Packing patterns in restricted permutations

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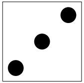
Permutations

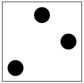
Definition

A *permutation* π of length n is an ordered list of the numbers $1, 2, \dots, n$.
$\mathcal{S}_n$ is the set of all permutations of length n.

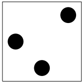
π is often visualized by plotting the points (i, π_i) in the Cartesian plane.



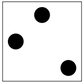
123



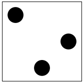
132



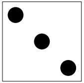
213



231

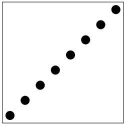


312

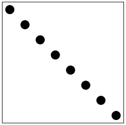


321

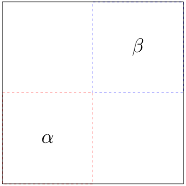
Permutation Constructions



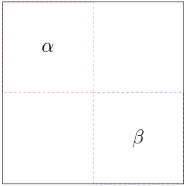
$I_n = 12 \cdots n$



$J_n = n \cdots 21$



$\alpha \oplus \beta$



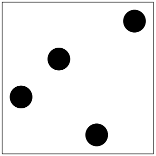
$\alpha \ominus \beta$

Permutation Patterns

Definition

$\pi \in \mathcal{S}_n$ *contains* $\rho \in \mathcal{S}_m$ as a pattern if there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\pi_{i_a} < \pi_{i_b}$ iff $\rho_a < \rho_b$.
If π doesn't contain ρ, we say π *avoids* ρ and we write $\pi \in \mathcal{S}_n(\rho)$.

Example:



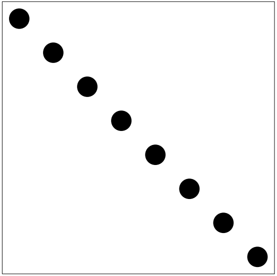
$\pi = 2314 \in \mathcal{S}_4(321)$.

Pattern Avoidance

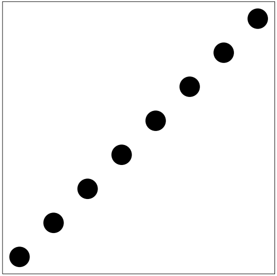
Let $\mathrm{s}_n(\rho) = |\mathcal{S}_n(\rho)|$.

Theorem

For $n \geq 0$, $\mathrm{s}_n(12) = \mathrm{s}_n(21) = 1$.



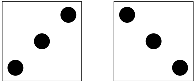
$\mathcal{S}_8(12)$



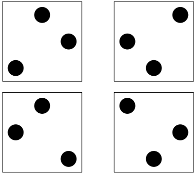
$\mathcal{S}_8(21)$

Pattern Avoidance Symmetries

$$\mathrm{s}_n(123) = \mathrm{s}_n(321)$$



$$\mathrm{s}_n(132) = \mathrm{s}_n(213) = \mathrm{s}_n(231) = \mathrm{s}_n(312)$$



Pattern Avoidance

Theorem

If $\rho \in \mathcal{S}_3$, then $\mathrm{s}_n(\rho) = \frac{\binom{2n}{n}}{n+1}$.

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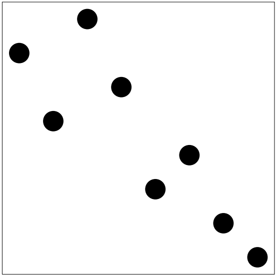
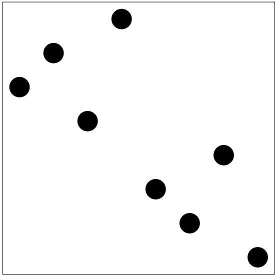


A member of $\mathcal{S}_8(123)$



A member of $\mathcal{S}_8(132)$

Pattern Avoidance

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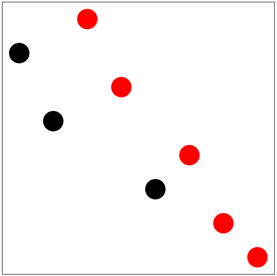
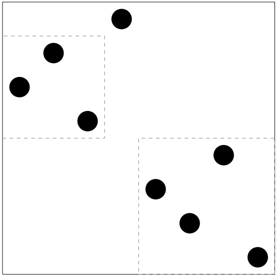
If $\rho \in \mathcal{S}_3$, then $\mathrm{s}_n(\rho) = \frac{\binom{2n}{n}}{n+1}$.



