 1$ layers of size 2

$\frac{n-1}{2}$ layers of size 2

Copies of 123 can use:

three layers of size 2	three layers of size 2
two layers of size 2	two layers of size 2
one layer of size 2	

Counting Sequences

Let $a(n)$ be the number of copies of 123 in $1\ 32\ 54\ 76\cdots$.

$$a(n) = \begin{cases} 2(\frac{n}{2}-1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$

Counting Sequences

Let $a(n)$ be the number of copies of 123 in $1\ 32\ 54\ 76 \cdots$.

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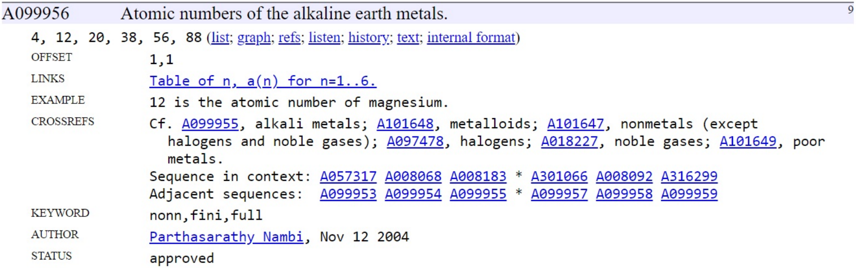


A099956 Atomic numbers of the alkaline earth metals. 9

4, 12, 20, 38, 56, 88 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

LINKS Table of n, a(n) for n=1..6.

EXAMPLE 12 is the atomic number of magnesium.

CROSSREFS Cf. A099955, alkali metals; A101648, metalloids; A101647, nonmetals (except halogens and noble gases); A097478, halogens; A018227, noble gases; A101649, poor metals.

Sequence in context: A057317 A008068 A008183 * A301066 A008092 A316299

Adjacent sequences: A099953 A099954 A099955 * A099957 A099958 A099959

KEYWORD nonn,fini,full

AUTHOR Parthasarathy Nambi, Nov 12 2004

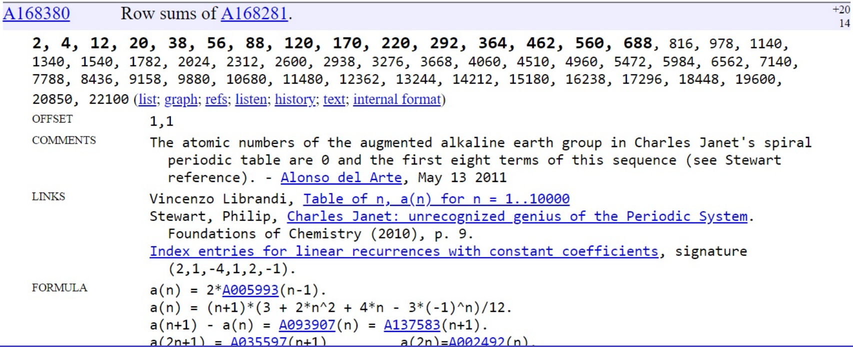
STATUS approved

Counting Sequences

Let $a(n)$ be the number of copies of 123 in $1\ 32\ 54\ 76 \cdots$.

$$a(n) = \begin{cases} 2(\frac{n}{2} - 1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, . . .



A168380 Row sums of A168281. +20 14

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, 1140, 1340, 1540, 1782, 2024, 2312, 2600, 2938, 3276, 3668, 4060, 4510, 4960, 5472, 5984, 6562, 7140, 7788, 8436, 9158, 9880, 10680, 11480, 12362, 13244, 14212, 15180, 16238, 17296, 18448, 19600, 20850, 22100 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS The atomic numbers of the augmented alkaline earth group in Charles Janet's spiral periodic table are 0 and the first eight terms of this sequence (see Stewart reference). - Alonso del Arte, May 13 2011

LINKS Vincenzo Librandi, Table of n, a(n) for n = 1..10000

Stewart, Philip, Charles Janet: unrecognized genius of the Periodic System. Foundations of Chemistry (2010), p. 9.

Index entries for linear recurrences with constant coefficients, signature (2,1,-4,1,2,-1).

