OEIS meets UGR

Lara Pudwell

faculty.valpo.edu/lpudwell

DIMACS Conference on Challenges of Identifying Integer Sequences
DIMACS Center, Rutgers University
October 9, 2014
OEIS meets UGR?

Online Encyclopedia of Integer Sequences meets UGR
OEIS meets UGR?

Online Encyclopedia of Integer Sequences meets Undergraduate Research
Identifying Integer Sequences...

- 2, 4, 6, 8, 10, 12, 14, …
- 3, 9, 27, 81, 243, 729, …

Undergraduates have a much smaller list of sequences they recognize – OEIS helps bridge the gap.
Identifying Integer Sequences...

- 2, 4, 6, 8, 10, 12, 14, ...
- 3, 9, 27, 81, 243, 729, ...
- 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- 2, 1, 3, 4, 7, 11, 18, 29, 47, ...

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- 1, 2, 4, 9, 21, 51, 127, 323, 835, ...
- 1, 2, 6, 22, 90, 394, 1806, 8558, ...
- 1, 2, 5, 15, 52, 203, 877, 4140, ...
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Undergraduates have a much smaller list of sequences they recognize – OEIS helps bridge the gap.
Definitions/Notation

- $S_n$ is the set of permutations of length $n$.

\[ S_3 = \{123, 132, 213, 231, 312, 321\} \]

- $\sigma_1 \cdots \sigma_n$ contains $\rho_1 \cdots \rho_m$ if there exist $1 \leq i_1 < \cdots < i_m \leq n$ so that $\sigma_{i_a} \leq \sigma_{i_b}$ iff $\rho_a \leq \rho_b$. Otherwise $\sigma$ avoids $\rho$.

14235 contains 132. (e.g. 14\textcolor{red}{2}35)
14235 avoids 4321.
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  14235 contains 132. (e.g. 14235)

  14235 avoids 4321.

- $D_n = \{\pi \pi \mid \pi \in S_n\}$.
  
  $D_3 = \{123123, 132132, 213213, 231231, 312312, 321321\}$.

- $D_n(\rho) = \{\sigma \mid \sigma \in D_n \text{ and } \sigma \text{ avoids } \rho\}$. 

Warmup

- $D_n(1) = \emptyset$ for $n \geq 1$.
- $D_n(12) = D_n(21) = \emptyset$ for $n \geq 2$.

1423514235 →
Warmup

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\[ 1423514235 \rightarrow \]

- $|D_n(\rho)| = |D_n(\rho^r)| = |D_n(\rho^c)|$
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- $\mathcal{D}_n(1) = \emptyset$ for $n \geq 1$.
- $\mathcal{D}_n(12) = \mathcal{D}_n(21) = \emptyset$ for $n \geq 2$.

1423514235 $\rightarrow$

- $|\mathcal{D}_n(\rho)| = |\mathcal{D}_n(\rho^r)| = |\mathcal{D}_n(\rho^c)|$
- $\mathcal{D}_n(123) = \{n \cdots 1 n \cdots 1\}$ for $n \geq 3$.
- $\mathcal{D}_n(132) = \begin{cases} 
\{11\} & n = 1 \\
\{1212, 2121\} & n = 2 \\
\{231231\} & n = 3 \\
\emptyset & n \geq 4
\end{cases}$
Length 4 Trivial Wilf Classes

| Pattern $\rho$ | $\{ |D_n(\rho)| \}^{10}_{n=1}$ |
|---------------|----------------------------------|
| 1342, 2431, 3124, 4213 | 1, 2, 6, 12, 15, 15, 15, 15, 15, 15 |
| 2143, 3412 | 1, 2, 6, 12, 13, 14, 16, 18, 20, 22 |
| 1423, 2314, 3241, 4132 | 1, 2, 6, 12, 17, 23, 27, 30, 33, 36 |
| 1432, 2341, 3214, 4123 | 1, 2, 6, 12, 17, 23, 31, 40, 50, 61 |
| 1243, 2134, 3421, 4312 | 1, 2, 6, 12, 19, 25, 34, 44, 55, 67 |
| 2413, 3142 | 1, 2, 6, 12, 18, 29, 47, 76, 123, 199 |
| 1324, 4231 | 1, 2, 6, 12, 21, 38, 69, 126, 232, 427 |
| 1234, 4321 | 1, 2, 6, 12, 27, 58, 121, 248, 503, 1014 |

