An Introduction to Enumeration Schemes

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For example, the reduction of \( 26745 \) is \( 1\bullet2\bullet \).
Definitions

Reduction

- Given a string of numbers $p = p_1 \cdots p_n$, the reduction of $p$ is the string obtained by replacing the $i^{th}$ smallest number of $p$ with $i$.

- For example, the reduction of $26745$ is $1\bullet\bullet23$. 
Given a string of numbers $p = p_1 \cdots p_n$, the reduction of $p$ is the string obtained by replacing the $i^{th}$ smallest number of $p$ with $i$.

For example, the reduction of $26745$ is $14\circ23$. 
Given a string of numbers \( p = p_1 \cdots p_n \), the reduction of \( p \) is the string obtained by replacing the \( i^{th} \) smallest number of \( p \) with \( i \).

For example, the reduction of 26745 is 14523.
Pattern Avoidance in Permutations

- Given $p \in S_n$ and $q \in S_m$, we say $p$ contains $q$ if there are $1 \leq i_1 < \cdots < i_m \leq n$ such that $p_{i_1} \cdots p_{i_m}$ reduces to $q$. Otherwise, $p$ avoids $q$.

- $p = 21354$ contains 132.
  (since 21354 reduces to 132.)

- $p = 21354$ avoids 321.
  (since $p$ has no decreasing subsequence of length 3.)
Counting Results

Two Questions

Easy: Given \( p \in S_n \), what patterns does \( p \) contain?

Hard: Given \( q \in S_m \),

- Let \( S_n(q) = \{ p \in S_n \mid p \text{ avoids } q \} \).
- Find an expression for \( |S_n(q)| \).
Patterns of length 1, 2, and 3

$$|S_n(1)| = \begin{cases} 
1 & n = 0 \\
0 & n > 0 
\end{cases}$$
Patterns of length 1, 2, and 3

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\end{cases} \quad |S_n(12)| = |S_n(21)| = 1, \ n \geq 0 \]
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Graph of \( p \in S_n(132) \)
Patterns of length 1, 2, and 3

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Pattern-Avoiding Permutations

Enumeration Schemes

Summary

Counting Results

Patterns of length 1, 2, and 3

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Graph of \( p \in S_n(132) \)

So, \[ |S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| \]
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Graph of \( p \in S_n(132) \)

So, \[ |S_n(132)| = \sum_{i=1}^{n} |S_{i-1}(132)| \cdot |S_{n-i}(132)| = \frac{(2n)}{n+1} = C_n \]

In fact, \( |S_n(q)| = C_n \) if \( q \) is any permutation of length 3.
Patterns of Length 4

There are 24 patterns of length 4.

Using several clever bijections, we can narrow our work to 3 cases:
$S_n(1342), S_n(1234),$ and $S_n(1324)$. 
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<td>23</td>
<td>103</td>
<td>513</td>
<td>2762</td>
<td>15793</td>
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Most techniques studying $|S_n(q)|$ finds formulas for a specific $q$.

1998: Zeilberger’s prefix enumeration schemes, i.e. a system of recurrences to count $|S_n(q)|$.

2005: Vatter’s modified schemes automate the enumeration of $|S_n(q)|$ for even more patterns $q$. 
**Refinement Notation**

Goal: Divide $S_n(q)$ into subsets.

$$S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \mid \begin{array}{l} \pi \text{ avoids } q \\ \pi \text{ has prefix } p_1 \cdots p_l \end{array} \right\}$$

and

$$S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \mid \begin{array}{l} \pi \text{ avoids } q \\ \pi \text{ has prefix } p_1 \cdots p_l \\ \pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n \end{array} \right\}$$

If $p_1 \cdots p_{l-1}$ reduces to $p_1^* \cdots p_{l-1}^*$, then $p = p_1 \cdots p_l$ is called a **refinement** if $p^*$.

e.g $p = 2413$ is a refinement of $p^* = 231$. 
For any pattern $q$, $S_n(q) = S_n(q; 1) = S_n(q; 12) \cup S_n(q; 21)$.

or graphically:
Reversibly Deletable Positions

Given a pattern $q$ and a prefix $p$, position $r$ is reversibly deletable if

- Deleting $p_r$ from $\pi \in S_n(q; p_1 \cdots p_l)$ produces a $q$-avoiding permutation of length $n - 1$, and
- Inserting $p_r$ into $\pi \in S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)$ produces a $q$ avoiding permutation of length $n$.

