Enumeration Schemes for Permutations Avoiding Barred Patterns

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Given a string of numbers $q = q_1 \cdots q_n$, the reduction of $q$ is the string obtained by replacing the $i^{th}$ smallest number of $q$ with $i$. 
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Given a string of numbers $q = q_1 \cdots q_n$, the reduction of $q$ is the string obtained by replacing the $i^{th}$ smallest number of $q$ with $i$.

For example, the reduction of $26795$ is $1\bullet\bullet2$. 
Given a string of numbers \( q = q_1 \cdots q_n \), the reduction of \( q \) is the string obtained by replacing the \( i^{th} \) smallest number of \( q \) with \( i \).

For example, the reduction of \( 26795 \) is \( 13\cdot\cdot2 \).
Definitions

Reduction

- Given a string of numbers $q = q_1 \cdots q_n$, the reduction of $q$ is the string obtained by replacing the $i^{th}$ smallest number of $q$ with $i$.

- For example, the reduction of $26795$ is $134\bullet 2$. 
Given a string of numbers $q = q_1 \cdots q_n$, the reduction of $q$ is the string obtained by replacing the $i^{th}$ smallest number of $q$ with $i$.

For example, the reduction of $26795$ is $13452$. 
Given permutations $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$,

- $p$ contains $q$ as a pattern if there is $1 \leq i_1 < \cdots < i_m \leq n$ so that $p_{i_1} \cdots p_{i_m}$ reduces to $q$;
- otherwise $p$ avoids $q$.

For example,

- 4576213 contains 312 (4576213).
- 4576213 avoids 1234.

Notation: $S_n(q) = \{\pi \in S_n | \pi \text{ avoids } q\}$. For example, $|S_n(12)| = 1$ for $n \geq 0$. 
Consider *pairs* of permutations $q_*, q^*$ such that $q_*$ is contained in $q^*$.  
*E.g.* $q_* = 123$ and $q^* = 15342$

Choose one instance of $q_*$ in $q^*$.  
*E.g.* $q_* = 123$ and $q^* = 15342$

Write $q$ by taking the letters of $q^*$ and putting a bar over letters *not* in $q_*$.  
For $q_* = 123$ and $q^* = 15342$, we have $q = 1\bar{5}3\bar{4}2$.  

Barred Pattern Avoidance/Containment

\( p \) avoids \( q \) if every instance of \( q^* \) in \( p \) is part of an instance of \( q^* \) in \( p \).

Some nice examples include:

\[ |S_n(\overline{132})| = (n - 1)! \]
Barred Pattern Avoidance/Containment

$p$ avoids $q$ if every instance of $q_*$ in $p$ is part of an instance of $q^*$ in $p$.

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$$|S_n(132)| = (n - 1)!$$
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$p$ avoids $q$ if every instance of $q^\ast$ in $p$ is part of an instance of $q^\ast$ in $p$.

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$$|S_n(\overline{132})| = (n - 1)!$$

$$= |\{\pi \in S_n | \pi_1 = 1\}|$$
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$$|S_n(1423)| = B_n \text{ (Bell numbers)}$$

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\[ = |\{ \pi \in S_n | \pi_1 = 1 \}| \]
Barred Pattern Avoidance/Containment

A pattern \( p \) avoids a pattern \( q \) if every instance of \( q^* \) in \( p \) is part of an instance of \( q^* \) in \( p \).

Some nice examples include:

\[
|S_n(132)| = (n - 1)!
\]

\[
|S_n(1423)| = B_n \text{ (Bell numbers)}
\]

\[= |\{\pi \in S_n | \pi_1 = 1\}|\]
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\( p \) avoids \( q \) if every instance of \( q^* \) in \( p \) is part of an instance of \( q^* \) in \( p \).

Some nice examples include:

\[ |S_n(\overline{132})| = (n - 1)! \]

\[ |S_n(1\overline{423})| = B_n \text{ (Bell numbers)} \]

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$p$ avoids $q$ if every instance of $q_*$ in $p$ is part of an instance of $q^*$ in $p$.

Some nice examples include:

\[ |S_n(132)| = (n - 1)! \]

\[ |S_n(1\overline{4}23)| = B_n \text{ (Bell numbers)} \]

\[ = |\{ \pi \in S_n | \pi_1 = 1 \}| \]
Barred Patterns Avoidance/Containment

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$$= |\{\pi \in S_n | \pi_1 = 1\}|$$
Results

Previous work with Particular Barred Patterns

- West, 1990: A permutation is \textit{2-stack sortable} if and only if it avoids 2341 and 3\textoverline{5}241.

