On the Edge Set of Graphs of Lattice Paths

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Introduction

A classic combinatorical problem, presented in nearly every introductory text, is enumerating the number of distinct paths on an m x n rectangular lattice. For the purposes of this paper, we let m denote the number of rows and n denote the number of columns of rectangular cells in the lattice.

This problem is often modeled as walking along a rectangular grid of square city blocks. Following Gillman [2], we say two paths are essentially the same, or $(k+1)$-equivalent, if they share more than k steps (or, conversely they are k-distinct if they share no more than k steps). Paths are denoted as a sequence of m North steps and n East steps on the lattice (N and E respectively). Also, WLOG, allow $m \geq n$ due to the symmetry of the lattice. For simplicity, we let C denote $\binom{m+n}{n}$ and will let $\{P_1, P_2, ..., P_C\}$ denote the set of all paths on the $m \times n$ lattice, with paths listed in reverse lexicographic order.

The set of all paths on the m x n lattice, denoted as L(m, n), can be viewed as the vertices of a graph. The edges of the graph will connect those paths that are $(k + 1)$-equivalent. This graph is denoted as G(m, n, k).

Barrier Paths

This section describes two complete subgraphs of $G(m, n, k)$.

Definition 1 *In $L(m,n)$, we call the paths of the form $N^{k+1}E^nN^{m-(k+1)}$ and $E^{k+1}N^mE^{n-(k+1)}$ barrier paths and denote them as P_{y_k} and P_{x_k} respectively when the set of paths of the $m \times n$ lattice are listed in reverse lexicographic order.*

Result 1 *The index y_k is found by*

$$y_k = \binom{m+n-k-1}{n}.$$

Proof: P_{y_k} is of the form $N^{k+1}E^nN^{m-(k+1)}$ and encloses an $(m-(k+1)) \times n$ sub-lattice of $L(m,n)$. Since P_{y_k} is the final path in this sub-lattice, $y_k = \binom{m+n-k-1}{n}$. □

Result 2 *The index x_k is found recursively by*

$$x_k = \binom{m-k+n-1}{m-1} + x_{k-1} \text{ with } x_0 = \binom{m+n-1}{m-1} + 1.$$

Proof: P_{x_0} immediately follows the path P_{y_0}. Since $y_0 = \binom{m+n-1}{n}$ from Result 1, it is easy to see that $x_0 = \binom{m+n-1}{n} + 1$.

P_{x_k} follows any path of the form $E^kNE^{n-k}N^{m-1}$. This means that $P_{x_{k-1}}$ and P_{x_k-1} enclose an $(m-k) \times (n-1)$ sub-lattice of $L(m,n)$, and $x_k - x_{k-1} = \binom{m-k+n-1}{m-1}$. Thus, $x_k = \binom{m-k+n-1}{m-1} + x_{k-1}$. □

Theorem 1 *The sets $A = \{P_1, ..., P_{y_k}\}$ and $B = \{P_{x_k}, ..., P_C\}$ induce complete sub-graphs of $G(m,n,k)$.*

Proof: Any path in A begins with $(k+1)$ N steps. Thus, it follows that every path in A is adjacent to every other path in the set. Therefore, A forms a complete sub-graph of $G(m,n,k)$.

Likewise, any path in B begins with $(k+1)$ E steps. Thus, it follows that every path in B is adjacent to every other path in the set. Therefore, B forms a complete sub-graph of $G(m,n,k)$. □

It is important to note that neither of the induced sub-graphs on A or B is necessarily maximal. Consider the paths $P = E^k(NE)^{m-1}N$ and $Q = N^k(EN)^{n-1}E$. P is adjacent to every path in B and Q is adjacent to every path in A so neither sub-graph is maximal.

Some Special Cases

Brewer et al [1] determined the size of $G(m,1,k)$ in the following theorem.

Theorem 2 *If* $0 \le k \le m-1$,

$$|E(m,1,k)| = \binom{m-1}{2} - \binom{k+2}{2}.$$

Since the size of $G(m,1,k)$ has been determined, we turn our attention to the size of $G(m,2,k)$ and begin by considering the extreme cases.