In other words, deleting and re-inserting $p_r$ gives a bijection between $S_n(q; p_1 \cdots p_l)$ and $S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)$, and

$$|S_n(q; p_1 \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|.$$
Reversibly Deletable Example

Graphically:
Reversibly Deletable Example

Graphically:
Gap Vectors

Gap vectors give a condition for which choices of $i_1, \ldots, i_l$ yield

$$\left| S_n \left( q; \begin{array}{c} p_1 \cdots p_r \cdots p_l \\ i_1 \cdots i_r \cdots i_l \end{array} \right) \right| = 0.$$ 

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Graphically:

$$\emptyset \quad \downarrow \quad 1 \quad \left\downarrow d_1 \quad \begin{array}{c} 12 \\ \geq (0, 1, 0) \end{array} \quad 21$$
Gap vectors give a condition for which choices of \( i_1, \ldots, i_l \) yield

\[
\left| S_n \left( q; \frac{p_1 \cdots p_r \cdots p_l}{i_1 \cdots i_r \cdots i_l} \right) \right| = 0.
\]

Graphically:
An enumeration scheme is a set of triples \([p_i, R_i, G_i]\) such that for each triple:

- \(p_i\) is a reduced prefix of length \(n\)
- \(R_i\) a subset of \(\{1, \ldots, n\}\)
- \(G_i\) is a set of vectors of length \(n + 1\)

and either \(R_i\) is non-empty or all refinements of \(p_i\) are also in the scheme.
An enumeration scheme is a set of triples \([p_i, R_i, G_i]\) such that for each triple

- \(p_i\) is a reduced prefix of length \(n\) (prefix)
- \(R_i\) a subset of \(\{1, \ldots, n\}\) (reversibly deletable positions)
- \(G_i\) is a set of vectors of length \(n + 1\) (gap vectors)

and

- either \(R_i\) is non-empty or all refinements of \(p_i\) are also in the scheme.
**Avoid(12) and Avoid(123)**

\[
S_n(12)
\]

- \(\emptyset\)
- \(d_1\)
- 1
- \(\geq (0, 1)\)

\[
S_n(123)
\]

- \(\emptyset\)
- \(d_1\)
- 1
- 12
- 21
- \(\geq (0, 0, 1)\)

\[
d_2
\]
Pattern-Avoiding Permutations

Enumeration Schemes

Summary

Enumeration Schemes

**Avoid(1234)**

\[ S_n(1234) \]

\[ \emptyset \]

\[ 1 \]

\[ 21 \]

\[ 12 \]

\[ d_1 \]

\[ 123 \]

\[ \geq (0, 0, 0, 1) \]

\[ 132 \]

\[ \geq (0, 0, 0, 1) \]

\[ 231 \]

\[ d_{1,2} \]

\[ 3412 \]

\[ 2413 \]

\[ 2314 \]

\[ 3421 \]

\[ d_3 \]

\[ d_4 \]

\[ d_2 \]
There are few techniques to count large classes of pattern-avoiding permutations.

Extending Zeilberger’s and Vatter’s schemes gives a good success rate for counting the elements of $S_n(q)$.

Enumeration schemes have also been successfully used to count:

- pattern-avoiding words (strings with repeated letters)
- permutations avoiding barred patterns (permutations that avoid a particular pattern unless that pattern is part of an even larger specified pattern)
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