Stacking Blocks and Counting Permutations

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In this article we will explore two seemingly unrelated enumeration questions, both of which are answered by the same formula. In the first section, we find the surface area of a solid formed from unit cubes. In second section, we enumerate multiset permutations with a specified set of restrictions. In the final sections, we give two different bijections between the two problems. First, however, it is worth explaining how this paper came about.

The author received an email from David Harris while he was helping his 12-year-old daughter complete a project for her math class. Together Harris and his daughter computed the surface areas of a sequence of constructions of cubes, and hoped to find a formula for the surface area of their nth construction. This project and its solution are described in the next section. At the time of his correspondence, Harris and his daughter had deduced several facts about the construction but were unable to find a formula for the surface area in general. When they searched for the first few terms in their sequence, Google returned only one hit: a Maple data file on the author's website.

The sequence that Harris discovered online was originally generated in the context of pattern-avoiding words and permutations. His web search produced a conjecture that gives a nice geometric interpretation of a permutation pattern question. Although the methods in this paper are elementary, the novelty is not in the techniques, but in the surprising and beautiful connection between a geometry problem and an enumeration problem. This serendipitous discovery illustrates how attractive new results may sometimes appear in such a surprising place as an elementary homework exercise.

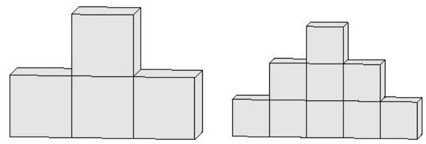

Figure 1: Harris's second and third solids

The Surface Area of Cubes

We begin with Harris's original geometry question. We first describe a construction involving unit cubes, and then compute the surface area of the nth solid in this construction.

The first solid is a unit cube, which has surface area 6.

To construct the nth solid, first form a row of $2n-1$ cubes. Then, center the $(n-1)$st construction on top of this row. For example, the second solid is shown in figure 1. It has surface area 18. The third solid is also shown. It has surface area 34.

Now, we wish to compute the surface area SA_n of the nth solid. We have already computed SA_1, SA_2, and SA_3 above.

Notice that to construct SA_n, we glue together a solid of surface area SA_{n-1} together with a rectangular prism of surface area $4 \cdot (2n-1) + 2 = 8n - 2$. However, there are $2n-3$ squares which overlap, and are now on the interior of the shape. Thus, the surface area only increases by $(8n - 2) - 2(2n-3) = 4n + 4$ units; that is $SA_n - SA_{n-1} = 4n + 4$. Since the difference sequence for SA_n is linear, we know that SA_n is quadratic. In fact, there is a unique quadratic equation that fits the three data points we have already computed. Let $SA_n = an^2 + bn + c$. We easily see that $SA_n - SA_{n-1} = (2a)n + (b-a)$. Thus $2a = 4$ and $b - a = 4$, or $a = 2$ and $b = 6$. Together with the fact that $SA_1 = 6$, we see that $SA_n = 2n^2 + 6n - 2$.

Permutation Patterns

We have just seen that the surface area of Harris's nth solid is $2n^2 + 6n - 2$. We now give the necessary definitions to produce a set of permutations with $2n^2 + 6n - 2$ elements.

Given a string of numbers s, the *reduction* of s is the string obtained by replacing the ith smallest number(s) of s with i. For example, the reduction of 2671165 is 2451143. Now, given strings of numbers $p = p_1 \cdots p_n$ and $q = q_1 \cdots q_m$, we say that p *contains* q as a pattern if there exist indices $1 \leq i_1 < i_2 < \cdots i_m \leq n$ such that $p_{i_1} \cdots p_{i_m}$ reduces to q. Otherwise, we say that p *avoids* q. For example, 2671165 contains the pattern 2321 because it contains the subsequence 6765, which reduces to 2321. However, 2671165 avoids the pattern 1234 because it has no strictly increasing subsequence of length 4. Permutations which avoid other permutations have been actively studied since the seminal paper of Simion and Schmidt [7]. They aid in the study of a number of combinatorial objects. A friendly introduction to permutation patterns can be found in [2]. More detailed work with pattern avoidance involving multiset permutations can be found in [1], [3], [5], and [6].