Definition 2 *Let $p(m,n,k)$ denote the number of pairs of paths in $L(m,n)$ that share exactly k steps. For $k \ge 1$, we have*

$$p(m,n,k) = |E(m,n,k-1)| - |E(m,n,k)|.$$

Theorem 3 *If* $m \ge 2$,

$$|E(m,2,0)| = 3\binom{m+3}{4} - \binom{m+1}{2} - 2(m-1).$$

Proof: Let $|E'_A(B)|$ denote the number of pairs of paths, one in set A and the other in set B, that do not share any edges. Note that the index of the barrier paths $y_0 = \binom{m+1}{1} = m+1$ and $x_0 = y_0 + 1 = m+2$. So let $A = \{P_1, ..., P_{m+1}\}$ and $B = \{P_{m+2}, ..., P_C\}$.

There are $\binom{C}{2}$ possible edges in $G(m,n,k)$. Since $p(m,2,0) = |E'_A(A)| + |E'_B(B)| + |E'_A(B)|$, that is, the total number of pairs of paths in $L(m,2)$ that are disjoint, then $|E(m,2,0)| = \binom{C}{2} - p(m,2,0)$. It is clear that $|E'_A(A)| = |E'_B(B)| = 0$ by Theorem 1, so we are left to find $|E'_A(B)|$.

P_1 does not share any steps with any path of the form EN^iEN^{m-i} for $0 \le i \le m-1$. Therefore, $|E'_A(P_1)| = m$.

For $A^* = \{P_x | 2 \le x \le m\}$, P_x is of the form $N^{m-x+1}EN^{x-1}E$.

P_x does not share any steps with any path in the $m-(x-1) \times 1$ sublattice of paths of the form $E \cdots N^{m-x+1}$ of $L(m,2)$. Thus,

$|E'_B(P_x)| = m - x + 2$ and

$$|E'_B(A^*)| = \sum_{x=2}^{m}(m - x + 2) = (m-1)(m+2) - \binom{m+1}{2} + 1.$$

P_{m+1} does not share any steps with any path of the form $N^i E^2 N^{m-i}$ for $1 \leq i \leq m-1$. Therefore, $|E'_B(P_{m+1})| = m - 1$.

The above cases have accounted for every path in A so

$$\begin{aligned} p(m,2,0) &= |E'_A(B)| \\ &= m + (m-1)(m+2) - \binom{m+1}{2} + 1 + m - 1 \\ &= \binom{m+1}{2} + 2(m-1). \end{aligned}$$

Since $|E(m,2,0)| = \binom{C}{2} - p(m,2,0)$, through algebraic manipulation we have

$$|E(m,2,0)| = 3\binom{m+3}{4} - \binom{m+1}{2} - 2(m-1). \qquad \square$$

Lemma 1 *There are*

$$\sum_{i=1}^{\binom{m+2}{2}-1} (i - |E(m,2,0)|)$$

pairs of disjoint paths in an $m \times 2$ lattice.

Proof: There are $\binom{m+2}{2}$ vertices in $G(m,2,k)$. The complete graph on these $\binom{m+2}{2}$ vertices has an edge for every pair of vertices, i.e. for all pairs of paths in $L(m,2)$. There are $\sum_{i=1}^{\binom{m+2}{2}-1} i$ edges in the complete graph on $\binom{m+2}{2}$ vertices.

Further, $|E(m,2,0)|$ is the number of edges in $G(m,2,k)$ representing pairs of paths that share at least 1 edge, i.e. not disjoint.

