Beautiful Bijections for Permutation Patterns

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Joint Mathematics Meetings
MAA Invited Paper Session on
Clever Counting or Beautiful Bijection
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Outline

Permutations

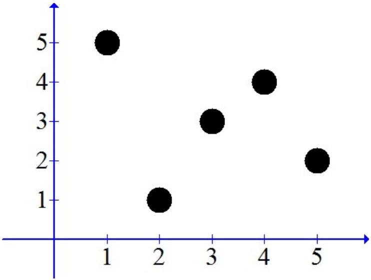
- A permutation of length n is an ordered list of the numbers $\{1, 2, \ldots, n\}$.
- There are $n!$ permutations of length n.
- Example: the 6 permutations of $\{1, 2, 3\}$ are 123, 132, 213, 231, 312, 321.

Graphs of Permutations

Consider the permutation $\pi = \pi_1\pi_2\cdots\pi_n$ as a function from $\{1,2,\ldots,n\}$ to $\{1,2,\ldots,n\}$.

Example, $\pi = 51342$

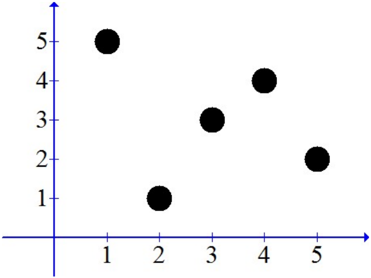
- $\pi_1 = 5$
- $\pi_2 = 1$
- $\pi_3 = 3$
- $\pi_4 = 4$
- $\pi_5 = 2$



Containment/Avoidance

$\pi = \pi_1 \cdots \pi_n$ **contains** $\rho = \rho_1 \cdots \rho_k$ as a pattern if there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi_{i_a} < \pi_{i_b}$ if and only if $\rho_a < \rho_b$. Otherwise π **avoids** the pattern ρ.

Picture definition:



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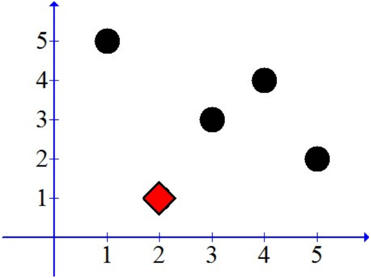

$\pi = 51342$ contains

1,

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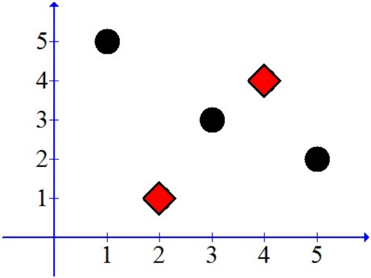

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12,

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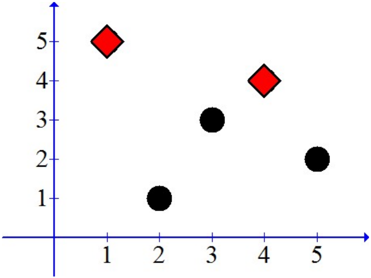

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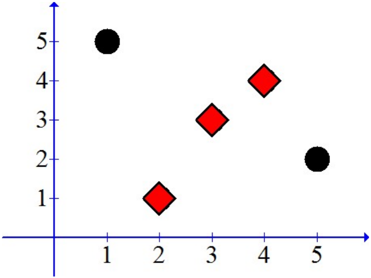

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12, 21,
123,

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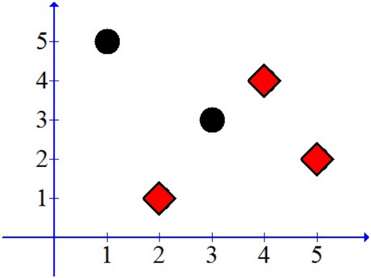

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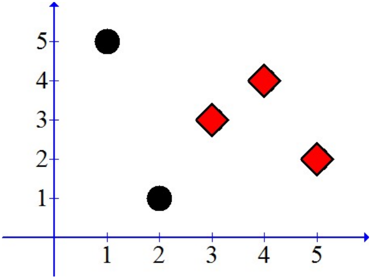