Finally, we introduce a bit of notation. Let $S_n(Q)$ denote the set of permutations of length n avoiding all patterns in the set Q. For example, $S_n(\{21\}) = \{12 \cdots n\}$ for $n \geq 1$. In this paper we are actually concerned with multiset permutations with *two* copies of each letter. So, let $S_n^{(2)}(Q)$ denote the set of permutations of two 1s, two 2s, ..., and two ns avoiding all patterns in the set Q. For example $S_2^{(2)}(\{112\}) = \{1221, 2121, 2211\}$.

We now have the machinery necessary to state and prove a useful lemma. This lemma is a special case of a result of Burstein [3].

Lemma 1 *$\left|S_{n+1}^{(2)}(\{132, 231, 213\})\right| = 2n + 4$ for $n \geq 1$.*

Proof. Since we will only consider permutations that avoid the set of patterns $\{132, 231, 213\}$ in this proof, we will write A_n instead of $S_n^{(2)}(\{132, 231, 213\})$.

Notice first that $A_2 = \{1122, 1212, 1221, 2112, 2121, 2211\}$, so $|A_2| = 6$, as desired.

We proceed by induction. Consider $p \in A_n$. Let p' be the multiset permutation formed by deleting the two copies of n in p. For example if $p = 312123$, then $p' = 1212$. Notice that since $p \in A_n$, we have that $p' \in A_{n-1}$.

Now, given $p' \in A_{n-1}$, we consider all the ways to insert two copies of n into p' to obtain a multiset permutation in A_n. Notice that if n is inserted *between* two letters of p', we have necessarily created either a 132 pattern or a 231 pattern. Thus, the ns can be inserted in one of only 3 ways: (i) both ns are prepended to the beginning of p', (ii) both ns are appended to the end of p', or (iii) one n is prepended to the beginning of p' and the other n is appended to the end of p'. Clearly, (i) will always produce a member of $S_n^{(2)}$, however, (ii) and (iii) must be considered more carefully. In particular, appending an n to the end of p' will only produce a 213-avoiding multiset permutation if p' avoids the pattern 21, i.e. if p' is weakly increasing. Thus, $|A_n| = |A_{n-1}| + 2$, since we may prepend two ns to the beginning of any member of A_{n-1}, but we may also append two ns to the end of the unique increasing permutation of A_{n-1}, or we may prepend an n to the beginning of it and append an n to the end of it.

Finally, since $|A_n| - |A_{n-1}| = 2$, we know that $|A_n|$ grows linearly, and use the fact that $A_2 = 6$ to compute the formula $|A_{n+1}| = 2n + 4$. ■

This lemma is key to our main theorem, which gives a set of multiset permutations whose size has a familiar formula. This theorem was first observed using the method of enumeration schemes found in [6].

Theorem 1 $\left|S_{n+1}^{(2)}(\{132, 231, 2134\})\right| = 2n^2 + 6n - 2$ *for* $n \geq 1$.

Proof. Since we will only consider permutations that avoid the set of patterns $\{132, 231, 2134\}$ in this proof, we will now use B_n to denote $S_n^{(2)}(\{132, 231, 2134\})$.

As in the lemma, $|B_2| = 6$, as desired.

We again proceed by induction. Given $p' \in B_{n-1}$, we consider all the ways to insert two copies of n into p' to obtain a multiset permutation in

B_n. As before the ns can be inserted in one of only 3 ways: (i) both ns are prepended to the beginning of p', (ii) both ns are appended to the end of p', or (iii) one n is prepended to the beginning of p' and the other n is appended to the end of p'. (i) will always produce a member of B_n, but (ii) and (iii) must be considered in more detail. In particular, appending n to the end of p' may induce a copy of a forbidden 2134 pattern if p' contains a 213 pattern.

We showed in the lemma that $\left|S^{(2)}_{n+1}(\{132, 231, 213\})\right| = 2n + 4$. Thus, there are $2(n-2)+4 = 2n$ permutations in B_{n-1} for which we can perform construction (ii) or construction (iii) and obtain a permutation in B_n. Also, for *all* members of B_{n-1}, we can perform construction (i). This indicates that $|B_n| = |B_{n-1}| + 2 \cdot (2n) = |B_{n-1}| + 4n$.