- Bousquet-Melou and Butler, 2006: A permutation is \textit{forest-like} if and only if it avoids 1324 and 2\textoverline{1}354.

- Claesson, Dukes, and Kitaev, 2008: \((2 + 2)\)-free posets are in bijection with permutations which avoid 3\textoverline{1}52\textoverline{4}.

- Burstein and Lankham, 2006: A permutation is a \textit{reverse patience word} if and only if it avoids 3\textoverline{–1–42}. 
### Patterns of length \( \leq 5 \)

<table>
<thead>
<tr>
<th>Length</th>
<th>Bars</th>
<th>No. Sequences</th>
<th>Possible Sequences</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1, ((n - 1)!)</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>1, ((n - 2)!)</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1, ((n - 3)!)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>Catalan</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>4</td>
<td>Catalan, Bell, A051295, &amp; A137533</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>17</td>
<td>A110447, A117106, &amp; 15 more</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>3</td>
<td>A005802, A061552, A022558</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>13</td>
<td>A006789, A047970, A098569, A122993, &amp; 9 more</td>
</tr>
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Patterns of length $\leq 5$

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There are at least 24 new sequences obtained by counting $S_n(q)$, where $q$ is a barred pattern of length 5.
Main Goal

Automate a method to count $S_n(q)$.

The method of enumeration schemes is a divide-and-conquer approach to finding recurrences that has been applied to pattern-avoiding permutations (Vatter, Zeilberger), and to pattern-avoiding words (P.).
### Notation

\[
S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \middle| \begin{array}{l}
\pi \text{ avoids } q \\
\pi \text{ has prefix } p_1 \cdots p_l
\end{array} \right\}
\]

\[
S_n\left(q; p_1 \cdots p_l \atop i_1 \cdots i_l\right) := \left\{ \pi \in S_n \middle| \begin{array}{l}
\pi \text{ avoids } q \\
\pi \text{ has prefix } p_1 \cdots p_l \\
\pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n
\end{array} \right\}
\]

For example,

\[
S_3(132; 12) = \{123, 231\}
\]

\[
S_3\left(132; \begin{array}{c}12 \\ 23\end{array}\right) = \{231\}\]
### Notation

\[
S_n(q; p_1 \cdots p_l) := \left\{ \pi \in S_n \mid \begin{array}{l}
\pi \text{ avoids } q \\
\pi \text{ has prefix } p_1 \cdots p_l
\end{array} \right\}
\]

\[
S_n(q; i_1 \ldots i_l) := \left\{ \pi \in S_n \mid \begin{array}{l}
\pi \text{ avoids } q \\
\pi \text{ has prefix } p_1 \cdots p_l \\
\pi = i_1 \cdots i_l \pi_{l+1} \cdots \pi_n
\end{array} \right\}
\]

For any pattern \( q \), we have

\[
S_n(q) = S_n(q; 1) = S_n(q; 12) \cup S_n(q; 21),
\]

etc.
Given $q$ and $p$, a position $r$ is said to be *reversibly deletable* if

1. deleting $p_r$ from a $q$-avoiding permutation beginning with $p_1 \cdots p_l$ always produces a $q$ avoiding permutation and

2. inserting $p_r$ into a $q$-avoiding permutation beginning with $p_1 \cdots p_{r-1} p_{r+1} \cdots p_l$ always produces a $q$-avoiding permutation.

If $r$ is reversibly deletable, we have a natural bijection between $S_n(q; p_1 \cdots p_r \cdots p_l)$ and $S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)$ so that:

$$|S_n(q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$$
Spacing Vectors

Given $q$ and $p$ (of length $l$) let $v$ be a vector in $\mathbb{N}^{l+1}$. Then, $S_n(q; p; v)$ denotes the set of permutations of length $n$, avoiding $q$, beginning with prefix $p$ with exactly $v_1$ letters smaller than “1”, $v_j$ letters greater than “$j - 1$” and smaller than “$j$”, and exactly $v_{l+1}$ letters greater than “$l$”.