Contrast: For large $n$, $|S_n(1342)| < |S_n(1234)| < |S_n(1324)|$. 
\[ D_n(1342) \]
\[
(1, 2, 6, 12, 15, 15, 15, 15, 15, 15, \ldots)
\]
$D_n(1342)$

$(1, 2, 6, 12, 15, 15, 15, 15, 15, 15, \ldots)$
$D_n(1234)$

$(1, 2, 6, 12, 27, 58, 121, 248, 503, 1014, \ldots)$

Picture of 1234-avoiding double lists:
$D_n(1234)$

(1, 2, 6, 12, 27, 58, 121, 248, 503, 1014, ...)

Picture of 1234-avoiding double lists:

For $n \geq 4$, in OEIS, “Number of different permutations of a deck of $n$ cards that can be produced by a single shuffle”.

1. Begin with ordered deck $n \cdots 1$.
2. Cut.
3. Each card either comes from upper or lower partial deck.
$D_n(1234)$

(1, 2, 6, 12, 27, 58, 121, 248, 503, 1014, \ldots)

Picture of 1234-avoiding double lists:

For $n \geq 4$, in OEIS, “Number of different permutations of a deck of $n$ cards that can be produced by a single shuffle”.

1. Begin with ordered deck $n \cdots 1$.

2. Cut.

3. Each card either comes from upper or lower partial deck.

There are $2^n$ strings on $\{U, L\}^n$, but the $(n + 1)$ decks of the form $U \cdots UL \cdots L$ are all equivalent to the original deck.

$$|D_n(1234)| = 2^n - n \text{ for } n \geq 4.$$
## Summary

| Pattern $\rho$ | $|\mathcal{D}_n(\rho)|$ for sufficiently large $n$ | OEIS          |
|---------------|--------------------------------------------------|---------------|
| 1342, 2431, 3124, 4213 | 15                                              | A010854       |
| 2143, 3412   | $2n + 2$                                         | A005843       |
| 1423, 2314, 3241, 4132 | $3n + 6$                                        | A008585       |
| 1432, 2341, 3214, 4123 | $\frac{1}{2}n^2 + \frac{3}{2}n - 4$            | A052905       |
| 1243, 2134, 3421, 4312 | $\frac{1}{2}n^2 + \frac{5}{2}n - 8$            | (soon!)       |
| 2413, 3142   | $L_{n+2}$                                       | A000032       |
| 1324, 4231   | (Tribonacci recurrence)                         | (soon!)       |
| 1234, 4321   | $2^n - n$                                        | A000325       |
Key Question

Our trees are:
- rooted (root vertex at top)
- ordered (left child and right child are distinct)
- full binary (each vertex has exactly 0 or 2 children)

$\mathbb{T}_n$ is the set of $n$-leaf binary trees.

Question: How many trees in $\mathbb{T}_n$ avoid a given tree pattern?
Tree patterns

Noncontiguous tree pattern

Tree $T$ contains tree $t$ if and only if there exists a sequence of edge contractions of $T$ (by pairs) that produces $t$.

