Enumeration Schemes for Permutations Avoiding Barred Patterns

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- For example, the reduction of `26795` is `13452`.

Pattern Avoidance/Containment

Given permutations $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$,

- p contains q as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $p_{i_1} \cdots p_{i_m}$ reduces to q;
- otherwise p avoids q.

For example,

- 4576213 contains 312 (4576213).
- 4576213 avoids 1234.

Notation: $S_n(q) = \{\pi \in S_n | \pi \text{ avoids } q\}$.
For example, $|S_n(12)| = 1$ for $n \geq 0$.

Barred Patterns

- Consider *pairs* of permutations q_*, q^* such that q_* is contained in q^*.
  e.g. $q_* = 123$ and $q^* = 15342$
- Choose one instance of q_* in q^*.
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- Write q by taking the letters of q^* and putting a bar over letters *not* in q_*.
  For $q_* = 123$ and $q^* = 15342$, we have $q = 1\bar{5}34\bar{2}$.

Barred Pattern Avoidance/Containment

p avoids q if every instance of q_* in p is part of an instance of q^* in p.

Some nice examples include:

$$\left|S_n(\bar{1}32)\right| = (n-1)!$$

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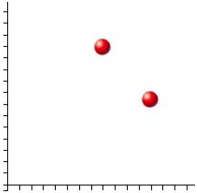

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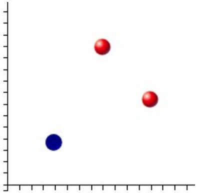

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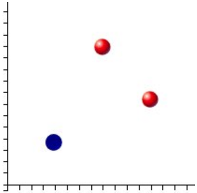

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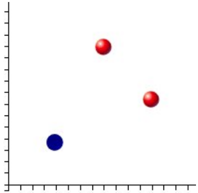

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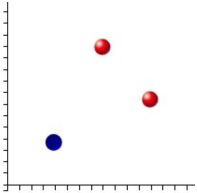

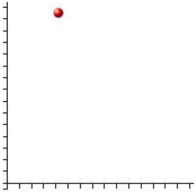

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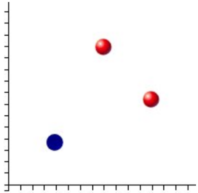

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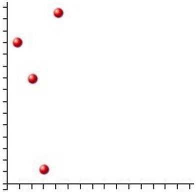

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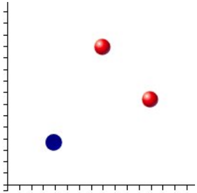

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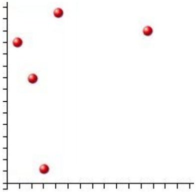

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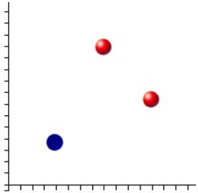

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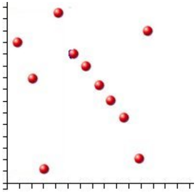

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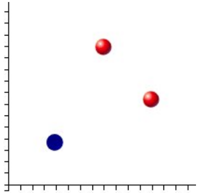

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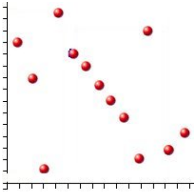

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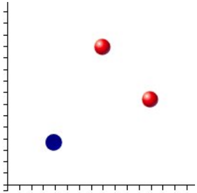

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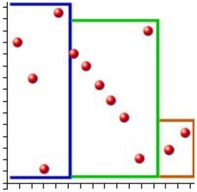

Previous work with Particular Barred Patterns

- West, 1990:
  A permutation is 2-stack sortable
  if and only if it avoids 2341 and $3\bar{5}241$.
- Bousquet-Melou and Butler, 2006:
  A permutation is forest-like
  if and only if it avoids 1324 and $21\bar{3}54$.
- Claesson, Dukes, and Kitaev, 2008:
  $(2+2)$-free posets
  are in bijection with permutations which avoid $3\bar{1}52\bar{4}$.
- Burstein and Lankham, 2006:
  A permutation is a reverse patience word
  if and only if it avoids $3-\bar{1}-42$.