For example,
Spacing Vectors

Given $q$ and $\rho$ (of length $l$) let $\nu$ be a vector in $\mathbb{N}^{l+1}$. Then, $S_n(q; \rho; \nu)$ denotes the set of permutations of length $n$, avoiding $q$, beginning with prefix $\rho$ with exactly $\nu_1$ letters smaller than “1”, $\nu_j$ letters greater than “$j-1$” and smaller than “$j$”, and exactly $\nu_{l+1}$ letters greater than “$l$”.

For example,

$$S_n(132; 12; \langle 0, 1, 0 \rangle) = \{\}.$$
A spacing vector \( v \) is a gap vector for \([q, p]\) if there are no permutations avoiding \( q \) with prefix \( p \) and spacing vector \( \geq v \) (componentwise).

Gap vectors give a condition for which choices of \( i_1, \ldots, i_l \) yield

\[
\left| S_n \left( q; p_1 \cdots p_r \cdots p_l \atop i_1 \cdots i_r \cdots i_l \right) \right| = 0.
\]
Enumeration Schemes

Conquer

Objectives

1. Given $q$ and $p$ find $r$ such that
   
   \[ |S_n(q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1}p_{r+1} \cdots p_l)| \]
Conquer

Objectives

1. Given $q$ and $p$ find $r$ such that
   $$|S_n(q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$$

2. Given $q$ and $p$, find $i_1, \ldots, i_l$ such that
   $$|S_n(q; p_1 \cdots p_r \cdots p_l)| = 0$$

\[ \begin{array}{c}
\emptyset \\
\downarrow \\
\emptyset \\
\downarrow \\
1 \\
\arrowsleft{d_r} \\
\downarrow \\
12 \\
\downarrow \\
12 \\
\geq \langle 0, 0, 1 \rangle
\end{array} \]
Conquer

Objectives

1. Given $q$ and $p$ find $r$ such that
   $$|S_n(q; p_1 \cdots p_r \cdots p_l)| = |S_{n-1}(q; p_1 \cdots p_{r-1} p_{r+1} \cdots p_l)|$$

2. Given $q$ and $p$, find $i_1, \ldots, i_l$ such that
   $$|S_n(q; p_1 \cdots p_r \cdots p_l)_{i_1 \cdots i_r \cdots i_l}| = 0$$
An enumeration scheme is a set of triples $[p_i, R_i, G_i]$ such that for each triple

- $p_i$ is a reduced prefix of length $n$
- $R_i$ is a subset of $\{1, \ldots, n\}$
- $G_i$ is a set of vectors of length $n + 1$

and

- either $R_i$ is non-empty or all refinements of $p_i$ are also in the scheme.
An enumeration scheme is a set of triples \([p_i, R_i, G_i]\) such that for each triple:

- \(p_i\) is a reduced prefix of length \(n\) (prefix)
- \(R_i\) is a subset of \(\{1, \ldots, n\}\) (reversibly deletable positions)
- \(G_i\) is a set of vectors of length \(n + 1\) (gap vectors)

and

- either \(R_i\) is non-empty or all refinements of \(p_i\) are also in the scheme.
# Success Rate