Example:

contains , , , and .
### Noncontiguous pattern data

<table>
<thead>
<tr>
<th>Pattern $t$</th>
<th>Number of $n$-leaf trees avoiding $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Pattern" /></td>
<td>0</td>
</tr>
<tr>
<td><img src="image2" alt="Pattern" /></td>
<td>$\begin{cases} 0 &amp; n &gt; 1 \ 1 &amp; n = 1 \end{cases}$</td>
</tr>
<tr>
<td><img src="image3" alt="Pattern" /></td>
<td>1</td>
</tr>
<tr>
<td><img src="image4" alt="Pattern" /></td>
<td>$2^{n-2}$</td>
</tr>
</tbody>
</table>
The Main Theorem

Notation

- Let $a_v t(n)$ be the number trees in $\mathbb{T}_n$ that avoid $t$ noncontiguously.
- Let $g_t(x) = \sum_{n=1}^{\infty} a_v t(n)x^n$.

Theorem

Fix $k \in \mathbb{Z}^+$. Let $t, s \in \mathbb{T}_k$. Then $g_t(x) = g_s(x)$. 
Given tree $t$,
- let $t_\ell$ be the subtree whose root is the left child of $t$’s root.
- let $t_r$ be the subtree whose root is the right child of $t$’s root.
(More) Notation

- Given tree $t$,
  - let $t_\ell$ be the subtree whose root is the left child of $t$’s root.
  - let $t_r$ be the subtree whose root is the right child of $t$’s root.

Notice

$$g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)$$
Given tree \( t \),
- let \( t_\ell \) be the subtree whose root is the left child of \( t \)'s root.
- let \( t_r \) be the subtree whose root is the right child of \( t \)'s root.

Notice

\[
g_t(x) = x + g_{t_\ell}(x) \cdot g_t(x) + g_t(x) \cdot g_{t_r}(x) - g_{t_\ell}(x) \cdot g_{t_r}(x)
\]

Solving...

\[
g_t(x) = \frac{x - g_{t_\ell}(x) \cdot g_{t_r}(x)}{1 - g_{t_\ell}(x) - g_{t_r}(x)}.
\]
A special case...

Let $c_k$ be the $k$-leaf left comb (the unique $k$-leaf binary tree where every right child is a leaf).

$c_1 = \cdot$, $c_2 = \wedge$, $c_3 = \text{tree with one leaf}$, $c_4 = \text{tree with two leaves}$, $c_5 = \text{tree with three leaves}$, etc.
A special case...

Let $c_k$ be the $k$-leaf left comb (the unique $k$-leaf binary tree where every right child is a leaf).

$c_1 = \cdot$, $c_2 = \ast$, $c_3 = \ast \ast$, $c_4 = \ast \ast \ast$, $c_5 = \ast \ast \ast \ast$, etc.

For $k \geq 2$, we have

$$g_{c_k}(x) = \frac{x - g_{c_{k-1}}(x) \cdot g_\ast(x)}{1 - g_{c_{k-1}}(x) - g_\ast(x)} = \frac{x}{1 - g_{c_{k-1}}(x)}.$$
Theorem

Fix \( k \in \mathbb{Z}^+ \). Let \( t \) and \( s \) be two \( k \)-leaf binary tree patterns. Then \( g_t(x) = g_s(x) \).

Proof sketch

Inductive step:

- Assume the theorem holds for tree patterns with \( j \) leaves where \( j < k \).
- Any \( j \)-leaf tree has avoidance generating function \( g_{c_j}(x) \).
- Consider tree \( t \) with \( j \) leaves to the left of its root and tree \( s \) with \( j - 1 \) leaves to the left of its root.
- Do algebra with previous work to show that \( g_t(x) = g_s(x) \).
## Generating functions