Patterns of length ≤ 5

Length	Bars	No. Sequences	Possible Sequences
2	0	1	1
3	1	2	1, $(n-1)!$
4	2	2	1, $(n-2)!$
5	3	2	1, $(n-3)!$
3	0	1	Catalan
4	1	4	Catalan, Bell, A051295, & A137533
5	2	17	A110447, A117106, & 15 more
4	0	3	A005802, A061552, A022558
5	1	13	A006789, A047970, A098569, A122993, & 9 more

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There are at least 24 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.

Goals

Main Goal

Automate a method to count $S_n(q)$.

The method of enumeration schemes is a divide-and-conquer approach to finding recurrences that has been applied to pattern-avoiding permutations (Vatter, Zeilberger), and to pattern-avoiding words (P.).

Divide

Notation

$$S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \;\middle|\; \begin{array}{l} \pi \text{ avoids } q \\ \pi \text{ has prefix } p_1 \cdots p_l \end{array} \right\}$$

$$S_n\left(q; \begin{matrix} p_1 \cdots p_l \\ i_1 \cdots i_l \end{matrix}\right) := \left\{ \pi \in S_n \;\middle|\; \begin{array}{l} \pi \text{ avoids } q \\ \pi \text{ has prefix } p_1 \cdots p_l \\ \pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n \end{array} \right\}$$

For example,

$$S_3(132; 12) = \{123, 231\}$$

$$S_3\left(132; \begin{matrix} 12 \\ 23 \end{matrix}\right) = \{231\}$$

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For *any* pattern q, we have

$$S_n(q) = S_n(q; 1)$$
$$= S_n(q; 12) \cup S_n(q; 21),$$

etc.

$$\begin{tikzcd} & \emptyset \arrow[d] & \\ & 1 \arrow[dl] \arrow[dr] & \\ 12 & & 21 \end{tikzcd}$$

Reversibly Deletable

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Given q and p, a position r is said to be *reversibly deletable* if

1. deleting p_r from a q-avoiding permutation beginning with $p_1 \cdots p_l$ always produces a q avoiding permutation and
2. inserting p_r into a q-avoiding permutation beginning with $p_1 \cdots p_{r-1} p_{r+1} \cdots p_l$ always produces a q-avoiding permutation.

If r is reversibly deletable, we have a natural bijection between $S_n(q; p_1 \cdots p_r \cdots p_l)$ and $S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)$ so that:

$$|S_n(q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$$

Spacing Vectors

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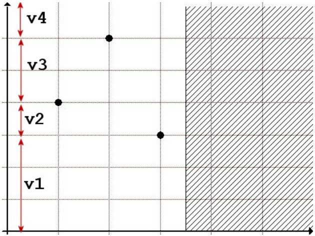
Given q and p (of length l) let v be a vector in $\mathbb{N}^{l+1}$. Then, $S_n(q; p; v)$ denotes the set of permutations of length n, avoiding q, beginning with prefix p with exactly v_1 letters smaller than "1", v_j letters greater than "$j-1$" and smaller than "j", and exactly v_{l+1} letters greater than "l".

For example,



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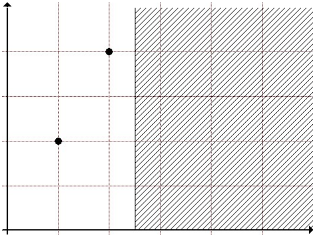

$$S_n(132; 12; \langle 0, 1, 0 \rangle) = \{\} .$$

Gap Vectors

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A spacing vector v is a *gap vector* for $[q, p]$ if there are no permutations avoiding q with prefix p and spacing vector $\geq v$ (componentwise).

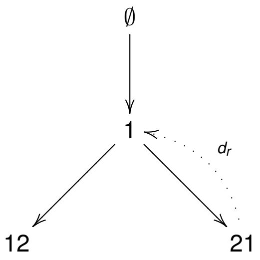
Gap vectors give a condition for which choices of $i_1, \ldots, i_l$ yield

$$\left| S_n \left(q; \begin{matrix} p_1 \cdots p_r \cdots p_l \\ i_1 \cdots i_r \cdots i_l \end{matrix} \right) \right| = 0.$$