Therefore, the number of pairs of disjoint paths in $L(m,2)$ is the difference of these which is equal to

$$\sum_{i=1}^{\binom{m+2}{2}-1} (i - |E(m,2,0)|). \qquad \square$$

Theorem 4

$$|E(m,2,1)| = |E(m,2,0)| - 2\binom{m}{2} - 2(3m-5).$$

Proof: Since $|E(m,2,0)|$ is the number of pairs of paths that share at least one step and $|E(m,2,1)|$ is the number of pairs of paths that share at least two steps in $L(m,2)$, $|E(m,2,0)| - |E(m,2,1)|$ is the number of pairs of paths that share exactly one step. Consider all the ways two paths can share exactly one edge in $L(m,2)$. Notice that any path must begin with either a N step or an E step and must end with either a N step or an E step. There is exactly one pair of disjoint paths that share either their first or last N step. Also, for every interior E step, of which there are $m-2$, there are two pairs of disjoint paths. Now we only have to find the number of pairs of disjoint paths in the $(m-1)\times 2$ sublattices consisting of paths that either begin or end with an E step. Combining this result with (4), we have

$$|E(m,2,0)| - |E(m,2,1)| = 2 + 2(m-2) + 2[\sum_{i=1}^{\binom{m+1}{2}-1} (i - |E(m-1,2,0)|)].$$

Through algebraic manipulation, we arrive at our final result

$$|E(m,2,1)| = |E(m,2,0)| - 2\binom{m}{2} - 2(3m-5). \qquad \square$$

Theorem 5 *If $m \geq 2$,*

$$|E(m,2,m-1)| = 2\binom{m+1}{2}.$$

Proof: Two vertices in $G(m,2,m-1)$ share an edge iff their corresponding paths in $L(m,2)$ differ by exactly one N-E transposition. We will count all such pairs of equivalent paths.

Case 1: $P = EEN^m$ or $P = N^mEE$. Each of these paths has exactly one corner at which to make an N-E transposition. Therefore, these paths yield two pairs of equivalent paths.

Case 2: $P = N^iEEN^{m-i}, 0 < i < m$ or $P = EN^mE$. Each of these paths has exactly two corners at which to make a N-E transposition. Further, notice that there are $m-1$ choices for i and only one path of the form EN^mE, yielding $2m$ new pairs of equivalent paths.

Case 3: $P = EN^iEN^{m-i}$ or $P = N^iEN^{m-i}E, 0 < i < m$. Each of these paths has exactly three corners at which to make a N-E transposition. Since there are $2(m-1)$ paths of this form, there are $6(m-1)$ new pairs of equivalent paths.

Case 4: Consider the remaining paths of the form $P = N^iEN^jEN^{m-i-j}$, for $i,j > 0$ and $i+j < m$. Each of these paths has exactly four corners at which to make a N-E transposition. Further, since there are $\binom{m+2}{2} - 2(m-1) - m - 2$ paths of this form, we have $4(\binom{m+2}{2} - 2(m-1) - m - 2)$ new pairs of equivalent paths.

Thus, we have accounted for every path in $L(m,2)$ in one of the cases above, and for each path, we have counted all its equivalent paths. However, each pair of paths has been counted exactly twice, one time for each path in the pairing. Therefore, the size of $G(m,2,m-1)$ is exactly half the number of pairs of equivalent paths counted above, i.e.

$$2|E(m,2,m-1)| = 2 + 2m + 6(m-1) + 4(\binom{m+2}{2} - 2(m-1) - m - 2),$$

and

$$\begin{aligned} |E(m,2,m-1)| &= 1 + m + 3(m-1) + 2(\binom{m+2}{2} - 2(m-1) - m - 2) \\ &= 2\binom{m+1}{2}. \qquad \square \end{aligned}$$

The Size of $G(m, 2, k)$

Now we are ready to develop a formula for the number of edges in $G(m, 2, k)$ for all values of k. We begin with the following definition.

Definition 3 *The number of pairs of paths, P_1 and P_2, in $L(m, n)$ that share exactly k steps given P_1 begins with a N step and P_2 begins with an E step is found by the function $g(m, n, k)$.*

This definition gives us the following relationship.

