OEIS meets UGR

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DIMACS Conference on Challenges of Identifying Integer Sequences
DIMACS Center, Rutgers University
October 9, 2014

OEIS meets UGR?

Online
Encyclopedia of
Integer
Sequences

meets

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Online

Encyclopedia of

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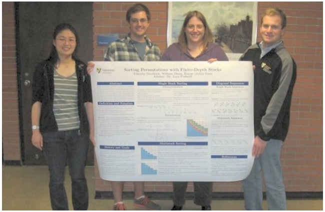
meets

Under-

Graduate

Research



Identifying Integer Sequences...

- $2, 4, 6, 8, 10, 12, 14, \ldots$
- $3, 9, 27, 81, 243, 729, \ldots$

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Undergraduates have a much smaller list of sequences they recognize – OEIS helps bridge the gap.

Definitions/Notation

- $\mathcal{S}_n$ is the set of permutations of length n.

$$\mathcal{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

- $\sigma_1 \cdots \sigma_n$ *contains* $\rho_1 \cdots \rho_m$ if there exist $1 \leq i_1 < \cdots < i_m \leq n$ so that $\sigma_{i_a} \leq \sigma_{i_b}$ iff $\rho_a \leq \rho_b$. Otherwise σ *avoids* ρ.

14235 contains 132. (e.g. $\boxed{1}\,\boxed{4}\,2\,\boxed{3}\,5$)
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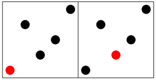
- $\mathcal{D}_n = \{\pi\pi \mid \pi \in \mathcal{S}_n\}$.
$\mathcal{D}_3 = \{123123, 132132, 213213, 231231, 312312, 321321\}$.

- $\mathcal{D}_n(\rho) = \{\sigma \mid \sigma \in \mathcal{D}_n \text{ and } \sigma \text{ avoids } \rho\}$.

Warmup

- $\mathcal{D}_n(1) = \emptyset$ for $n \geq 1$.
- $\mathcal{D}_n(12) = \mathcal{D}_n(21) = \emptyset$ for $n \geq 2$.

1423514235 $\rightarrow$ 

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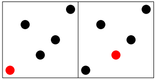
$1423514235 \rightarrow$ 

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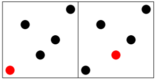
$1423514235 \rightarrow$ 

- $|\mathcal{D}_n(\rho)| = |\mathcal{D}_n(\rho^r)| = |\mathcal{D}_n(\rho^c)|$

- $\mathcal{D}_n(123) = \{n \cdots 1 n \cdots 1\}$ for $n \geq 3$.
- $$\mathcal{D}_n(132) = \begin{cases} \{11\} & n = 1 \\ \{1212, 2121\} & n = 2 \\ \{231231\} & n = 3 \\ \emptyset & n \geq 4 \end{cases}.$$

Length 4 Trivial Wilf Classes

Pattern ρ	$\{|\mathcal{D}_n(\rho)|\}_{n=1}^{10}$
1342, 2431, 3124, 4213	1, 2, 6, 12, 15, 15, 15, 15, 15, 15
2143, 3412	1, 2, 6, 12, 13, 14, 16, 18, 20, 22
1423, 2314, 3241, 4132	1, 2, 6, 12, 17, 23, 27, 30, 33, 36
1432, 2341, 3214, 4123	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
1243, 2134, 3421, 4312	1, 2, 6, 12, 19, 25, 34, 44, 55, 67
2413, 3142	1, 2, 6, 12, 18, 29, 47, 76, 123, 199
1324, 4231	1, 2, 6, 12, 21, 38, 69, 126, 232, 427
1234, 4321	1, 2, 6, 12, 27, 58, 121, 248, 503, 1014