Conquer

Objectives

1. Given q and p find r such that
$|S_n(q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$



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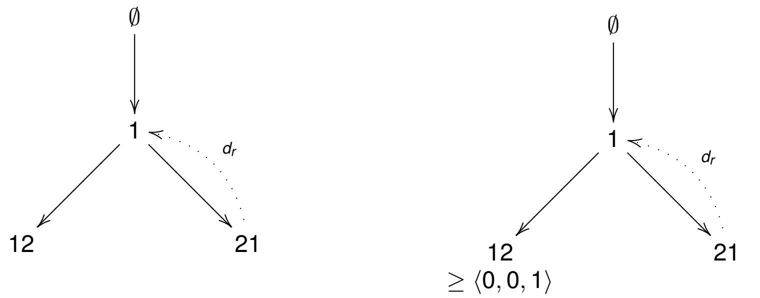

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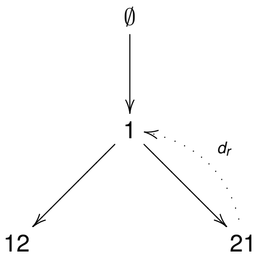

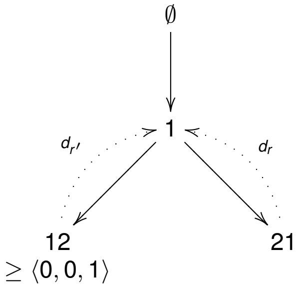

Enumeration Scheme

An enumeration scheme is a set of triples $[p_i, R_i, G_i]$ such that for each triple

- p_i is a reduced prefix of length n
- R_i is a subset of $\{1, \ldots, n\}$
- G_i is a set of vectors of length $n+1$
  and
- either R_i is non-empty or all refinements of p_i are also in the scheme.

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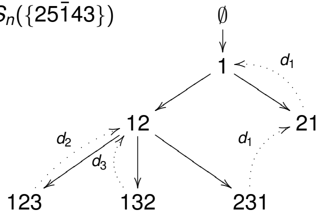
- p_i is a reduced prefix of length n (prefix)
- R_i is a subset of $\{1, \ldots, n\}$ (reversibly deletable positions)
- G_i is a set of vectors of length $n+1$ (gap vectors)
  and
- either R_i is non-empty or all refinements of p_i are also in the scheme.

Success Rate

Pattern Lengths	Success Rate	Pattern Lengths	Success Rate
[2,1]	1/1 (100%)	[3,0],[3,0],[3,1]	43/45 (95.6%)
[2,1],[2,0]	2/2 (100%)	[3,0],[3,0],[3,2]	45/45 (100%)
[2,1],[2,1]	2/2 (100%)	[3,0],[3,1],[3,1]	135/138 (97.8%)
		[3,0],[3,1],[3,2]	280/280 (100%)
[3,1]	4/4 (100%)	[3,0],[3,2],[3,2]	138/138 (100%)
[3,2]	4/4 (100%)	[3,1],[3,1],[3,1]	115/118 (97.5%)
[3,0],[3,1]	18/20 (90%)	[3,1],[3,1],[3,2]	378/378 (100%)
[3,0],[3,2]	20/20 (100%)	[3,1],[3,2],[3,2]	378/378 (100%)
[3,1],[3,1]	27/28 (96.4%)	[3,2],[3,2],[3,2]	118/118 (100%)
[3,1],[3,2]	50/50 (100%)		
[3,2],[3,2]	28/28 (100%)	[4,1]	12/16 (75%)
		[4,2]	25/26 (96.2%)
[3,1],[4,0]	59/71 (83.1%)	[4,3]	16/16 (100%)
[3,1],[4,1]	229/240 (95.4%)		
[3,1],[4,2]	355/364 (97.5%)	[5,1]	15/89 (16.9%)
[3,0],[4,1]	84/88 (95.5%)	[5,2]	(in progress)
[3,0],[4,2]	133/136 (97.8%)		
[4,0],[5,1]	(in progress)		

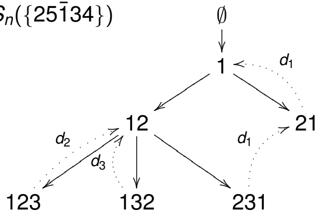
New Results: Length 5 with 1 Bar

$S_n(\{25\bar{1}43\})$



$\geq \langle 0, 1, 0, 0 \rangle$

$S_n(\{25\bar{1}34\})$

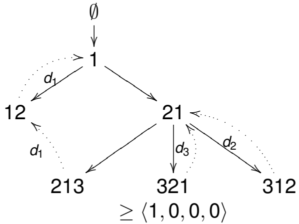


$\geq \langle 0, 0, 1, 0 \rangle$

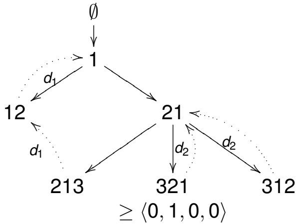
both give the sequence
1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810, 6325151, 49448313, 405298482, 3470885747, 30965656442
for $1 \leq n \leq 15$.