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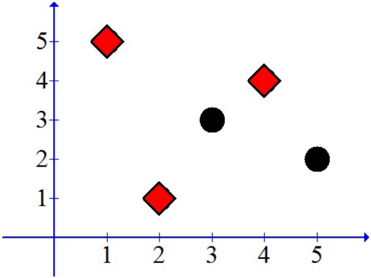

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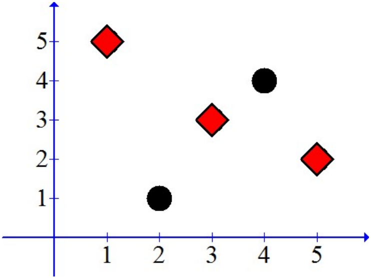

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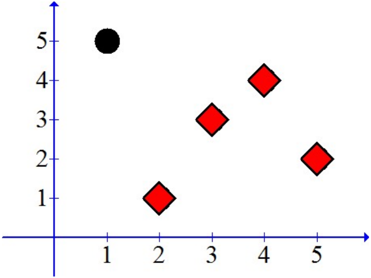

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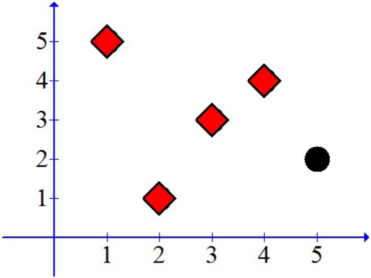

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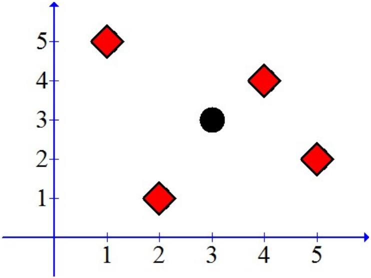

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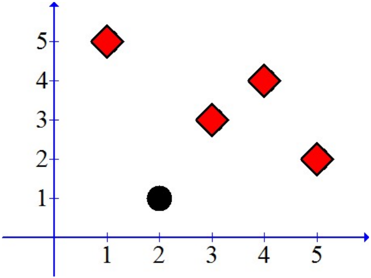

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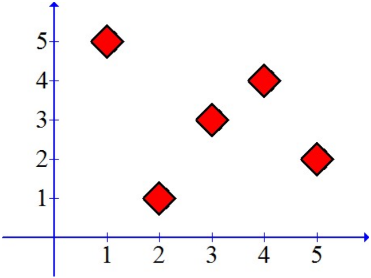

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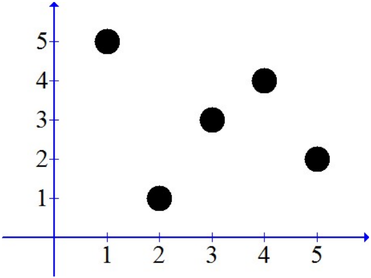

$\pi = 51342$ contains
1,
12, 21,
123, 132, 231, 312, 321,
1342, 4123, 4132, 4231,
51342.
$\pi = 51342$ avoids all other permutations.

A Family of Counting Problems

$\mathcal{S}_n(Q)$

$\mathcal{S}_n(Q)$ is the *set* of permutations of length n that avoid all permutations in Q.

$s_n(Q)$

$s_n(Q) = |\mathcal{S}_n(Q)|$

Problem

Given a list of permutations Q, describe the structure of $\pi \in \mathcal{S}_n(Q)$ and/or find an expression for $s_n(Q)$. (Generally Hard)

Strategy

1. Count (Find $s_n(Q)$ for as many Q as possible.)
2. Compare (Where have I seen this sequence before?)
3. Connect (*Why* did I get the same sequence twice?)

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2. Compare (Where have I seen this sequence before?)
3. Connect (*Why* did I get the same sequence twice?)

Argument 1

For a family of counting problems...
clever counting says "these things are the same", but
beautiful bijections explain "*why* things are the same".

Argument 2

Beautiful bijections allow us to translate what we know about one object to better understand another object.

Permutations and Compositions

Theorem

$s_n(123, 132)$ is equal to the number of compositions of n.

Recall: a composition of n is an ordered list of positive integers whose sum is n.

