Supplement to "Stacking Blocks and Counting Permutations"

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In the article "Stacking Blocks and Counting Permutations", I provide a bijection between the faces of a certain sequence of solids and a set of pattern-avoiding multiset permutations. For the interested reader, this file contains a second bijection between these same objects, but uses more complicated combinatorial machinery.

The Second Bijection

The bijection of the paper "Stacking Blocks and Counting Permutations" establishes a clear relationship between the Harrises' cube problem and the enumeration of permutations in $S_{n+1}(\{132, 231, 2134\})$. However, since this bijection was motivated primarily by the geometry of the Harrises' construction, it leaves much room for variations in labeling. Here we give an alternate bijection that is motivated more by the structure of the permutations of $S^{(2)}_{n+1}(\{132, 231, 2134\})$.

As a base case, we know that the unit cube (the Harrises' first construction) has 6 faces, and there are 6 permutations of the multiset $\{1, 1, 2, 2\}$, so we may assign each such permutation to a face of the cube in any way we please. Without loss of generality, we choose the labeling shown in Figure 1, where 1122 is on the top, 1212 is on the front, and 1221 is on the right side of the cube. Further, define the *complement* of a permutation of $\{1, 2, \ldots, n\}$ as the permutation formed by replacing i with $n + 1 - i$ for all letters in the permutation. Let each pair of opposite faces of the cube be labeled with complementary permutations. So the bottom of the cube is labeled 2211, the back is labeled 2121, and the left side is labeled 2112.

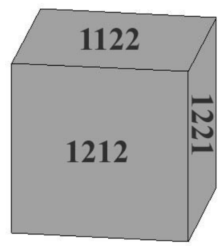

Figure 1: A unit cube labelled with the elements of $S_2^{(2)}(\{132, 231, 2134\})$

Given a permutation π of $\{1, 1, \ldots, k, k\}$ and a permutation ρ of $\{1, 1, \ldots, n, n\}$ $(n > k)$, we say that ρ is a *child* of π if deleting all letters of ρ that are in the set $\{k+1, k+1, \ldots, n, n\}$ yields the permutation π. For example 5534123124 is a child of 1212. If ρ is a child of π, then π is a *parent* of ρ. A child that is formed by adding copies of just one letter is an *immediate child*, and a parent that is formed by deleting copies of just one letter is an *immediate parent*.

Now, we describe a labeling of the faces the Harrises' nth solid where the children of 1212 lie on the front, the children of 1221 lie on the right side, the children of 2112 lie on the left of the solid, and so on, corresponding to our original labeling of the cube.

Our labeling will rely on the following two propositions

Proposition 1 *Let* $\rho \in \{1212, 1221, 2121, 2112, 2211\}$. *Then* ρ *has* $2n - 1$ *children in* $S_{n+1}^{(2)}(\{132, 231, 2134\})$, $n \geq 1$

Proof. Pick ρ from the set above. All $\{231, 132, 2134\}$-avoiding children of ρ must have each of the letters $\{3, 3, \ldots, n+1, n+1\}$ either prepended to ρ or appended to ρ because if some letter that is greater than or equal to 3 were inserted in the middle of ρ we will have created either a forbidden 132 pattern or a forbidden 231 pattern.

Now, notice that at most one value may be appended to the end of ρ. If the letters i and j ($i < j$) were appended to ρ to obtain $\rho \dots i \dots j$ then we would have a forbidden 2134 pattern. If they were appended to ρ to obtain $\rho \dots j \dots i$ then we would have a forbidden 132 pattern.

Therefore, in order to create a $\{231, 132, 2134\}$-avoiding child of ρ we may either (1) choose to append either 1 or 2 copies of some letter $3 \leq i \leq n+1$ to the end of ρ while the rest of $\{3, 3, \dots, n+1, n+1\}$ is prepended to ρ in decreasing order, or (2) simply prepend $(n+1)(n+1)(n)(n)\cdots(3)(3)$ to the beginning of ρ. Option (1) provides $2 \cdot (n-1)$ children, while option (2) provides 1 child for a total of $2 \cdot (n-1) + 1 = 2n - 1$ children. ■

Proposition 2 *The multiset permutation 1122 has $2n^2 - 4n + 3$ children in $S^{(2)}_{n+1}(\{132, 231, 2134\})$, $n \geq 1$.*

Proof. We saw in Theorem 1 of the original paper that

$$\left| S^{(2)}_{n+1}(\{132, 231, 2134\}) \right| = 2n^2 + 6n - 2.$$

Notice that every member of $S^{(2)}_{n+1}(\{132, 231, 2134\})$ is a child of exactly one of 1122, 1212, 1221, 2211, 2121, or 2112. We know from Proposition 1 that each of 1212, 1221, 2211, 2121, and 2112 have exactly $2n - 1$ children in $S^{(2)}_{n+1}(\{132, 231, 2134\})$. This means that the remaining $(2n^2 + 6n - 2) = 5 \cdot (2n - 1) = 2n^2 - 4n + 3$ members of $S^{(2)}_{n+1}(\{132, 231, 2134\})$ are children of 1122. ■