Because $|B_n| - |B_{n-1}|$ is linear, we know that $|B_n| = an^2 + bn + c$, for some $a, b, c \in \mathbb{R}$. This gives $|B_n| - |B_{n-1}| = (2a)n + (b - a) = 4n$, so we deduce that $a = 2$ and $b = 2$. Now use the fact that $|B_2| = 6$ to see that $c = -6$, so $|B_n| = 2n^2 + 2n - 6$, or equivalently $|B_{n+1}| = 2n^2 + 6n - 2$. ■

The First Bijection

We have now seen two constructions that produce the sequence $2n^2 + 6n - 2$ for $n \geq 1$. The surface area of the nth solid in Harris's construction produces this sequence as does the number of multiset permutations in $S^{(2)}_{n+1}(\{132, 231, 2134\})$. In this section, we provide a simple bijection that follows from an alternate recursive description of Harris's cube construction. In the next section, we will provide a more complex bijection that preserves some nice properties of the permutations of $S^{(2)}_{n+1}(\{132, 231, 2134\})$. In both cases, it suffices to associate each permutation in $S^{(2)}_{n+1}(\{132, 231, 2134\})$ with a unique unit square on the surface of Harris's nth solid.

To this end, we consider another description of Harris's construction. To construct the nth solid from the $(n-1)$st solid, we first remove the bottom face of the solid and move it one unit lower as in figure 2 (i). Next, we form a rectangular ring of $4n$ squares. This ring should be constructed so that it has two opposing sides of length 1 and two opposing sides of length $2n - 1$,

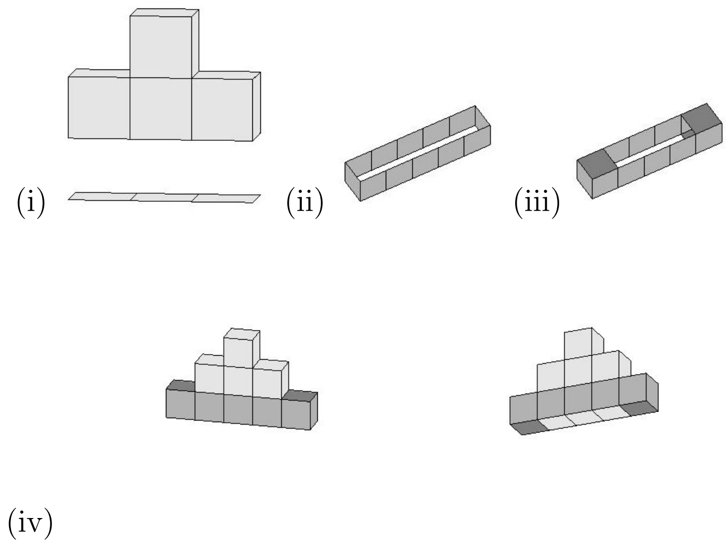

Figure 2: Constructing Harris's $n = 3$ solid from Harris's $n = 2$ solid

as shown in figure 2 (ii). Now, attach a new square to the top and bottom of each end of the ring, as shown in figure 2 (iii), for a total of $4n + 4$ new squares. We may glue the modified version of the $(n-1)$st solid together with this new modified ring of $4n + 4$ squares to form the nth solid. Two views of this gluing are shown in figure 2 (iv).

This alternate construction has a clear advantage. Although it is more complicated to explain, this revised description allows us to associate each square on the surface of the $(n-1)$st solid with squares on the nth solid, rather than "gluing" some squares into the interior.

Now, we may recursively define a bijection between the squares of the nth solid and the permutations of $S_{n+1}^{(2)}$.

To begin, since there are 6 faces in a unit cube, and 6 elements of $S_2^{(2)}$ we may assign each one of these permutations to a unique face of the cube.