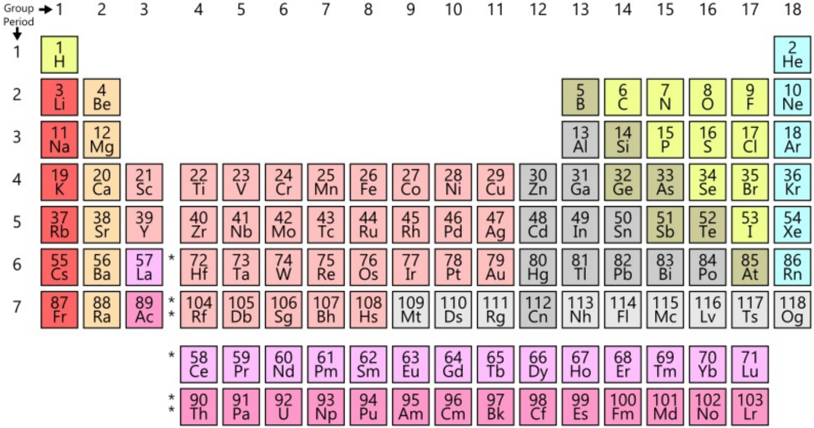
FORMULA a(n) = 2*A005993(n-1).

a(n) = (n+1)*(3 + 2*n^2 + 4*n - 3*(-1)^n)/12.

a(n+1) - a(n) = A093907(n) = A137583(n+1).

a(2n+1) = A035597(n+1)   a(2n)=A002492(n).

Alkaline Earth Metals (Group 2)



Group → Period ↓	1	2	3		4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc		22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y		40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	**	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				*	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				**	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

A little chemistry...

- *Quantum numbers* describe trajectories of electrons.
  - n (principal number) determines the electron shell

    $$n = 1, 2, 3, \dots$$

  - ℓ (orbital angular momentum) determines the shape of the orbital

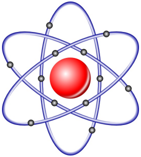
    $$0 \leq \ell \leq n-1$$

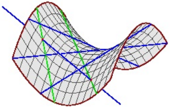
$\ell = 0$ $\ell = 1$ $\ell = 2$

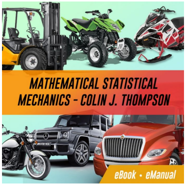
  - m (magnetic number) determines number of orbitals and orientation within shell

    $$-\ell \leq m \leq \ell$$

  - Two possible spin numbers for each choice of (n, ℓ, m)



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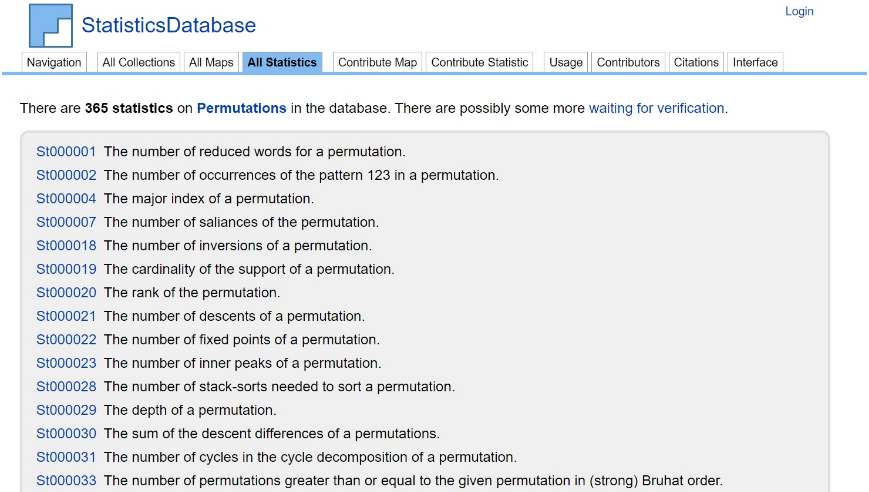

There are **365 statistics** on **Permutations** in the database. There are possibly some more waiting for verification.

St000001	The number of reduced words for a permutation.
St000002	The number of occurrences of the pattern 123 in a permutation.
St000004	The major index of a permutation.
St000007	The number of saliances of the permutation.
St000018	The number of inversions of a permutation.
St000019	The cardinality of the support of a permutation.
St000020	The rank of the permutation.
St000021	The number of descents of a permutation.
St000022	The number of fixed points of a permutation.
St000023	The number of inner peaks of a permutation.
St000028	The number of stack-sorts needed to sort a permutation.
St000029	The depth of a permutation.
St000030	The sum of the descent differences of a permutations.
St000031	The number of cycles in the cycle decomposition of a permutation.
St000033	The number of permutations greater than or equal to the given permutation in (strong) Bruhat order.

For further reading...

- Miklos Bóna, *Combinatorics of Permutations*, Chapman & Hall, 2004.
- Donald Knuth, *The Art of Computer Programming: Volume 1*, Addison Wesley, 1968.
- Lara Pudwell, From permutation patterns to the periodic table, *Notices of the American Mathematical Society*. **67.7** (2020), 994–1001.
- Lara Pudwell, The hidden and surprising structure of ordered lists, *Math Horizons*. **29.3** (February 2022), 5–7.
- The On-Line Encyclopedia of Integer Sequences at `oeis.org`.
- FindStat at `findstat.org`

Thanks for listening!

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