A member of $\mathcal{S}_8(123)$



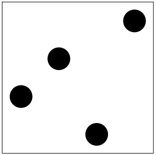
A member of $\mathcal{S}_8(132)$

Patterns

Definition

$\pi \in \mathcal{S}_n$ *contains* $\rho \in \mathcal{S}_m$ as a pattern if there exist $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ such that $\pi_{i_a} < \pi_{i_b}$ iff $\rho_a < \rho_b$.

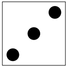
Example:



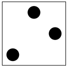
$\pi = 2314$ contains...
1 copy of 123
2 copies of 213
1 copy of 231

Pattern Packing

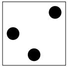
- $\nu(\rho, \pi)$ is the number of occurrences of ρ in π.
- Given n and ρ, consider $\max_{\pi \in \mathcal{S}_n} \nu(\rho, \pi)$
  Example: $n = 3$ and $\rho = 12$



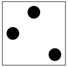
$\nu(12, 123) = 3$



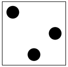
$\nu(12, 132) = 2$



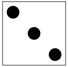
$\nu(12, 213) = 2$



$\nu(12, 231) = 1$



$\nu(12, 312) = 1$



$\nu(12, 321) = 0$

$$\max_{\pi \in \mathcal{S}_3} \nu(12, \pi) = 3$$

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- $\nu(\rho, \pi)$ is the number of occurrences of ρ in π.
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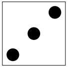
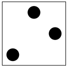
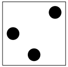
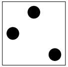
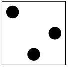
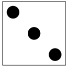
  Example: $n = 3$ and $\rho = 12$

$\nu(12, 123) = 3$ $\quad \nu(12, 132) = 2$ $\quad \nu(12, 213) = 2$

$\nu(12, 231) = 1$ $\quad \nu(12, 312) = 1$ $\quad \nu(12, 321) = 0$

$$\max_{\pi \in \mathcal{S}_3} \nu(12, \pi) = 3$$

- $d(\rho) = \lim\limits_{n \to \infty} \dfrac{\max_{\pi \in \mathcal{S}_n} \nu(\rho, \pi)}{\binom{n}{|\rho|}}$ (packing density)

Pattern Packing

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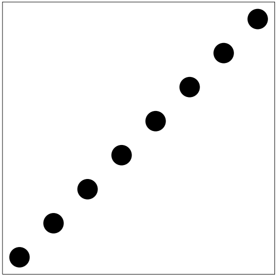
Known:

- $d(12\cdots m) = 1$ (Pack $12\cdots m$ into $12\cdots n$.)



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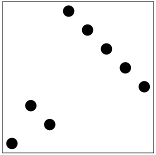
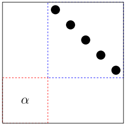
Known:

- $d(12\cdots m) = 1$ (Pack $12\cdots m$ into $12\cdots n$.)
- For all $\rho \in \mathcal{S}_m$, $d(\rho)$ exists.
- If ρ is layered, then $\max_{\pi\in\mathcal{S}_n} \nu(\rho, \pi)$ is achieved by a layered π.



Packing 132

Known: Since 132 is layered, then $\max_{\pi \in \mathcal{S}_n} \nu(132, \pi)$ is achieved by a layered π.



$\pi = \alpha \oplus J_i$

$$\nu(132, \pi) = \nu(132, \alpha) + (n - i) \cdot \binom{i}{2}$$

Packing 132

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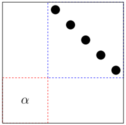


$\pi = \alpha \oplus J_i$

$$\nu(132, \pi) = \nu(132, \alpha) + (n - i) \cdot \binom{i}{2}$$

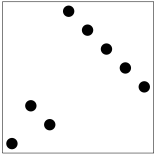
- $\frac{\nu(132,\pi)}{\binom{n}{3}}$ is maximized when $i = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right) n \approx 0.634n$
- Implies $d(132) = 2\sqrt{3} - 3 \approx 0.464$

Pattern Packing

- $\nu(\rho, \pi)$ is the number of occurrences of ρ in π.
- $d(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$ (packing density)

Known:

- $d(12\cdots m) = 1$ (Pack $12\cdots m$ into $12\cdots n$.)
- For all $\rho \in \mathcal{S}_m$, $d(\rho)$ exists.
- If ρ is layered, then $\max_{\pi\in\mathcal{S}_n} \nu(\rho,\pi)$ is achieved by a layered π.
- $d(132) = 2\sqrt{3} - 3 \approx 0.464$



Notation

- $\nu(\rho, \pi)$ is the number of occurrences of ρ in π.