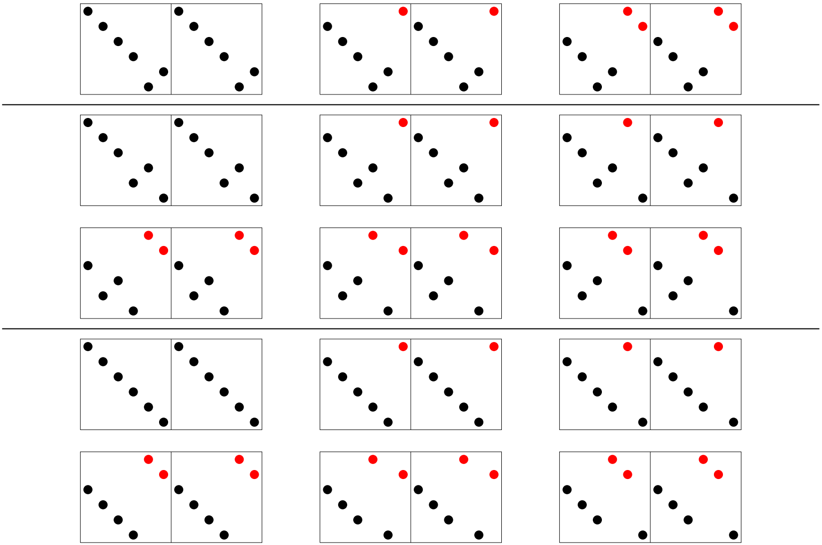
Contrast: For large n, $|\mathcal{S}_n(1342)| < |\mathcal{S}_n(1234)| < |\mathcal{S}_n(1324)|$.

$\mathcal{D}_n(1342)$

$(1, 2, 6, 12, 15, 15, 15, 15, 15, 15, \ldots)$

$\mathcal{D}_n(1342)$

$(1, 2, 6, 12, 15, 15, 15, 15, 15, 15, \ldots)$



$\mathcal{D}_n(1234)$

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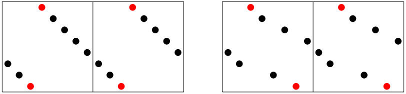
Picture of 1234-avoiding double lists:



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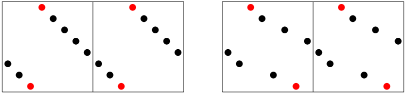

For $n \geq 4$, in OEIS, "Number of different permutations of a deck of n cards that can be produced by a single shuffle".

1. Begin with ordered deck $n \cdots 1$.
2. Cut.
3. Each card either comes from upper or lower partial deck.

$\mathcal{D}_n(1234)$

(1, 2, 6, 12, 27, 58, 121, 248, 503, 1014,...)

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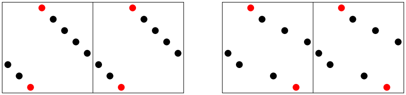

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1. Begin with ordered deck $n \cdots 1$.
2. Cut.
3. Each card either comes from upper or lower partial deck.

There are 2^n strings on $\{U, L\}^n$, but the $(n+1)$ decks of the form $U \cdots UL \cdots L$ are all equivalent to the original deck.

$$|\mathcal{D}_n(1234)| = 2^n - n \text{ for } n \geq 4.$$

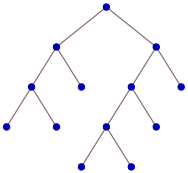
Summary

Pattern ρ	$\lvert\mathcal{D}_n(\rho)\rvert$ for sufficiently large n	OEIS
1342, 2431, 3124, 4213	15	A010854
2143, 3412	$2n+2$	A005843
1423, 2314, 3241, 4132	$3n+6$	A008585
1432, 2341, 3214, 4123	$\frac{1}{2}n^2+\frac{3}{2}n-4$	A052905
1243, 2134, 3421, 4312	$\frac{1}{2}n^2+\frac{5}{2}n-8$	(soon!)
2413, 3142	L_{n+2}	A000032
1324, 4231	(Tribonacci recurrence)	(soon!)
1234, 4321	2^n-n	A000325

Key Question

Our trees are:

- rooted (root vertex at top)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)



$\mathbb{T}_n$ is the set of n-leaf binary trees.

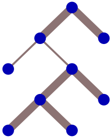
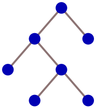
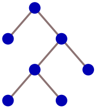
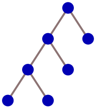
Question: How many trees in $\mathbb{T}_n$ avoid a given tree pattern?

