GRAPHS OF ESSENTIALLY EQUIVALENT LATTICE PATHS

Marjorie Brewer, Adam Hughes, Lara Pudwell
Department of Mathematics and Computer Science
Valparaiso University, Valparaiso, IN 46383

Introduction

One of the classic combinatorial problems, presented in nearly every introductory text, is enumerating the number of distinct paths on an m x n rectangular lattice from bottom left to top right. (For an example, see Exercise 42 on page 240 of [1].) For this paper, we define m as the number of rows and n as the number of columns of rectangular cells in the lattice.

This problem is frequently modeled as walking along a rectangular grid of square city blocks. Following Gillman [2], we will say the paths P_1 and P_2 are "essentially the same", or $(k + 1)$-equivalent, if they share more than k steps (or, conversely that they are k-distinct if the number of steps they share is no greater than k). Paths are denoted as a sequence of m North steps and n East steps on the lattice (N and E respectively). WLOG, allow $m \geq n$, due to the symmetry of the lattice. For ease of notation, we will let C denote $\binom{m+n}{n}$ and will let $\{P_1, P_2, \ldots, P_C\}$ denote the set of all paths on the m x n lattice.

The set of all paths on the m x n lattice, denoted as $L(m, n)$, can be viewed as the vertices of a graph. The edges of the graph will connect those paths that are $(k + 1)$-equivalent. This graph will be denoted as $G(m, n, k)$.

Connectedness and Trees

First, we investigate the connectedness of these graphs and determine which are trees. We begin with a containment lemma.

Lemma 1: $E(G(m, n, k + 1)) \subseteq E(G(m, n, k))$; that is, $G(m, n, k + 1)$ is a subgraph of $G(m, n, k)$.

Proof: Any pair of paths in $L(m, n)$ that share at least $k + 1$ steps will necessarily share at least k steps as well. Thus, every edge in $G(m, n, k + 1)$, is also an edge of $G(m, n, k)$. □

Now we will determine when these graphs are connected. We want to be able to transform one path into any other path. The Trotter-Johnson Algorithm accomplishes this and has the property that each path differs from its predecessor by only one adjacent transposition. Thus, any path in $L(m, n)$ can be changed into any other path by a series of adjacent N-E transpositions. [3] [4].

Theorem 1: For all m, n, $G(m, n, k)$ is connected iff $k \leq m + n - 3$.

Proof: Begin by noting that if two paths in $L(m, n)$ differ by *exactly* two steps (they share $m + n - 2$), then they differ by exactly one N-E transposition. Therefore, for all $k \leq m + n - 3$, these two paths are equivalent, and their corresponding vertices share an edge. It follows, by the Trotter-Johnson Algorithm, that each vertex in $G(m, n, k)$ will be connected to each of the others by at least one path. Therefore, $G(m, n, k)$ is a connected graph.

Further, since it is impossible for two distinct lattice paths to differ by exactly 0 or 1 steps, each pair of paths in $L(m, n)$ will share at most $m + n - 2$ steps. It follows that $G(m, n, k)$ is the empty graph on C vertices for all $k \geq m + n - 2$. □

Our next theorem characterizes when the graph is a tree.

Theorem 2: $G(m, n, k)$ is a tree iff $n = 1$, $m \geq 2$, and $k = m - 2$.

Proof: Suppose $n = 1$ and $k = m - 2$. The paths in $L(m, 1)$ are $P_0P_1\ldots P_m$ where each $P_i = N^iEN^{m-i}$. Clearly P_o is adjacent in $G(m, 1, m-2)$ only to P_1, P_m only to P_{m-1}, and P_i only to P_{i-1} and P_{i+1}, if $1 \le i \le m-1$. Thus, $G(m, 1, m-2)$ is a path, with $m + 1$ vertices.

As noted above, $G(m, n, k)$ is an empty graph if $k \ge m + n - 2$. In the case $k \le m + n - 4$, $(ENN)(N^{m-2}E^{n-1})$, $(NEN)(N^{m-2}E^{n-1})$, and $(NNE)(N^{m-2}E^{n-1})$ are the vertices of a triangle in $G(m, n, k)$, so $G(m, n, k)$ is not a tree.

The cases remaining are $k = m + n - 3$, $n > 1$. In these cases, there is a four-cycle in $G(m, n, k)$ with vertices

$NENE(N^{m-2}E^{n-2})$, $ENNE(N^{m-2}E^{n-2})$, $ENEN(N^{m-2}E^{n-2})$, and $NEEN(N^{m-2}E^{n-2})$. □

On the Size of G(*m*, *n*, *k*)

Since the order of $G(m, n, k)$ is C, then the maximum number of edges possible on $G(m, n, k)$ is $\binom{C}{2}$. However, since P_1 and P_C are always *k*-distinct, this maximum is never attained. Our next theorem gives the size of the graph when $n = 1$.

Theorem 3: If $0 \le k \le m - 1$,

$$|E(G(m, 1, k))| = \binom{m+1}{2} - \binom{k+2}{2}.$$

Proof: Let, as in Theorem 2, $P_i = N^iEN^{m-i}$, $i = 0, 1, \ldots m$. These are the distinct vertices of $G(m, 1, k)$. Clearly, if $0 \le i < j \le m$, P_i and P_j are adjacent in $G(m, 1, k)$ if and only if $i + (m - j) \ge k + 1$, or $j - i \le m - (k + 1)$. It is an elementary exercise to count the number of such pairs (i, j) and thus to verify the conclusion of this Theorem. □

This theorem may generalize. Tables 1, 2, and 3 show some of the data we collected for $n = 1, 2, 3$ respectively. Clearly, patterns can be seen on the number of edges generated. We can use the method of repeated differences to find a formula for the sequences of sizes shown, but we are not sure if this sequence continues to give the sizes of the graphs.

Table 1

m	n	k	size
2	1	0	2
3	1	0	5
4	1	0	9
5	1	0	14
6	1	0	20
3	1	1	3
4	1	1	7
5	1	1	12
6	1	1	18
7	1	1	25
4	1	2	4
5	1	2	9
6	1	2	15
7	1	2	22
8	1	2	30

Table 2

m	n	k	size
1	2	0	2
2	2	0	10
3	2	0	35
4	2	0	89
5	2	0	187
2	2	1	6
3	2	1	21
4	2	1	63
5	2	1	147
6	2	1	291
3	2	2	12
4	2	2	36
5	2	2	99
6	2	2	219
7	2	2	417

Table 3

m	n	k	size
3	3	0	150
4	3	0	493
5	3	0	1331
6	3	0	3110
7	3	0	6520
3	3	1	102
4	3	1	351
5	3	1	1009
6	3	1	2492
7	3	1	5456
3	3	2	54
4	3	2	222
5	3	2	678
6	3	2	1793
7	3	2	4172

Summary

This is only the beginning of an investigation of this family of graphs. There are many paths subsequent analysis may take; however, our own immediate focus will be the following points.

1. Determine the size of G when $n \ne 1$.
2. Explore relationships between $G(m, n, k)$ and $G(m, n + 1, k)$ and $G(m + 1, n, k)$.
3. Investigate the independence number.

References

[1] R. Brown, *Advanced Mathematics: Precalculus with Discrete Mathematics and Data Analysis*, Houghton-Mifflin Co., Boston, 1994.

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[3] S. Johnson, Generation of permutations by adjacent transpositions, *Math. Comp*. **17** (1963), 282-285.

[4] H. Trotter, Algorithm 115: Perm, *Comm. ACM* **5** (1962), 434-435.