Previous work:

$$d(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$$

Notation

- $\nu(\rho, \pi)$ is the number of occurrences of ρ in π.

Previous work:

$$d(\rho) = \lim_{n\to\infty} \frac{\max_{\pi \in \mathcal{S}_n} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$$

In this talk:

$$d_{\sigma}(\rho) = \lim_{n\to\infty} \frac{\max_{\pi \in \mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}} \qquad d_{A}(\rho) = \lim_{n\to\infty} \frac{\max_{\pi \in A_n} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$$

A_n is the set of *alternating permutations*,
i.e. those that avoid consecutive 123 patterns and consecutive 321 patterns.

Packing patterns of length 3

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi \in \mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho \backslash \sigma$	123	132	213	231	312	321	-
123							1
132							$2\sqrt{3}-3$

Packing patterns of length 3

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho\backslash\sigma$	123	132	213	231	312	321	-
123	0						1
132		0					$2\sqrt{3}-3$

Packing patterns of length 3

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho\backslash\sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132		0					$2\sqrt{3}-3$

- $I_n = 12\cdots n$ avoids $\sigma \in \mathcal{S}_3 \setminus \{123\}$.

Packing patterns of length 3

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho\backslash\sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132		0		$2\sqrt{3}-3$	$2\sqrt{3}-3$		$2\sqrt{3}-3$

- $I_n = 12\cdots n$ avoids $\sigma \in \mathcal{S}_3 \setminus \{123\}$.
- Layered permutations avoid 231 and 312.

Packing patterns of length 3

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$\rho \backslash \sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132	?	0	?	$2\sqrt{3}-3$	$2\sqrt{3}-3$	?	$2\sqrt{3}-3$

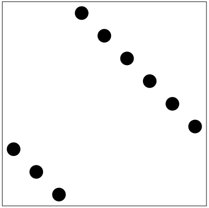
- $I_n = 12\cdots n$ avoids $\sigma \in \mathcal{S}_3 \setminus \{123\}$.
- Layered permutations avoid 231 and 312.
- New: $d_{123}(132)$, $d_{213}(132)$, and $d_{321}(132)$

Packing 132

...and avoiding 123

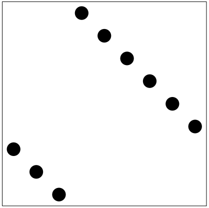
Packing 132

...and avoiding 123



Packing 132

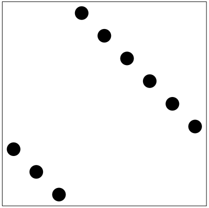
...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i\binom{n-i}{2}$ copies of 132. $(J_n = n \cdots 21)$

Packing 132

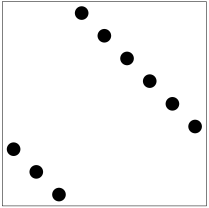
...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i\binom{n-i}{2}$ copies of 132. $(J_n = n \cdots 21)$
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.

Packing 132

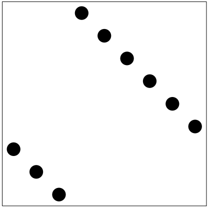
...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i\binom{n-i}{2}$ copies of 132. $(J_n = n \cdots 21)$
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.
- Implies $d_{123}(132) = \dfrac{4}{9}$.