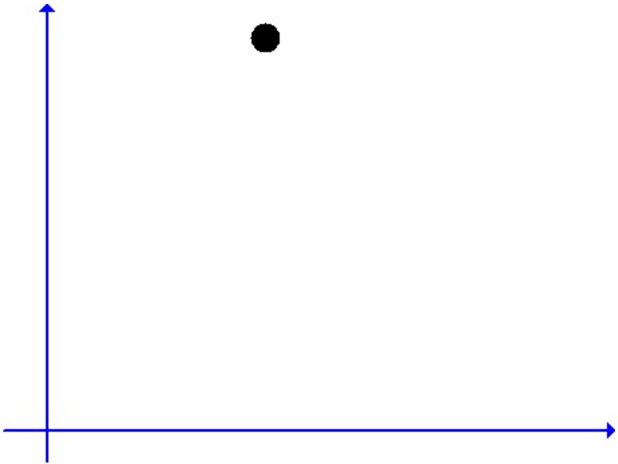
Example:
Members of $\mathcal{S}_4(123, 132)$ are:

$$3214, 3241, 3412, 3421, 4213, 4231, 4312, 4321$$

Compositions of 4 are:

$$4, 1+3, 3+1, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1$$

Structure of a permutation in $\mathcal{S}_n(123, 132)$



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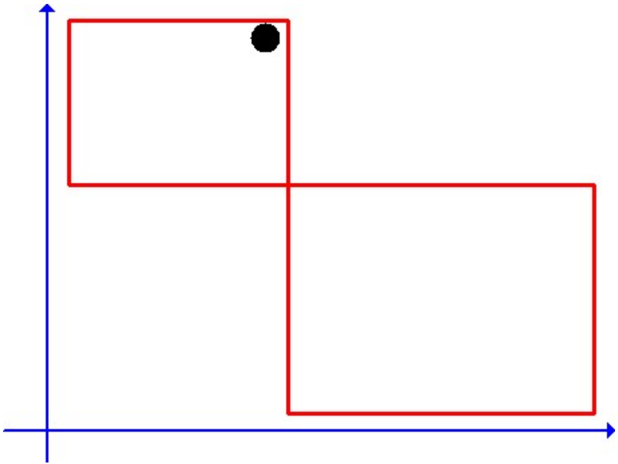

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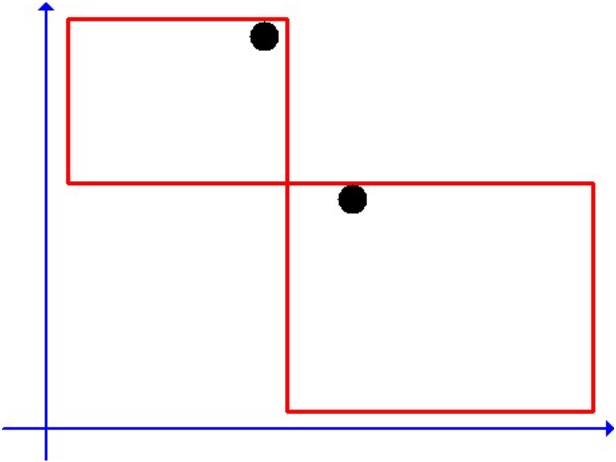
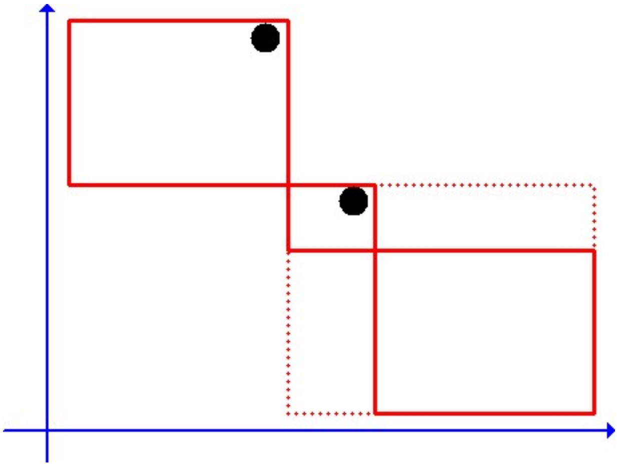
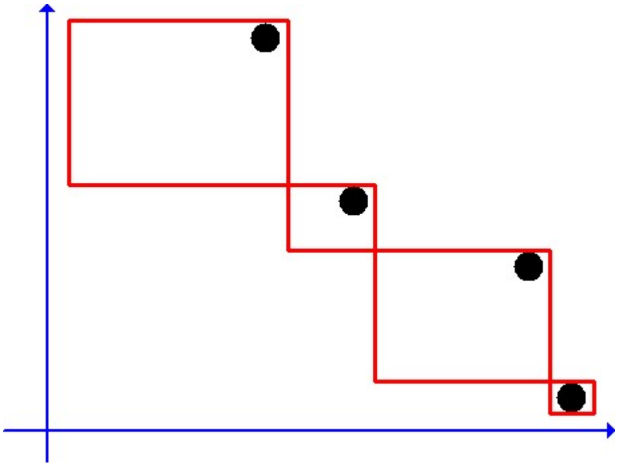


