# de Bruijn arrays for L-fillings

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**Abstract**

We use modular arithmetic to construct a de Bruijn array, a $k \times k^2$ array consisting of exactly one copy of each L (a $2 \times 2$ array with the upper right corner removed) with digits chosen from $\{0, \ldots, k-1\}$.

## 1 Introduction

Consider the sequence 00010111. If we take its values three at a time we get 000, 001, 010, 101, 011, 111, 110, and 100 – all eight possible binary sequences of length 3. (We consider the end of the sequence to be glued to the beginning.) Such a sequence is called a *de Bruijn sequence*. More generally, we may ask for a sequence made up from the $k$ digits $\{0, 1, 2, \ldots, k-1\}$ that contains all possible subsequences of length $n$, i.e., a $(k, n)$-de Bruijn sequence.

Such sequences are well-studied and have been used in applications ranging from robotics to developing card tricks. Diaconis and Graham give a delightful overview of such applications in [2]. Nicolaas Govert de Bruijn [1], for whom such sequences are named, and I.J. Good [3] independently proved that $(k, n)$-de Bruijn sequences exist for every $k, n \geq 2$, and there are $\frac{k!^{k^{n-1}}}{k^n}$ of them.

More recently, mathematicians have analyzed a 2-dimensional generalization. Given $k \geq 2$ and $m, n, r, s \in \mathbb{Z}^+$ a $(k, r, s, m, n)$-*de Bruijn torus* is an $r \times s$ array that contains each of the $k^{mn} = rs$ fillings of an $m \times n$ array with entries from $\{0, \ldots, k-1\}$ exactly once. (Here, the left side of the array is glued to the right side, and the top of the array is glued to the bottom.) However, much is still unknown about general de Bruijn tori. Hurlbert and Isaak [4] showed that $(k, r, s, m, n)$-de Bruijn tori always exist

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 |
| 0 | 1 | 0 | 0 |

Figure 1: A (2,4,4,2,2)-de Bruijn torus

| | |
|---|---|
| 0 | |
| 0 | 0 |

| | |
|---|---|
| 0 | |
| 0 | 1 |

| | |
|---|---|
| 0 | |
| 1 | 0 |

| | |
|---|---|
| 0 | |
| 1 | 1 |

| | |
|---|---|
| 1 | |
| 0 | 0 |

| | |
|---|---|
| 1 | |
| 0 | 1 |

| | |
|---|---|
| 1 | |
| 1 | 0 |

| | |
|---|---|
| 1 | |
| 1 | 1 |

Figure 2: All possible binary L fillings

when $r = s > m = n$. An example of a $(2, 4, 4, 2, 2)$-de Bruijn torus is shown in Figure 1.

In this note, we consider a similar problem. Rather than compactly arranging all possible sequences or all possible rectangles, we consider using an L-shape, that is, a $2 \times 2$ array with the upper right corner removed. Since the L-shape has 3 entries, there are $k^3$ fillings of the L with digits from $\{0, \ldots, k-1\}$. The $2^3 = 8$ fillings of an L with a binary alphabet are shown in Figure 2. We wish to find a $k \times k^2$ array (with the left side glued to the right side an the top glued to the bottom) that contains each of the $k^3$ different L's exactly once. To distinguish from the tori in the previous paragraph, we call such an array a *$k$-de Bruijn L-array*. An example of a 2-de Bruijn L-array is shown in Figure 3.

It turns out that such arrays always exist. The rest of this note is aimed constructing one.

Suppose the alphabet size is $k$, and consider a $k \times k^2$ array, numbering

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 |

Figure 3: A 2-de Bruijn L-array

| | | | |
|---|---|---|---|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| 0 | 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 |
| 0 | 2 | 1 | 1 | 0 | 2 | 2 | 1 | 0 |

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 | 3 | 3 |
| 0 | 1 | 2 | 3 | 1 | 2 | 3 | 0 | 2 | 3 | 0 | 1 | 3 | 0 | 1 | 2 |
| 0 | 2 | 0 | 2 | 1 | 3 | 1 | 3 | 2 | 0 | 2 | 0 | 3 | 1 | 3 | 1 |
| 0 | 3 | 2 | 1 | 1 | 0 | 3 | 2 | 2 | 1 | 0 | 3 | 3 | 2 | 1 | 0 |

Figure 4: A 2-de Bruijn L-array, a 3-de Bruijn L-array, and a 4-de Bruijn L-array

both the rows and columns starting with 0. Towards filling the array, define the function $f(r,c) = (s+re)\%k$ where $0 \leq r \leq k-1$ is the row, $c = sk+e$ is the column with $0 \leq s, e, \leq k-1$, and $\%k$ means to take this value modulo $k$. For example, suppose $k = 4$ and we wish to compute $f(2,13)$. Since $13 = 3 \cdot 4 + 1$, we have $f(2,13) = (3 + 2 \cdot 1)\%4 = 1$. Our main result is:

**Theorem 1.** *Placing the value $f(r,c)$ into row $r$, column $c$ in a $k \times k^2$ array produces a $k$-de Bruijn L-array.*

The 2-de Bruijn L-array, the 3-de Bruijn L-array, and the 4-de Bruijn L-array produced by this formula are shown in Figure 4.