Packing 132

...and avoiding 123



...and avoiding 213

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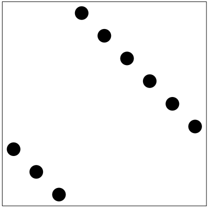
Packing 132

...and avoiding 123



- $J_i \oplus J_{n-i}$ has $i\binom{n-i}{2}$ copies of 132. $(J_n = n \cdots 21)$
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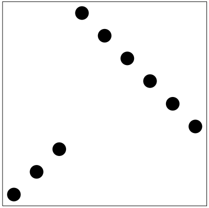
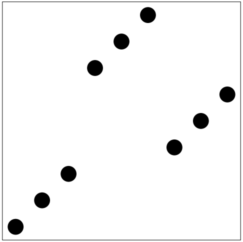
- Implies $d_{123}(132) = \dfrac{4}{9}$.

...and avoiding 213



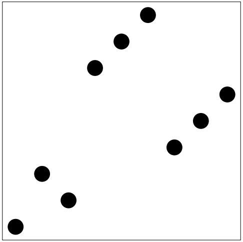
- $I_i \oplus J_{n-i}$ has $i\binom{n-i}{2}$ copies of 132.
- Maximized when $i = \lfloor \frac{n}{3} \rfloor$.
- Implies $d_{213}(132) = \dfrac{4}{9}$.

Packing 132 and Avoiding 321



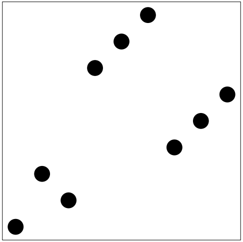
- $I_a \oplus (I_b \ominus I_c)$ has $a \cdot b \cdot c$ copies of 132.

Packing 132 and Avoiding 321



- $I_a \oplus (I_b \ominus I_c)$ has $a \cdot b \cdot c$ copies of 132.
- Replace initial I_a with a 132-optimizer of length a to get more copies.

Packing 132 and Avoiding 321



- $I_a \oplus (I_b \ominus I_c)$ has $a \cdot b \cdot c$ copies of 132.
- Replace initial I_a with a 132-optimizer of length a to get more copies.
- Optimized when $a = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) n$, $b = c = \left(\frac{3}{4} - \frac{\sqrt{3}}{4}\right) n$.
- Implies $d_{321}(132) = \sqrt{3} - \frac{3}{2}$.

Recap:

$$d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$$

$\rho\backslash\sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132	$\frac{4}{9}$	0	$\frac{4}{9}$	$2\sqrt{3}-3$	$2\sqrt{3}-3$	$\sqrt{3}-\frac{3}{2}$	$2\sqrt{3}-3$

Recap:

$$d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$$

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123	0	1	1	1	1	1	1
132	$\frac{4}{9}$	0	$\frac{4}{9}$	$2\sqrt{3}-3$	$2\sqrt{3}-3$	$\sqrt{3}-\frac{3}{2}$	$2\sqrt{3}-3$

Or approximately...

$\rho\backslash\sigma$	123	132	213	231	312	321	-
123	0	1	1	1	1	1	1
132	0.444	0	0.444	0.464	0.464	0.232	0.464

Packing patterns of length 4

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho\backslash\sigma$	123	132	213	231	312	321	-
1234							1
1432							α
2143							$\frac{3}{8}$
1243							$\frac{3}{8}$
1324							≈ 0.244
1342							≈ 0.19658
2413							≈ 0.10474

α is the real root of $x^3 - 12x^2 + 156x - 64$ (≈ 0.42357)

Packing patterns of length 4

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi \in \mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho \backslash \sigma$	123	132	213	231	312	321	-
1234	0						1
1432		0				0	α
2143		0	0				$\frac{3}{8}$
1243	0	0					$\frac{3}{8}$
1324	0	0	0				≈ 0.244
1342	0	0		0			≈ 0.19658
2413		0	0	0	0		≈ 0.10474

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1234	0	1	1	1	1	1	1
1432		0				0	α
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1243	0	0					$\frac{3}{8}$
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1342	0	0		0			≈ 0.19658
2413		0	0	0	0		≈ 0.10474

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$\rho \backslash \sigma$	123	132	213	231	312	321	-
1234	0	1	1	1	1	1	1
1432		0		α	α	0	α
2143		0	0	$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1243	0	0		$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1324	0	0	0	β	β		≈ 0.244 (β)
1342	0	0		0			≈ 0.19658
2413		0	0	0	0		≈ 0.10474