We now use Proposition 1 to label part of the Harrises' nth solid. Pick $\rho \in \{1212, 1221, 2121, 2112, 2211\}$. We know that ρ has $2n - 1$ children in $S^{(2)}_{n+1}(\{132, 231, 2134\})$. In particular, given the $2n - 3$ children of ρ on $\{1, 1, \dots, n, n\}$, prepend $(n+1)(n+1)$ to the front of each child. Additionally, consider the permutation $\rho'_{(n)} = (n)(n)(n-1)(n-1)\cdots 33\rho$. We can obtain two additional children: $(n+1)\rho'_{(n)}(n+1)$ and $\rho'_{(n)}(n+1)(n+1)$ for a total of $2n - 1$ children of ρ. We will place all $2n - 1$ children on a single row of squares. In the first solid ρ is its own only child, so it lies on a single square. For subsequent constructions, copy all children of ρ from the $(n-1)$st

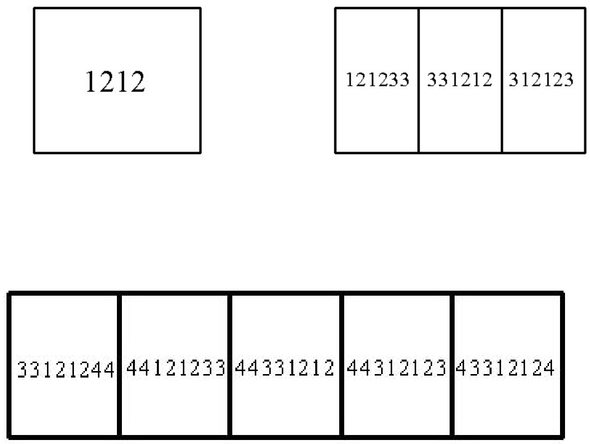

Figure 2: The children of $\rho = 1212$ on Harris's first three solids

construction to the middle $2n-3$ squares of the row, and prepend $(n+1)(n+1)$ to each of them. Then, without loss of generality put $\rho'_{(n)}(n+1)(n+1)$ on the first square of the row and $(n+1)\rho'_{(n)}(n+1)$ on the last square of the row. The first three labelings of a row of squares with the children of $\rho = 1212$ are shown in Figure 2.

We now have a well-defined way to label a row of $2n-1$ squares in the Harrises' nth solid with the children of any $\rho \in \{1212, 1221, 2121, 2112, 2211\}$.

This allows us to label $5 \cdot (2n-1)$ of the squares of the Harrises' nth solid. Analogous to the labeling of the unit cube given at the beginning of this section, let the row corresponding to 1212 be the labels for the bottom row of the front face of the nth solid, the row corresponding to 1221 be the labels for the right side, the row corresponding to 2121 the bottom row of the back face, the row corresponding to 2112 the left side, and the row corresponding to 2211 the bottom face. We have now labeled the squares that lie in the dark gray area of the Harrises' nth solid in Figure 3.

It now remains the label the light gray squares of Figure 3 with the

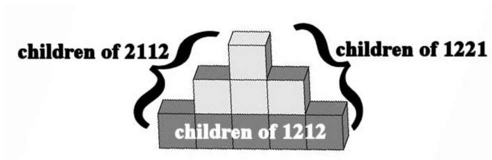

Figure 3: Labelled and unlabbeled portions of the Harrises' third solid

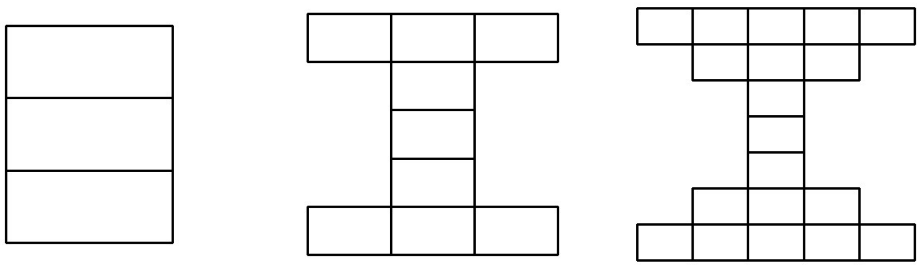

Figure 4: Region of Harris's first three solids to be filled with the children of 1122

children of 1122. Notice that if we remove these lighter unlabeled squares from the Harrises' nth solid and flatten them, we need to fill a shape formed by $2n-1$ rows of cubes where the lengths of the rows are $2n-3, 2n-5, \ldots, 3, 1, 1, 1, 3, \ldots, 2n-5, 2n-3$ for $n \geq 2$. Indeed, the number of squares in this unlabeled portion of Harris's nth construction is $1+2\cdot\sum_{i=2}^{n}(2i-3) = 2n^2-4n+3$, and we know from Proposition 2 that there are $2n^2-4n+3$ children of 1122 in $S_{n+1}^{(2)}(\{132, 231, 2134\})$. The shapes to be filled for $n = 2, 3,$ and 4 are shown in Figure 4.