Lemma 2

$$p(m, 2, k) = p(m - 1, 2, k - 1) + p(m, 1, k - 1) + g(m, 2, k).$$

Proof: $p(m-1, 2, k-1)$ is the number of pairs of paths that share exactly k steps given both paths begin with a N step, $p(m, 1, k - 1)$ is the number of pairs of paths that share exactly k steps given both paths begin with a E step, and by definition, $g(m, 2, k)$ is the number of pairs of paths that share exactly k steps given one path begins with a N step and the other begins with E step. Thus, all pairs of paths have been accounted for. □

Lemma 3

$$g(m, 2, k) = m + (2m - 1)(m - k - 1) - 3\binom{m - k - 1}{2}.$$

Proof: Begin by noting that $g(m - 1, 2, k - 1)$ is the number of pairs of paths in $L(m, 2)$ sharing exactly k steps, with one beginning with an N step and the other beginning with a E step, but both paths ending with a N step. There are $(m - k - 2)$ ways for two paths to share exactly k N steps on the interior of the lattice, and each of these sets of N steps results in two pairs of paths that share exactly k N steps and are disjoint elsewhere. Also, there is one pair of paths that share exactly k steps such that they share a final E step. This accounts for all possible paths in $g(m, 2, k)$ and verifies that

$$g(m, 2, k) = g(m - 1, 2, k - 1) + 2(m - k) - 3.$$

Also, from Lemma 2, Theorem 3, and Theorem 4,

$$\begin{aligned} g(m,2,1) &= p(m,2,1) - p(m-1,2,0) - p(m,1,0) \\ &= m^2 + 5m - 10 - \frac{1}{2}((m-1)^2 + 5(m-1) - 4) - 1 \\ &= \frac{1}{2}(m^2 + 7m - 14). \end{aligned}$$

Thus, by back substitution on $g(m,2,k) - g(m,2,k-1)$, we have

$$g(m,2,k) = m + (2m-1)(m-k-1) - 3\binom{m-k-1}{2}. \qquad \square$$

Theorem 6

$$p(m,2,k) = \frac{k+1}{2}(m^2 + 5m - (k+1)(k+4)).$$

Proof: (by induction) $p(m,2,1) = m^2 + 5m - 10$ is a direct result of Theorem 4. Now assume $p(m,2,i-1) = \frac{i}{2}(m^2 + 5m - (i)(i+3))$. From Lemma 2,

$$\begin{aligned} p(m,2,i) &= p(m-1,2,i-1) + i + g(m,2,i) \\ &= \frac{i+1}{2}(m^2 + 5m - (i+1)(i+4). \end{aligned}$$

Thus,

$$p(m,2,k) = \frac{k+1}{2}(m^2 + 5m - (k+1)(k+4)). \qquad \square$$

This theorem leads directly to the result that we want, a general formula for $|E(m,2,k)|$, which we now present as a corollary.

Corollary 1 *If $m \geq 2$ and $k < m$,*

$$|E(m,2,k)| = |E(m,2,0)| + 3\binom{k+3}{4} + 2k(k+2) + \frac{2k^2(k+1) - k(k+3)m(m+5)}{4}.$$

Proof: From Theorem 6,
$p(m,2,k) = \frac{k+1}{2}(m^2+5m-(k+1)(k+4))$. Thus,

$$|E(m,2,k)| = |E(m,2,0)| + \sum_{i=1}^{k-1}(\frac{i+1}{2}(m^2+5m-(i+1)(i+4)))$$

$$= |E(m,2,0)| + 3\binom{k+3}{4} + 2k(k+2) + \frac{2k^2(k+1) - k(k+3)m(m+5)}{4}. \quad \square$$

A Small Generalization

The following theorem suggests the increasing complexity of the recursion formula as n increases.

Theorem 7

$$|E(m,n,1)| = |E(m,n,0)|$$

$$-2(\binom{\binom{m+n-1}{m}}{2} - |E(m,n-1,0)) - 2(\binom{\binom{m+n-1}{n}}{2} - |E(m-1,n,0)|$$

$$-\sum_{j=1}^{m-1}\sum_{i=1}^{n-2}(2(\binom{\binom{i+j}{i}}{2} - |E(i,j,0)|)(\binom{\binom{m-j+n-i-1}{m-j}}{2} - |E(m-j,n-i-1,0)|))$$

$$-\sum_{j=1}^{m-2}\sum_{i=1}^{n-1}(2(\binom{\binom{i+j}{i}}{2} - |E(i,j,0)|)(\binom{\binom{m-j+n-i-1}{n-i}}{2} - |E(m-j-1,n-i,0)|)).$$

Proof: Notice that $|E(m,n,0)| - |E(m,n,1)|$ is the number of distinct paths in $L(m,n)$ that share exactly one edge.