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$\rho\backslash\sigma$	123	132	213	231	312	321	-
1234	0	1	1	1	1	1	1
1432		0		α	α	0	α
2143	$\frac{3}{8}$	0	0	$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1243	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1324	0	0	0	β	β		$\approx 0.244\ (\beta)$
1342	0	0		0			≈ 0.19658
2413		0	0	0	0		≈ 0.10474

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Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \frac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

$\rho\backslash\sigma$	123	132	213	231	312	321	-
1234	0	1	1	1	1	1	1
1432		0		α	α	0	α
2143	$\frac{3}{8}$	0	0	$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1243	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$		$\frac{3}{8}$
1324	0	0	0	β	β		≈ 0.244 (β)
1342	0	0		0			≈ 0.19658
2413		0	0	0	0		≈ 0.10474

α is the real root of $x^3 - 12x^2 + 156x - 64$ (≈ 0.42357)

Packing patterns of length 4

Recall: $d_\sigma(\rho) = \lim_{n\to\infty} \dfrac{\max_{\pi\in\mathcal{S}_n(\sigma)} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

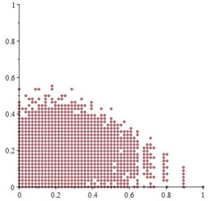
$\rho\backslash\sigma$	123	132	213	231	312	321	-
1234	0	1	1	1	1	1	1
1432	$\frac{27}{64}$	0	$\frac{27}{64}$	α	α	0	α
2143	$\frac{3}{8}$	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\geq \frac{3}{32}$	$\frac{3}{8}$
1243	0	0	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\geq \frac{3}{16}$	$\frac{3}{8}$
1324	0	0	0	β	β	$\geq \gamma$	≈ 0.244 (β)
1342	0	0	$\geq \frac{3}{16}$	0	$\geq \frac{3}{16}$	$\geq \delta$	≈ 0.19658
2413	$\geq \frac{3}{32}$	0	0	0	0	$\geq \frac{3}{32}$	≈ 0.10474

α is the real root of $x^3 - 12x^2 + 156x - 64$ (≈ 0.42357)

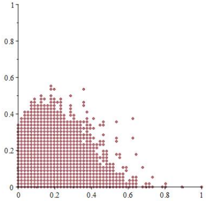
$\frac{3}{16} = 0.1875$, $\frac{3}{32} = 0.09375$, $\frac{27}{64} = 0.421875$

$\gamma \approx 0.09450$, $\delta \approx 0.18825$

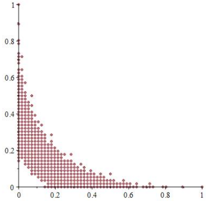
Joint Distributions



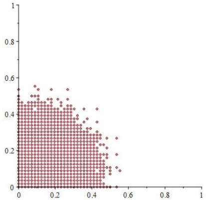
123 vs. 132



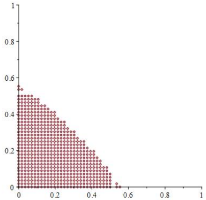
123 vs. 231



123 vs. 321

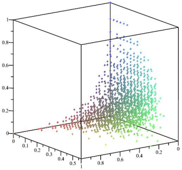


132 vs. 213

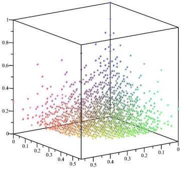


132 vs. 231

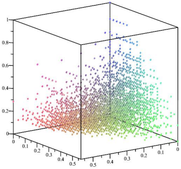
More Joint Distributions



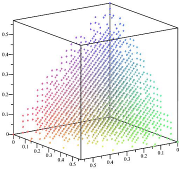
123, 312, 321



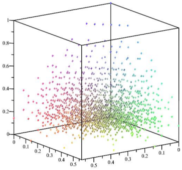
132, 213, 321



213, 231, 321



132, 213, 231



231, 312, 321

Alternating Permutations

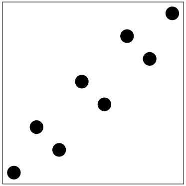
A_n is the set of permutations of length n avoiding 123 and 321 consecutively.

1324	1423	2314	2413	3412
4231	4132	3241	3142	2143

Goal: Find $d_A(\rho) = \lim_{n\to\infty} \frac{\max_{\pi \in A_n} \nu(\rho,\pi)}{\binom{n}{|\rho|}}$

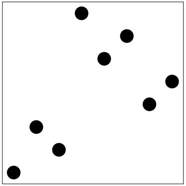
Alternating packing densities

Packing 123



- $1 \oplus 21 \oplus \cdots \oplus 21\,(\oplus 1)$
- Implies $d_A(123) = 1$.