Structure of a permutation in $\mathcal{S}_n(123, 132)$



Structure of a permutation in $\mathcal{S}_n(123, 132)$



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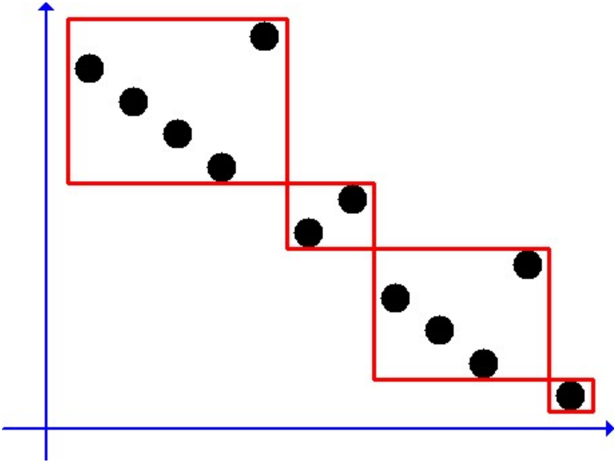

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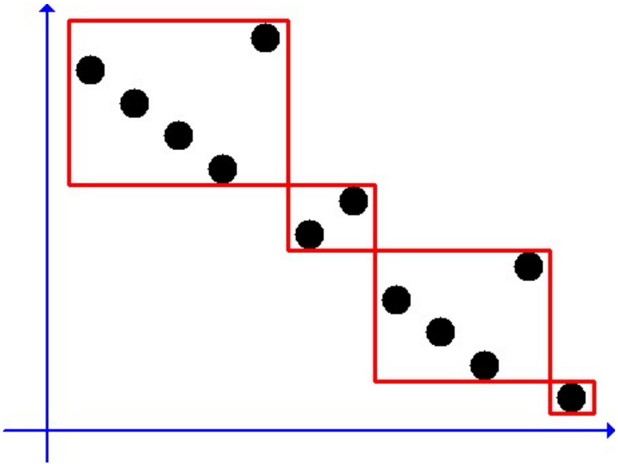
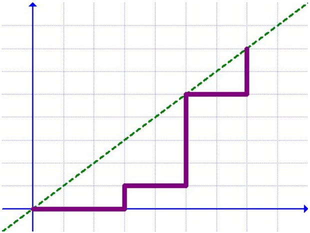


$\rightarrow 5 + 2 + 4 + 1$

Theorem

$s_n(321)$ is equal to the number of Dyck paths of length $2n$.

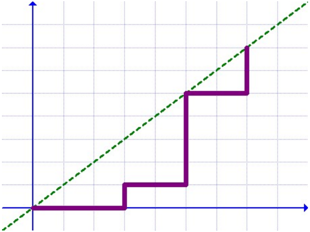
Recall: a Dyck path of length $2n$ is
a path in the xy-plane from $(0,0)$ to (n,n) that
(a) only uses the steps $\langle 1,0\rangle$ and $\langle 0,1\rangle$ and
(b) never goes above the line $y=x$.

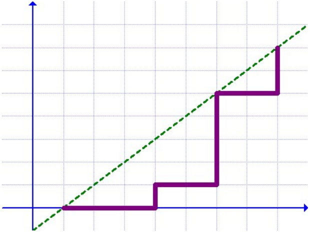


Theorem

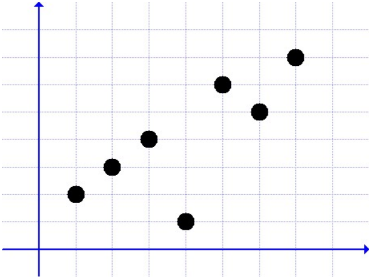
$s_n(321)$ is equal to the number of Dyck paths of length $2n$.