Now, consider the nth solid, constructed as described in this section. In the $(n-1)$st solid, each of the light gray squares was associated with some permutation $p \in S_n^{(2)}$. Let each such square now be associated with the permutation $(n+1)(n+1)p \in S_{n+1}^{(2)}$.

We must now account for the four dark gray squares (the tops and bottoms of the left and right cubes in the bottom row of the solid) and the $4n$ medium gray squares (the side faces of all cubes in the bottom row of the solid). Clearly, these must correspond to the permutations of $S_{n+1}^{(2)}$ that either begin and end with $(n+1)$ or that end with two copies of $(n+1)$. Notice that each of *these* permutations was formed by taking one of the $2n+2$ permutations in $S_n^{(2)}(\{132, 231, 213\})$ and inserting two (n+1)s in one of the two ways already described. Two of the permutations of $S_n^{(2)}(\{132, 231, 213\})$ were formed by taking the increasing permutation $1122 \cdots (n-1)(n-1)$ and appending two ns to it in one of two ways. Appending two $(n+1)$s to *these* permutations so that at least one $(n+1)$ is at the end of the permutation results in four new permutations of $S_{n+1}^{(2)}$, corresponding to the dark gray squares in the figure. Finally, the other $2n$ members of $S_n^{(2)}(\{132, 231, 213\})$ may either have an $(n+1)$ added to the beginning and to the end or they may have two $(n+1)$s appended to the end. These account for the $4n$ medium gray squares. We now have established a recursive bijection between our two problems.

The Second Bijection

The bijection of the previous section establishes a clear relationship between Harris's cube problem and the enumeration of permutations in $S_{n+1}(\{132, 231, 2134\})$. However, since this bijection was motivated primarily by the geometry of Harris's construction, it leaves much room for variations in labeling. Here we give an alternate bijection that preserves different permutation statistics.

As a base case, we know that there are 6 faces to the unit cube (Harris's first construction), and there are 6 permutations of the multiset $\{1, 1, 2, 2\}$, so we may assign each such permutation to a face of the cube in any way we please. Without loss of generality, we choose the labeling shown in figure 3, where 1122 is on the top, 1212 is on the front, and 1221 is on the right

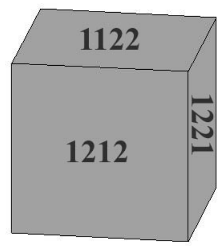

Figure 3: A unit cube labeled with the elements of $S_2^{(2)}(\{132, 231, 2134\})$

side of the cube. Further, recall that the complement of a permutation of $\{1, 2, \ldots, n\}$ is the permutation formed by replacing i with $n+1-i$ for all letters in the permutation. Let each pair of opposite faces of the cube be labeled with complementary permutations. So the bottom of the cube is labeled 2211, the back is labeled 2121, and the left side is labeled 2112.

Given a permutation π on $\{1, 1, \ldots, k, k\}$ and a permutation ρ on $\{1, 1, \ldots, n, n\}$ $(n > k)$, we say that ρ is a *child* of π if deleting all letters of ρ that are in the set $\{k+1, k+1, \ldots, n, n\}$ yields the permutation π. For example 5534123124 is a child of 1212. If ρ is a child of π, then π is a *parent* of ρ. A child that is formed by adding copies of just one letter is an *immediate child*, and a parent that is formed by deleting copies of just one letter is an *immediate parent*.

Now, we describe a labeling of the faces Harris's nth solid where the children of 1212 lie on the front, the children of 1221 lie on the right side, the children of 2112 lie on the left of the solid, etc., corresponding to our original labeling of the cube.

Our labeling will rely on the following two propositions

Proposition 1 *Let* $\rho \in \{1212, 1221, 2121, 2112, 2211\}$. *Then* ρ *has* $2n-1$ *children in* $S_{n+1}^{(2)}(\{132, 231, 2134\})$, $n \geq 1$

Proof. Pick ρ from the set above. All $\{231, 132, 2134\}$-avoiding children must have each of the letters $\{3, 3, \ldots, n+1, n+1\}$ either prepended to ρ or appended to ρ because if some letter that is greater than or equal to 3 were inserted in the middle of ρ we will have created either a forbidden 132 pattern or a forbidden 231 pattern.