Packing 132



- Use same ratios for "alternating layers" as 132-optimizer in $\mathcal{S}_n$.
- Implies $d_A(132) = 2\sqrt{3} - 3$.

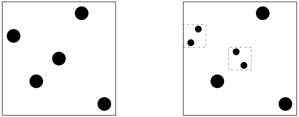
In fact...

Proposition

For all $\rho \in \mathcal{S}_k$, $d(\rho) = d_A(\rho)$.

Fix τ_n such that $\lim_{n\to\infty} \frac{\nu(\rho,\tau_n)}{\binom{n}{k}} = d(\rho)$ and let $m \geq 1$.

Let σ_n be obtained by inflating each point of τ_n with an alternating permutation of length m or $m+1$.



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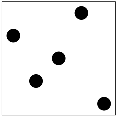

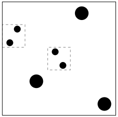

$$\nu(\rho, \tau_n) \cdot m^k \leq \nu(\rho, \sigma_n)$$

In fact...

Proposition

For all $\rho \in \mathcal{S}_k$, $d(\rho) = d_A(\rho)$.

Fix τ_n such that $\lim_{n\to\infty} \frac{\nu(\rho,\tau_n)}{\binom{n}{k}} = d(\rho)$ and let $m \geq 1$.

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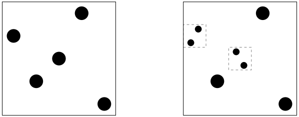

$$\lim_{n\to\infty} \frac{\nu(\rho,\tau_n) \cdot m^k \binom{n}{k}}{\binom{n}{k}} \leq \lim_{n\to\infty} \nu(\rho,\sigma_n)$$

In fact...

Proposition

For all $\rho \in \mathcal{S}_k$, $d(\rho) = d_A(\rho)$.

Fix τ_n such that $\lim_{n\to\infty} \frac{\nu(\rho,\tau_n)}{\binom{n}{k}} = d(\rho)$ and let $m \geq 1$.

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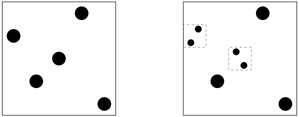


$$\lim_{n\to\infty} \frac{\nu(\rho,\tau_n)\cdot m^k\binom{n}{k}}{\binom{n}{k}\binom{mn}{k}} \leq \lim_{n\to\infty} \frac{\nu(\rho,\sigma_n)}{\binom{mn}{k}}$$

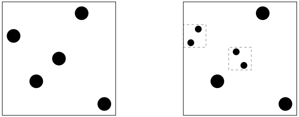
In fact...

Proposition

For all $\rho \in \mathcal{S}_k$, $d(\rho) = d_A(\rho)$.

Fix τ_n such that $\displaystyle\lim_{n\to\infty} \frac{\nu(\rho,\tau_n)}{\binom{n}{k}} = d(\rho)$ and let $m \geq 1$.

Let σ_n be obtained by inflating each point of τ_n with an alternating permutation of length m or $m+1$.



$$\lim_{n\to\infty} \frac{d(\rho)\cdot m^k \binom{n}{k}}{\binom{mn}{k}} \leq \lim_{n\to\infty} \frac{\nu(\rho,\sigma_n)}{\binom{mn}{k}}$$

In fact...

Proposition

For all $\rho \in \mathcal{S}_k$, $d(\rho) = d_A(\rho)$.

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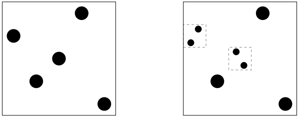

$$d(\rho) \leq \lim_{n\to\infty} \frac{\nu(\rho,\sigma_n)}{\binom{mn}{k}}$$

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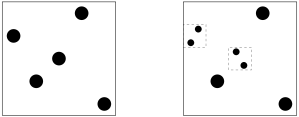
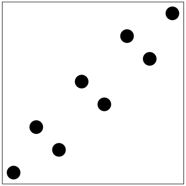


$$d(\rho) \leq \lim_{n\to\infty} \frac{\nu(\rho, \sigma_n)}{\binom{mn}{k}} \leq d(\rho)$$

Packing 123



- $1 \oplus 21 \oplus \cdots \oplus 21\,(\oplus 1)$
- Implies $d_A(123) = 1$.
- $\binom{n}{3}$ subsequences of length 3.
- $\approx c \cdot \binom{n}{2}$ are not 123 patterns.