In $L(m,n)$, label an edge in terms of its distance from the lower left corner (i.e. set the origin as the lower left corner).

First, we will count the pairs of paths that share exactly one E step.

(1). If two paths share an E step with label $(0,j)$ and $j > 0$, they necessarily share at least one N step as well. Thus, there are no pairs of paths that share exactly one E step of this form. Similarly for sharing an E step with label $(n-1,j)$ and $j < m$ and for (i,j) with ($i \neq 0$ or $n-1$) and ($j = 0$ or m).

(2). Now, count all pairs of paths that share exactly the E step $(0,0)$. Clearly, this is the same as the number of disjoint paths in $L(m,n-1)$. That is, $\binom{\binom{m+n-1}{m}}{2} - |E(m,n-1,0))$.

$\binom{m+n-1}{m}$ is the number of paths in $L(m,n-1)$, and $\binom{\binom{m+n-1}{m}}{2}$ in $L(m,n-1)$. $|E(m,n-1,0)|$ is the number of pairs of paths that share at least one step in $L(m,n-1)$. Thus, this difference is what we claim it is. Similarly for pairs of paths that share the E step $(n-1,m)$. Thus, there are $2(\binom{\binom{m+n-1}{m}}{2} - |E(m,n-1,0)))$ that share exactly the E step $(0,0)$ or $(n-1,m)$.

(3). Now, count all pairs of paths that share exactly an E step (i,j) that have not already been considered. There are $m-1$ choices for i and $n-2$ choices for j. Then, for each (i,j) pair, the number of pairs of paths that share this E step is twice the number of disjoint pairs of paths in $L(i,j)$ times the number of disjoint pairs of paths in $L(m-j,n-i-1)$. Yielding

$$\sum_{j=1}^{m-1}\sum_{i=1}^{n-2}(2(\binom{\binom{i+j}{i}}{2} - |E(i,j,0)|)(\binom{\binom{m-j+n-i-1}{m-j}}{2} - |E(m-j,n-i-1,0)|))$$

additional pairs of paths that share exactly one more E step.

Thus, there are

$$2(\binom{\binom{m+n-1}{m}}{2} - |E(m,n-1,0)))$$
$$+2\sum_{j=1}^{m-1}\sum_{i=1}^{n-2}(2(\binom{\binom{i+j}{i}}{2} - |E(i,j,0)|)(\binom{\binom{m-j+n-i-1}{m-j}}{2} - |E(m-j,n-i-1,0)|))$$

pairs of paths that share exactly one E step.

A similar argument shows that there are exactly

$$2(\binom{\binom{m+n-1}{n}}{2} - |E(m,n-1,0)))$$
$$+2\sum_{j=1}^{m-2}\sum_{i=1}^{n-1}(2(\binom{\binom{i+j}{i}}{2} - |E(i,j,0)|)(\binom{\binom{m-j+n-i-1}{n-i}}{2} - |E(m-j-1,n-i,0)|))$$

pairs of paths in $L(m,n)$ that share exactly one N step.

Combining the above two statements with some algebraic manipulation gives the stated theorem. $\square$

Conclusion

There are many other properties of these graphs that we could investigate, but our immediate attention will be on finding $|E(m,n,k)|$ in general and in a closed form, and of the other invariants for this family of graphs. Of particular interest is the independence number, which was the original question posed in Gillman [2].

References

[1] M. Brewer, A. Hughes, L. Pudwell, Graphs of Essentially Equivalent Lattice Paths, to appear in *Geombinatorics*.

[2] R. Gillman, Enumerating and constructing Essentially Equivalent Lattice Paths, *Geombinatorics*. 11 Oct 2001. 37-42.