As an equivalent way to describe the entries, we index rows and columns starting with 0, but now we partition the $k^2$ columns of the array into $k$ *squares*. Columns $0, 1, \ldots, k-1$ make up square 0; columns $k, \ldots, 2k-1$ make up square 1; in general, columns $sk, \ldots, (s+1)k-1$ make up square $s$. Like the row number, an entry's square number $s$ must have $0 \leq s \leq k-1$. An entry in column $c = sk+e$ is actually in the $e$th column of square $s$.

As an example, consider the $3 \times 9$ array in Figure 5. $a$ is in position $r = 0$, $e = 0$, $s = 0$. $b$ is in $r = 0$, $e = 0$, $s = 1$. $c$ is in $r = 1$, $e = 0$, $s = 2$. $d$ is in $r = 2$, $e = 2$, $s = 2$. This coordinate system uniquely identifies each of the $k^3$ entries in a $k \times k^2$ array with a 3-tuple in $\{0, \ldots, k-1\}^3$, and the

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| a | | | b | | | | | |
| | | | | | | c | | |
| | | | | | | | | d |

Figure 5: A $3 \times 9$ array

definition of $f(r, c)$ given above is equivalent to placing $(s + re)\%k$ in row $r$, column $e$ of square $s$.

# 2 Proof of Theorem 1

Consider the $k \times k^2$ array with $f(r, c)$ in row $r$, column $c$. To prove Theorem 1, we need to show that any two L-fillings in different positions in the array are distinct.

So suppose $\begin{array}{|c|c|}\hline a_1 & \\ \hline b_1 & d_1 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline a_2 & \\ \hline b_2 & d_2 \\ \hline\end{array}$ are two L-fillings in different locations. Clearly if $a_1 \neq a_2$ or $b_1 \neq b_2$, then these fillings are distinct, so suppose $a_1 = a_2 = a$ and $b_1 = b_2 = b$. We must show $d_1 \neq d_2$. To specify, say $a_1$ is in row $r$, column $c$ with $c = sk + e$ and $a_2$ is in row $R$, column $C$ with $C = Sk + E$. By construction, if $b$ appears immediately below $a$ in column $c = sk + e$, then $e = (b - a)\%k$. Thus, $e = E$.

If $r = R$ we see that $(s + re) \bmod k \equiv (S + Re) \bmod k$ and, after subtracting $re$ from both sides, $s \equiv S \bmod k$. This means both fillings have the same row and column numbers, which contradicts these two fillings being in different locations. It must be the case that $r \neq R$.

We now have two cases. If $e = k - 1$, we see by construction that $s \neq S$. Hence $d_1 \equiv (s + 1) \bmod k$ and $d_2 \equiv (S + 1) \bmod k$ cannot be equal and the fillings are distinct.

So suppose $e \neq k - 1$. Expanding $d_1 \equiv (s + (r + 1)(e + 1)) \bmod k$ and substituting $a \equiv (s + re) \bmod k$ shows $d_1 \equiv (a + e + r + 1) \bmod k$. We similarly see that $d_2 \equiv (a + e + R + 1) \bmod k$. Because $r \neq R$, it must be that $d_1 \neq d_2$, and so again we have distinct fillings.

A similar computation shows that the array with $f(r, c)$ in row $r$ and column $c$ is a $k$-de Bruijn L-array for each of the other 3 orientations of the L, namely: $\begin{array}{|c|c|}\hline \phantom{a} & \phantom{a} \\ \hline \phantom{a} & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$, $\begin{array}{c|c|}\cline{2-2} \phantom{a} & \phantom{a} \\ \hline \multicolumn{1}{|c|}{\phantom{a}} & \phantom{a} \\ \hline\end{array}$, and $\begin{array}{|c|c|}\hline \phantom{a} & \phantom{a} \\ \hline \multicolumn{1}{c|}{} & \phantom{a} \\ \cline{2-2}\end{array}$.

## 3 For Further Exploration

Although we have constructed a $k$-de Bruijn L-array for every $k \geq 2$, many interesting questions remain for de Bruijn L-arrays.

For example, when $k = 2$, a case analysis shows that (up to rotation) there are precisely two 2-de Bruijn L-arrays; one is shown in Figure 3 and the other is shown in Figure 4. A computer search for other L-arrays when $k = 3$ yields dozens more solutions. Some have evident symmetry while others do not; two examples are given in Figure 6. A computer search for other L-arrays when $k \geq 4$ is prohibitive, so less naive approaches are needed. In particular, for $k > 2$ it is unknown how many $k$-de Bruijn L-arrays exist.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2 |
| 1 | 0 | 0 | 2 | 1 | 1 | 0 | 2 | 2 |
| 1 | 2 | 1 | 2 | 0 | 2 | 0 | 1 | 0 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 | 1 | 2 | 2 | 1 |
| 0 | 0 | 1 | 1 | 2 | 1 | 0 | 0 | 2 |
| 0 | 2 | 0 | 0 | 2 | 2 | 1 | 2 | 2 |

Figure 6: Two 3-de Bruijn L-arrays

Other two-dimensional shapes merit further investigation. One could explore de Bruijn arrays for fillings of staircase shapes or for fillings of the intersection of a longer column with a longer row. The construction of this paper is particular to the 3-square L discussed above, though, and it is not easily generalized.

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## References

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