New Results: Length 5 with 1 Bar

$S_n(\{43\bar{5}21\})$



$S_n(\{43\bar{5}12\})$



both also give the sequence
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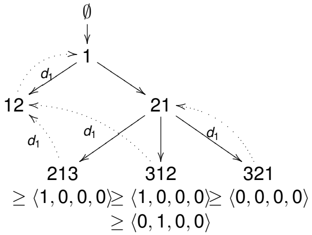
for $1 \leq n \leq 15$.

New Results: Length 5 with 2 Bars

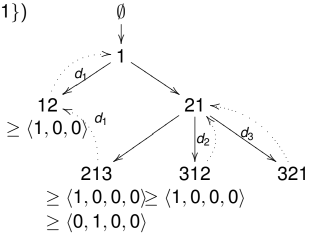
$S_n(\{5\bar{1}\bar{2}43\})$



gives the new sequence
1, 2, 5, 14, 43, 143, 511, 1950, 7903, 33848, 152529, 720466, 3555715, 18285538, 97752779

for $1 \leq n \leq 15$.

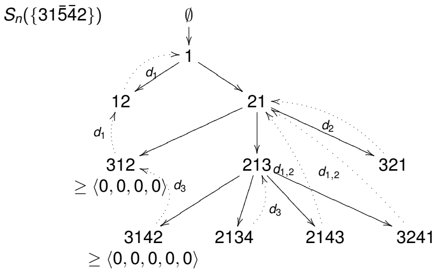
$S_n(\{\bar{5}\bar{4}231\})$



gives the new sequence
1, 2, 5, 14, 43, 146, 561, 2518, 13563, 88354, 686137, 6191526, 63330147, 720314930, 8985750097

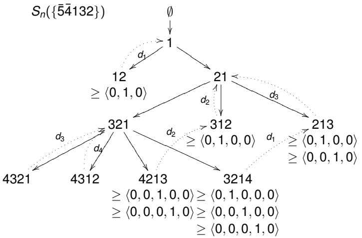
for $1 \leq n \leq 15$.

New Results: Length 5 with 2 Bars



gives the new sequence
1, 1, 2, 5, 14, 43, 144, 522, 2030, 8398, 36714, 168793, 813112, 4091735, 21451972, 116891160

for $1 \leq n \leq 15$.



gives the new sequence
1, 1, 2, 5, 14, 43, 147, 575, 2648, 14617, 96696, 754585, 6794015, 69116493, 781266266, 9688636317

for $1 \leq n \leq 15$.

Summary

- The method of enumeration schemes confirms many known results for barred patterns and generates new results for $S_n(25\bar{1}43)$, $S_n(25\bar{1}34)$, $S_n(43\bar{5}21)$, $S_n(43\bar{5}12)$, $S_n(5\bar{1}\bar{2}43)$, $S_n(\bar{5}\bar{4}231)$, $S_n(31\bar{5}\bar{4}2)$, $S_n(\bar{5}\bar{4}132)$.
- It remains to find other ways to count permutations avoiding barred patterns.
- There are at least 24 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.

Summary

- The method of enumeration schemes confirms many known results for barred patterns and generates new results for $S_n(25\bar{1}43)$, $S_n(25\bar{1}34)$, $S_n(43\bar{5}21)$, $S_n(43\bar{5}12)$, $S_n(5\bar{1}\bar{2}43)$, $S_n(\bar{5}\bar{4}231)$, $S_n(31\bar{5}\bar{4}2)$, $S_n(\bar{5}\bar{4}132)$.
- It remains to find other ways to count permutations avoiding barred patterns.
- There are at least 19 new sequences obtained by counting $S_n(q)$, where q is a barred pattern of length 5.