<table>
<thead>
<tr>
<th>$k$</th>
<th>$g_{c_k}(x)$</th>
<th>OEIS number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>trivial</td>
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<tr>
<td>2</td>
<td>$x$</td>
<td>trivial</td>
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<tr>
<td>3</td>
<td>$\frac{x}{1-x}$</td>
<td>trivial</td>
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<tr>
<td>4</td>
<td>$\frac{x-x^2}{1-2x}$</td>
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<tr>
<td>5</td>
<td>$\frac{x-2x^2}{1-3x+x^2}$</td>
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</tr>
<tr>
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<td>$\frac{x-3x^2+x^3}{1-4x+3x^2}$</td>
<td>A007051</td>
</tr>
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<td>7</td>
<td>$\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3}$</td>
<td>A080937</td>
</tr>
<tr>
<td>8</td>
<td>$\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3}$</td>
<td>A024175</td>
</tr>
<tr>
<td>9</td>
<td>$\frac{x-6x^2+10x^3-4x^4}{1-7x+15x^2-10x^3+x^4}$</td>
<td>A080938</td>
</tr>
</tbody>
</table>
Coefficient sightings...

\[
\frac{x}{1} \\
\frac{x}{1-x} \\
\frac{x-x^2}{1-2x} \\
\frac{x-2x^2}{1-3x+x^2} \\
\frac{x-3x^2+x^3}{1-4x+3x^2} \\
\frac{x-4x^2+3x^3}{1-5x+6x^2-x^3} \\
\frac{x-5x^2+6x^3-x^4}{1-6x+10x^2-4x^3} \\
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\]
## Coefficient Sightings...

<p>| | | | | | | |</p>
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<tr>
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<tr>
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<td>4</td>
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</tr>
<tr>
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<td>10</td>
<td>5</td>
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<td>1</td>
<td>6</td>
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<td>15</td>
<td>6</td>
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</tr>
<tr>
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<td>7</td>
<td>21</td>
<td>35</td>
<td>35</td>
<td>21</td>
<td>7</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
<td>28</td>
<td>56</td>
<td>70</td>
<td>56</td>
<td>28</td>
</tr>
</tbody>
</table>
Coefficient sightings...

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 \\
\end{array}
\]
An explicit formula

**Theorem**

Let $k \in \mathbb{Z}^+$ and let $t \in \mathbb{T}_k$. Then

$$g_t(x) = \frac{\sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} (-1)^i \cdot \binom{k-(i+2)}{i} \cdot x^{i+1}}{\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^i \cdot \binom{k-(i+1)}{i} \cdot x^i}.$$
Stack sorting

Theorem (Knuth)

$\pi \in S_n$ is 1-stack-sortable if and only if $\pi$ avoids the pattern 231.
Stack sorting

Theorem (Knuth)

\( \pi \in S_n \) is 1-stack-sortable if and only if \( \pi \) avoids the pattern 231.
Finite stacks

Question: What if the stack has a finite capacity $d$?

Theorem (Atkinson, Chow, West,...)

$\pi \in S_n$ is 1-stack-sortable in a depth $d$ (\(d \geq 1\)) stack if and only if $\pi$ avoids the patterns 231 and $(d+1)d \cdots 1$. 
Finite stacks

Question: What if the stack has a finite capacity \( d \)?

\[
\begin{align*}
15423 & \rightarrow 5423 & \rightarrow 23 & \rightarrow 3 & \rightarrow 3 & \rightarrow \cdots & \rightarrow \\
\hline
\end{align*}
\]

Theorem (Atkinson, Chow, West,...)

\( \pi \in S_n \) is 1-stack-sortable in a depth \( d \) \( (d \geq 1) \) stack if and only if \( \pi \) avoids the patterns 231 and \( (d + 1)d \cdots 1 \).
Generating functions

- $a_d(n)$ is the number of permutations of length $n$ sortable in a depth $d$ stack.
- $f_d(x) = \sum_{n \geq 1} a_d(n)x^n$. 
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Generating functions

- $a_d(n)$ is the number of permutations of length $n$ sortable in a depth $d$ stack.
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Summary

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- team trees (Mike Dairyko, Samantha Tyner, Casey Wynn)
- team stacks (Timothy Goodrich, Will Olson, Julia Yuan)

(slides at http://faculty.valpo.edu/lpudwell)
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- you, for listening!

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