Recall: a (shifted) Dyck path of length $2n$ is
a path in the xy-plane from $(1, 0)$ to $(n + 1, n)$ that
(a) only uses the steps $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ and
(b) never goes above the line $y = x - 1$.





From $\pi \in \mathcal{S}_n(321)$ to a (shifted) Dyck path



1. Delete all left-to-right maxima
2. Add the point $(n+1, n)$.
3. Start at $(1, 0)$. Move right until under a point, then up to the point, and repeat.

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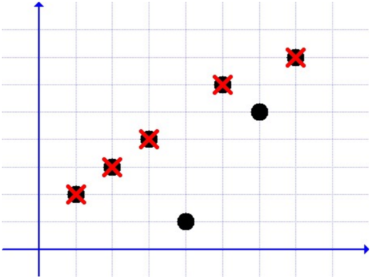

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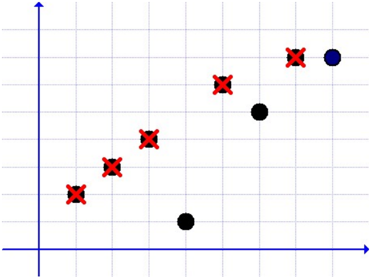

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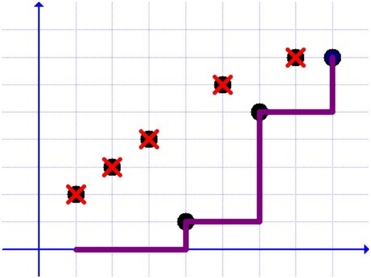

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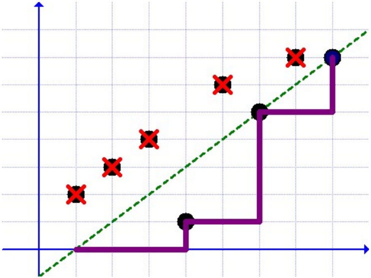

1. Delete all left-to-right maxima
2. Add the point $(n+1, n)$.
3. Start at $(1, 0)$. Move right until under a point, then up to the point, and repeat.

Notice:

1. All non-left-to-right-maxima are in increasing order.
2. For any point non-left-to-right-maxima (i, π_i), there are at least $1 + (\pi_i - 1) = \pi_i$ points to the left of (i, π_i), so $i \geq \pi_i + 1$, or $\pi_i \leq i - 1$, as desired.

Other Bijections

There exist bijections between various sets of pattern-avoiding permutations and...

- set partitions,
- trees,
- faces in certain geometric solids,

... and more!

Symmetry Bijections

Argument 3

There are times when *clever counting* fails, but *beautiful bijections* succeed!

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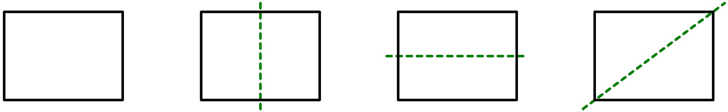

Symmetry Bijections

Argument 3

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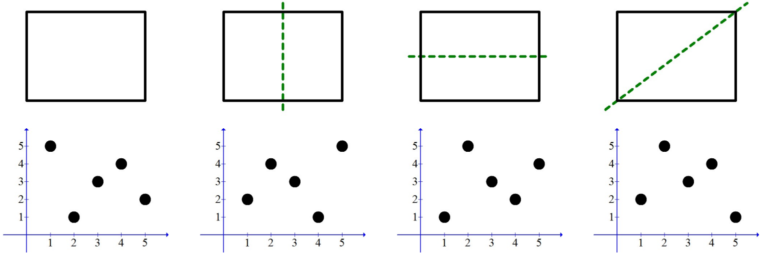

$$s_n(51342) = s_n(24315) = s_n(15324) = s_n(25341)$$

The Recap

While counting can be classy, bijections have some advantages. In particular...

- bijections explain “why”.
- bijections allow knowledge about one object to help discover new properties of another object.
- bijections may succeed even when counting fails!

Thank you for listening!