Tree patterns

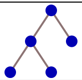
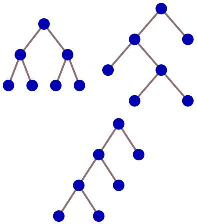
Noncontiguous tree pattern

Tree T contains tree t if and only if there exists a sequence of edge contractions of T (by pairs) that produces t.

Example:

 contains , , and .

Noncontiguous pattern data

Pattern t	Number of n-leaf trees avoiding t
	0
	$\begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$
	1
	2^{n-2}

The Main Theorem

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Notation

- Let $\mathrm{av}_t(n)$ be the number trees in $\mathbb{T}_n$ that avoid t noncontiguously.
- Let $g_t(x) = \sum_{n=1}^{\infty} \mathrm{av}_t(n)x^n$.

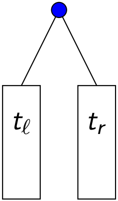
Theorem

Fix $k \in \mathbb{Z}^+$. Let $t, s \in \mathbb{T}_k$. Then $g_t(x) = g_s(x)$.

Notation and Computation

(More) Notation

- Given tree t,
  - let t_ℓ be the subtree whose root is the left child of t's root.
  - let t_r be the subtree whose root is the right child of t's root.



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Notice

$$g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)$$

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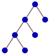
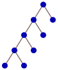
Solving...

$$g_t(x) = \frac{x - g_{t_\ell}(x) \cdot g_{t_r}(x)}{1 - g_{t_\ell}(x) - g_{t_r}(x)}.$$

A special case...

Let c_k be the k-leaf left comb
(the unique k-leaf binary tree where every right child is a leaf).

$c_1 =$ , $c_2 =$ , $c_3 =$ , $c_4 =$ , $c_5 =$ , etc.

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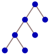
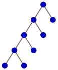
$c_1 = \bullet$, $c_2 =$ , $c_3 =$ , $c_4 =$ , $c_5 =$ ,etc.

For $k \geq 2$, we have

$$g_{c_k}(x) = \frac{x - g_{c_{k-1}}(x) \cdot g_{\bullet}(x)}{1 - g_{c_{k-1}}(x) - g_{\bullet}(x)} = \frac{x}{1 - g_{c_{k-1}}(x)}.$$

Back to the main result

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Theorem

Fix $k \in \mathbb{Z}^+$. Let t and s be two k-leaf binary tree patterns. Then $g_t(x) = g_s(x)$.

Proof sketch

Inductive step:

- Assume the theorem holds for tree patterns with j leaves where $j < k$.
- Any j-leaf tree has avoidance generating function $g_{c_j}(x)$.
- Consider tree t with j leaves to the left of its root and tree s with $j-1$ leaves to the left of its root.
- Do algebra with previous work to show that $g_t(x) = g_s(x)$.

Generating functions

k	$g_{c_k}(x)$	OEIS number
1	0	trivial
2	x	trivial
3	$\frac{x}{1-x}$	trivial
4	$\frac{x-x^2}{1-2x}$	A000079
5	$\frac{x-2x^2}{1-3x+x^2}$	A001519
6	$\frac{x-3x^2+x^3}{1-4x+3x^2}$	A007051
7	$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$	A080937
8	$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$	A024175
9	$\frac{x-6x^2+10x^3-4x^4}{1-7x+15x^2-10x^3+x^4}$	A080938

Coefficient sightings...

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

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Coefficient sightings...