Counting Sequences

Let $a_{123}(n)$ be the number of copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21\ (\oplus 1)$.

$$a_{123}(n) = \begin{cases} 2(\frac{n}{2} - 1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

Counting Sequences

Let $a_{123}(n)$ be the number of copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21\,(\oplus 1)$.

$$a_{123}(n) = \begin{cases} 2(\frac{n}{2}-1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$

Counting Sequences

Let $a_{123}(n)$ be the number of copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21\,(\oplus 1)$.

$$a_{123}(n) = \begin{cases} 2(\frac{n}{2} - 1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

$$2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, \ldots$$

A099956 Atomic numbers of the alkaline earth metals. 9

4, 12, 20, 38, 56, 88 (list; graph; refs; listen; history; text; internal format)

OFFSET	1,1
LINKS	Table of n, a(n) for n=1..6.
EXAMPLE	12 is the atomic number of magnesium.
CROSSREFS	Cf. A099955, alkali metals; A101648, metalloids; A101647, nonmetals (except halogens and noble gases); A097478, halogens; A018227, noble gases; A101649, poor metals. Sequence in context: A057317 A008068 A008183 * A301066 A008092 A316299 Adjacent sequences: A099953 A099954 A099955 * A099957 A099958 A099959
KEYWORD	nonn,fini,full
AUTHOR	Parthasarathy Nambi, Nov 12 2004
STATUS	approved

Counting Sequences

Let $a_{123}(n)$ be the number of copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21\ (\oplus 1)$.

$$a_{123}(n) = \begin{cases} 2(\frac{n}{2}-1) + 8\binom{\frac{n}{2}-1}{2} + 8\binom{\frac{n}{2}-1}{3} & n \text{ even} \\ 4\binom{\frac{n-1}{2}}{2} + 8\binom{\frac{n-1}{2}}{3} & n \text{ odd} \end{cases}$$

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, . . .

A168380 Row sums of A168281. +20 14

2, 4, 12, 20, 38, 56, 88, 120, 170, 220, 292, 364, 462, 560, 688, 816, 978, 1140, 1340, 1540, 1782, 2024, 2312, 2600, 2938, 3276, 3668, 4060, 4510, 4960, 5472, 5984, 6562, 7140, 7788, 8436, 9158, 9880, 10680, 11480, 12362, 13244, 14212, 15180, 16238, 17296, 18448, 19600, 20850, 22100 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS The atomic numbers of the augmented alkaline earth group in Charles Janet's spiral periodic table are 0 and the first eight terms of this sequence (see Stewart reference). - Alonso del Arte, May 13 2011

LINKS Vincenzo Librandi, Table of n, a(n) for n = 1..10000
Stewart, Philip, Charles Janet: unrecognized genius of the Periodic System. Foundations of Chemistry (2010), p. 9.
Index entries for linear recurrences with constant coefficients, signature (2,1,-4,1,2,-1).

FORMULA a(n) = 2*A005993(n-1).
a(n) = (n+1)*(3 + 2*n^2 + 4*n - 3*(-1)^n)/12.
a(n+1) - a(n) = A093907(n) = A137583(n+1).
a(2n+1) = A035597(n+1). a(2n)=A002492(n).

Alkaline Earth Metals (Group 2)

Group → / Period ↓	1	2	3		4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc		22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y		40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	**	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				*	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				**	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

Permutation packing and electrons

A little chemistry...

- *Quantum numbers* describe trajectories of electrons.

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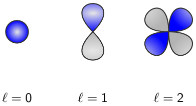

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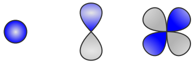

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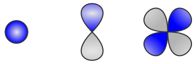

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  - m (magnetic number) determines number of orbitals and orientation within shell
  $$-\ell \leq m \leq \ell$$
  - Two possible spin numbers for each choice of (n, ℓ, m)

Notation for copies of 123 in $1 \oplus 21 \oplus \cdots \oplus 21\,(\oplus 1)$

Observation: copies of 123 come in pairs.