<table>
<thead>
<tr>
<th>Pattern Lengths</th>
<th>Success Rate</th>
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</tr>
</thead>
<tbody>
<tr>
<td>[2,1]</td>
<td>1/1 (100%)</td>
<td>[3,0],[3,0],[3,1]</td>
<td>43/45 (95.6%)</td>
</tr>
<tr>
<td>[2,1],[2,0]</td>
<td>2/2 (100%)</td>
<td>[3,0],[3,0],[3,2]</td>
<td>45/45 (100%)</td>
</tr>
<tr>
<td>[2,1],[2,1]</td>
<td>2/2 (100%)</td>
<td>[3,0],[3,1],[3,1]</td>
<td>135/138 (97.8%)</td>
</tr>
<tr>
<td>[3,1]</td>
<td>4/4 (100%)</td>
<td>[3,0],[3,1],[3,2]</td>
<td>138/138 (100%)</td>
</tr>
<tr>
<td>[3,2]</td>
<td>4/4 (100%)</td>
<td>[3,1],[3,1],[3,1]</td>
<td>115/118 (97.5%)</td>
</tr>
<tr>
<td>[3,0],[3,1]</td>
<td>18/20 (90%)</td>
<td>[3,1],[3,1],[3,2]</td>
<td>378/378 (100%)</td>
</tr>
<tr>
<td>[3,0],[3,2]</td>
<td>20/20 (100%)</td>
<td>[3,1],[3,2],[3,2]</td>
<td>378/378 (100%)</td>
</tr>
<tr>
<td>[3,1],[3,1]</td>
<td>27/28 (96.4%)</td>
<td>[3,2],[3,2],[3,2]</td>
<td>118/118 (100%)</td>
</tr>
<tr>
<td>[3,1],[3,2]</td>
<td>50/50 (100%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[3,2],[3,2]</td>
<td>28/28 (100%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[3,1],[4,0]</td>
<td>59/71 (83.1%)</td>
<td>[4,1]</td>
<td>12/16 (75%)</td>
</tr>
<tr>
<td>[3,1],[4,1]</td>
<td>229/240 (95.4%)</td>
<td>[4,2]</td>
<td>25/26 (96.2%)</td>
</tr>
<tr>
<td>[3,1],[4,2]</td>
<td>355/364 (97.5%)</td>
<td>[4,3]</td>
<td>16/16 (100%)</td>
</tr>
<tr>
<td>[3,0],[4,1]</td>
<td>84/88 (95.5%)</td>
<td>[5,1]</td>
<td>15/89 (16.9%)</td>
</tr>
<tr>
<td>[3,0],[4,2]</td>
<td>133/136 (97.8%)</td>
<td>[5,2]</td>
<td>(in progress)</td>
</tr>
<tr>
<td>[4,0],[5,1]</td>
<td>(in progress)</td>
<td></td>
<td></td>
</tr>
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</table>
New Results: Length 5 with 1 Bar

both give the sequence
1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810, 6325151, 49448313, 405298482, 3470885747, 30965656442
for $1 \leq n \leq 15$. 
New Results: Length 5 with 1 Bar

both also give the sequence
1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810, 6325151, 49448313, 405298482, 3470885747, 30965656442
for $1 \leq n \leq 15$. 
New Results: Length 5 with 2 Bars

\[ S_n(\{5\bar{1}243\}) \]
\[ S_n(\{\bar{5}4231\}) \]

\[
\begin{align*}
\text{gives the new sequence} & \quad 1, 2, 5, 14, 43, 143, 511, 1950, 7903, \\
& \quad 33848, 152529, 720466, 3555715, \\
& \quad 18285538, 97752779
\end{align*}
\]
\[ \text{for } 1 \leq n \leq 15. \]

\[
\begin{align*}
\text{gives the new sequence} & \quad 1, 2, 5, 14, 43, 146, 561, 2518, 13563, \\
& \quad 88354, 686137, 6191526, 63330147, \\
& \quad 720314930, 8985750097
\end{align*}
\]
\[ \text{for } 1 \leq n \leq 15. \]
New Results: Length 5 with 2 Bars

\[ S_n(\{31\bar{5}42\}) \]

\[ S_n(\{\bar{5}4132\}) \]

gives the new sequence
\[ 1, 1, 2, 5, 14, 43, 147, 575, 2648, 14617, 96696, 754585, 6794015, 69116493, 781266266, 9688636317 \]
for \(1 \leq n \leq 15\).
The method of enumeration schemes confirms many known results for barred patterns and generates new results for $S_n(25\bar{1}43)$, $S_n(25\bar{1}34)$, $S_n(43\bar{5}21)$, $S_n(43\bar{5}12)$, $S_n(5\bar{1}\bar{2}43)$, $S_n(\bar{5}\bar{4}231)$, $S_n(31\bar{5}\bar{4}2)$, $S_n(\bar{5}\bar{4}132)$.

It remains to find other ways to count permutations avoiding barred patterns.

There are at least 24 new sequences obtained by counting $S_n(q)$, where $q$ is a barred pattern of length 5.
Summary

- The method of enumeration schemes confirms many known results for barred patterns and generates new results for \( S_n(25\bar{1}43), S_n(25\bar{1}34), S_n(43\bar{5}21), S_n(43\bar{5}12), S_n(5\bar{1}\bar{2}43), S_n(\bar{5}\bar{4}231), S_n(31\bar{5}\bar{4}2), S_n(\bar{5}\bar{4}132). \)

- It remains to find other ways to count permutations avoiding barred patterns.

- There are at least 19 new sequences obtained by counting \( S_n(q) \), where \( q \) is a barred pattern of length 5.