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

$$\frac{x}{1}$$

$$\frac{x}{1-x}$$

$$\frac{x-x^2}{1-2x}$$

$$\frac{x-2x^2}{1-3x+x^2}$$

$$\frac{x-3x^2+x^3}{1-4x+3x^2}$$

$$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$$

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$$\frac{x-6x^2+10x^3-4x^4}{1-7x+15x^2-10x^3+x^4}$$

An explicit formula

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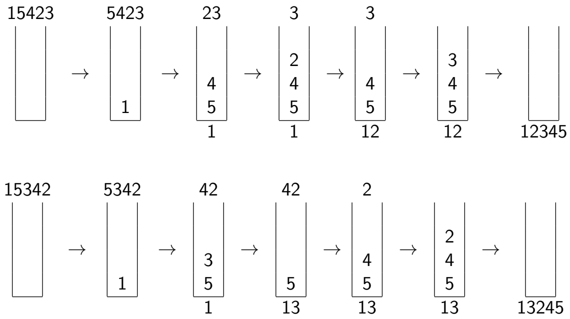
Theorem

Let $k \in \mathbb{Z}^+$ and let $t \in \mathbb{T}_k$. Then

$$g_t(x) = \frac{\displaystyle\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot \binom{k-(i+2)}{i} \cdot x^{i+1}}{\displaystyle\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot \binom{k-(i+1)}{i} \cdot x^i}.$$

Stack sorting



Stack sorting

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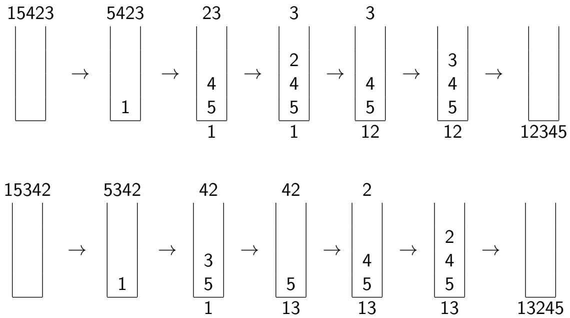
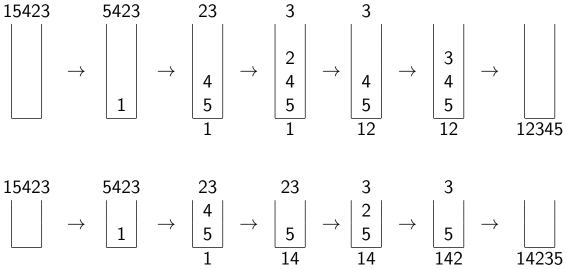


Theorem (Knuth)

$\pi \in \mathcal{S}_n$ is 1-stack-sortable if and only if π avoids the pattern 231.

Finite stacks

Question: What if the stack has a finite capacity d?



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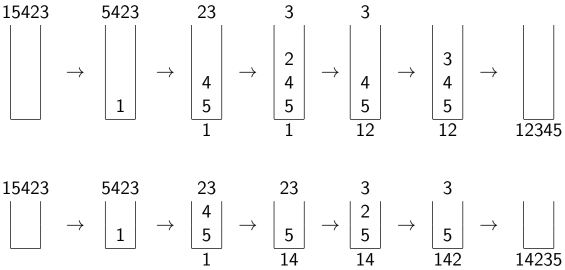

Theorem (Atkinson, Chow, West,...)

$\pi \in \mathcal{S}_n$ is 1-stack-sortable in a depth d ($d \geq 1$) stack if and only if π avoids the patterns 231 and $(d+1)d\cdots 1$.

Generating functions

- $a_d(n)$ is the number of permutations of length n sortable in a depth d stack.
- $f_d(x) = \sum_{n\geq 1} a_d(n)x^n$.

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7	$\frac{1-6x+10x^2-4x^3}{1-7x+15x^2-10x^3+x^4}$	A080938

Generating functions

- $a_d(n)$ is the number of permutations of length n sortable in a depth d stack.
- $f_d(x) = \sum_{n\geq 1} a_d(n)x^n = \frac{g_{c_{d+2}}(x)}{x}$.

d	$f_d(x)$	OEIS number
1	$\frac{1}{1-x}$	trivial
2	$\frac{1-x}{1-2x}$	A000079
3	$\frac{1-2x}{1-3x+x^2}$	A001519
4	$\frac{1-3x+x^2}{1-4x+3x^2}$	A007051
5	$\frac{1-4x+3x^2}{1-5x+6x^2-x^3}$	A080937
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- you, for listening!

(slides at http://faculty.valpo.edu/lpudwell)