First, notice that these shapes can be constructed recursively. Given one such shape, (i) replace the middle row of length 1 with three rows of length 1, and (ii), for all other rows, add one square to the beginning and one square to the end of the row. This process is illustrated in Figure 5

By construction the shape to be filled always has 3 rows of length 1 at the center, corresponding to the back, top, and front of the top cube

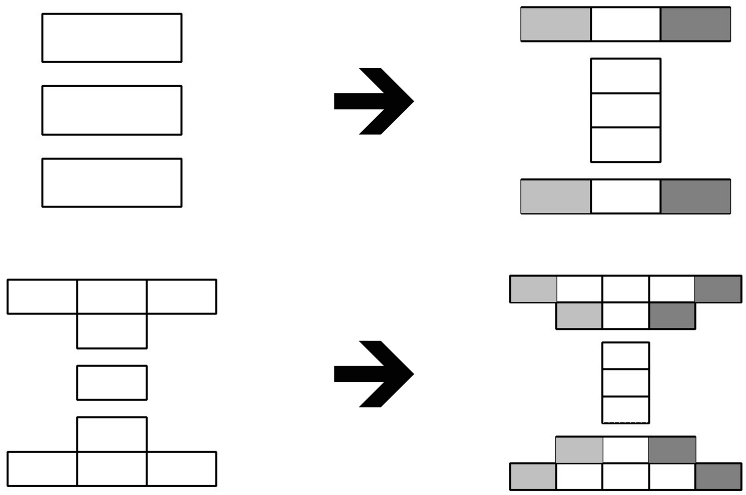

Figure 5: A recursive construction for the top, front, and back of the Harrises' solids

in the Harrises' construction. Conveniently, there are always 3 children of $\rho = 1122\cdots nn$. Place $(n+1)(n+1)\rho$ in the first, $\rho(n+1)(n+1)$ in the second, and $(n+1)\rho(n+1)$ in the third of these rows.

Now, consider the other members of $S^{(2)}_{n+1}(\{132, 231, 2134\})$. As we have seen, we can always obtain a member of $S^{(2)}_{n+1}(\{132, 231, 2134\})$ by prepending $(n+1)(n+1)$ to the front of some $\rho \in S^{(2)}_{n}(\{132, 231, 2134\})$. For each white square in the two triangles above, label the square with $(n+1)(n+1)\rho$ where ρ is the label of that square in the previous construction.

By construction the center permutation in each row is 213-avoiding, since it has the form $(n+1)(n+1)\cdots k112233\cdots l$ where $l = k$ or $l = k-1$. We may obtain two children from its immediate parent ρ on $\{1, 1, \ldots, n, n\}$ by taking $(n+1)\rho(n+1)$ and $\rho(n+1)(n+1)$. Let $\rho(n+1)(n+1)$ be the label of the light gray square at the left of the row with $(n+1)(n+1)\rho$ at the center, and let $(n+1)\rho(n+1)$ be the label of the dark gray square at the right of the row with $(n+1)(n+1)\rho$ at the center.

In this way, we have labeled all of the squares in the figure described

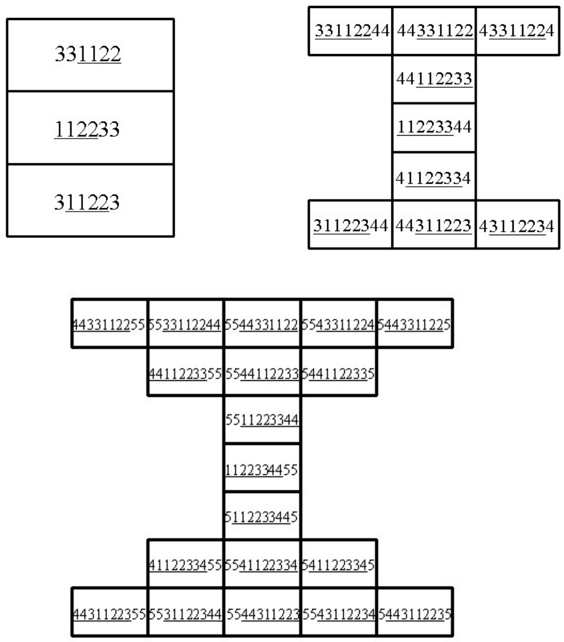

Figure 6: The children of $\rho = 1122$ on Harris's first three solids

above with the children of 1122. The first few examples of such labelings are given in Figure 6, with the immediate parent of each permutation underlined.

We now have a new way to label all squares on the surface of the Harrises' nth solid. The location of any permutation π indicates what its parent is in $S_2^{(2)}(\{132, 231, 2134\})$. This labeling also has the nice property that all 213-avoiding children of 1212 lie on the center column, and the number of columns a permutation is located away from the center column indicates how many immediate parents one would have to trace back to obtain a 213-avoiding permutation.

This establishes another clear bijection between the two problems in question, and further illustrates the nice and unexpected connection between a question of middle school geometry and enumerative combinatorics.