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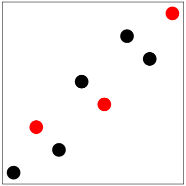

Given xyz embedding of 123 where y is even, $x(y+1)z$ is also a 123.

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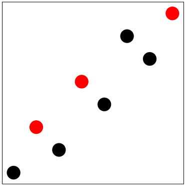

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Copies of 123 mapped to tuples

xyz corresponds to the tuple (n, ℓ, m) where...

- $|m|$ is the layer where x is found (count layers starting with 0).
- m is negative if we use the smaller entry in the layer as x, positive if we use the larger entry.
- ℓ is the layer *of size 2* where y is found (count layers starting with 0).
- $n + \ell + 3 = z$.

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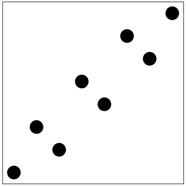
- $n + \ell + 3 = z$.

Example: $1 \oplus 21 \oplus 21 \oplus 21 = 1 \quad 32 \quad 54 \quad 76$

copies	tuple	copies	tuple	copies	tuple
124,134	(1,0,0)	146,156	(2,1,0)	147,157	(3,1,0)
125,135	(2,0,0)	246,256	(2,1,-1)	247,257	(3,1,-1)
126,136	(3,0,0)	346,356	(2,1,1)	347,357	(3,1,1)
127,137	(4,0,0)				

These tuples are valid quantum numbers



Layer 0 contains 1.
Layer L contains $2L$ and $2L+1$.
(For even length, the last layer has one point.)

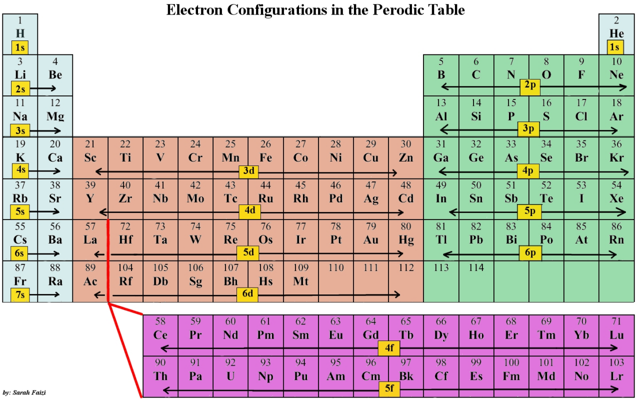
Construction: xyz is a 123 occurrence with...

- x in layer $|m|$, y in layer $\ell+1$, and z in layer $\ell+2$ or higher.

Consequences:

- $z \geq 2(\ell+2)$ and $n = z-\ell-3$ imply $n \geq \ell+1 \geq 1$ and so $\ell \leq n-1$.
- x in an earlier layer than y implies $|m|+1 \leq \ell+1$ and so $|m| \leq \ell$.
- A new ℓ value is introduced for each even permutation length.

Periodic Table, Take 2



Subshell is (n, ℓ) with ℓ given by s $(\ell = 0)$, p $(\ell = 1)$, d $(\ell = 2)$, f $(\ell = 3)$.

e.g. Calcium has subshells with

$$(n, \ell) \in \{(1,0), (2,0), (2,1), (3,0), (3,1), (4,0)\}.$$

123s to electrons

e.g. Calcium has subshells with
$(n, \ell) \in \{(1,0),(2,0),(2,1),(3,0),(3,1),(4,0)\}$.
Subshell: Know n and ℓ. Need all tuples (n, ℓ, m) where $-\ell \le m \le \ell$

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We saw the copies of 123 in $1 \oplus 21 \oplus 21 \oplus 21 = 1325476$ are:

copies	tuple	copies	tuple	copies	tuple
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126,136	(3,0,0)	346,356	(2,1,1)	347,357	(3,1,1)
127,137	(4,0,0)				

Future Directions

- Determine $d_\sigma(\rho)$ for other patterns.
- Joint distributions of patterns.
- Bijections between pattern embeddings and other combinatorial objects.

References

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Thanks for listening!

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