Now, notice that at most one value may be appended to the end of ρ. If the letters i and j $(i < j)$ were appended to ρ to obtain $\rho \ldots i \ldots j$ then we would have a forbidden 2134 pattern. If they were appended to ρ to obtain $\rho \ldots j \ldots i$ then we would have a forbidden 132 pattern.

Therefore, in order to create a $\{231, 132, 2134\}$-avoiding child of ρ we may either (1) choose to append either 1 or 2 copies of some letter $3 \leq i \leq n+1$ to the end of ρ while the rest of $\{3, 3, \ldots, n+1, n+1\}$ is prepended to ρ in decreasing order, or (2) simply prepend $(n+1)(n+1)(n)(n)\cdots(3)(3)$ to the beginning of ρ. Option (1) provides $2 \cdot (n-1)$ children, while option (2) provides 1 child for a total of $2 \cdot (n-1) + 1 = 2n - 1$ children. ■

Proposition 2 *The multiset permutation 1122 has* $2n^2 - 4n + 3$ *children in* $S_{n+1}^{(2)}(\{132, 231, 2134\})$*,* $n \geq 1$*.*

Proof. We saw in Theorem 1 that $\left|S_{n+1}^{(2)}(\{132, 231, 2134\})\right| = 2n^2 + 6n - 2$. Notice that every member of $S_{n+1}^{(2)}(\{132, 231, 2134\})$ is a child of exactly one of 1122, 1212, 1221, 2211, 2121, or 2112. We know from Proposition 1 that each of 1212, 1221, 2211, 2121, and 2112 have exactly $2n - 1$ children in $S_{n+1}^{(2)}(\{132, 231, 2134\})$. This means that the remaining $(2n^2 + 6n - 2) = 5 \cdot (2n - 1) = 2n^2 - 4n + 3$ members of $S_{n+1}^{(2)}(\{132, 231, 2134\})$ are children of 1122. ■

We now use Proposition 1 to label part of Harris's nth solid. Pick $\rho \in \{1212, 1221, 2121, 2112, 2211\}$. We know that ρ has $2n - 1$ children in $S_{n+1}^{(2)}(\{132, 231, 2134\})$. In particular, given the $2n - 3$ children of ρ on $\{1, 1, \ldots, n, n\}$, prepend $(n+1)(n+1)$ to the front of each child. Additionally, consider the permutation $\rho'_{(n)} = (n)(n)(n-1)(n-1)\cdots 33\rho$. We can obtain two additional children: $(n+1)\rho'_{(n)}(n+1)$ and $\rho'_{(n)}(n+1)(n+1)$ for a total of $2n - 1$ children of ρ. We will place all $2n - 1$ children on a single row of

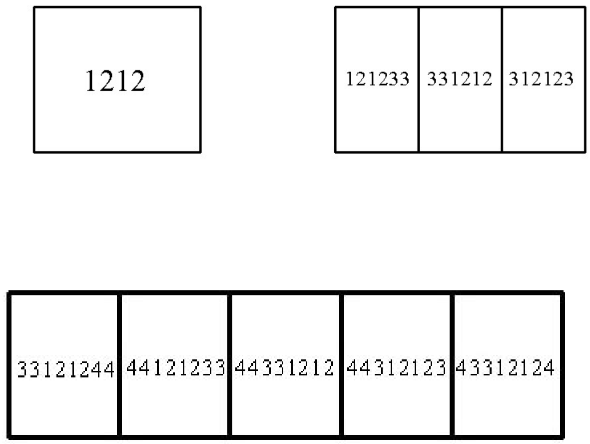

Figure 4: The children of $\rho = 1212$ on Harris's first three solids

squares. In the first solid ρ is its own only child, so it lies on a single square. For subsequent constructions, copy all children of ρ from the $(n-1)$st construction to the middle $2n-3$ squares of the row, and prepend $(n+1)(n+1)$ to each of them. Then, without loss of generality put $\rho'_{(n)}(n+1)(n+1)$ on the first square of the row and $(n+1)\rho'_{(n)}(n+1)$ on the last square of the row. The first three labelings of a row of squares with the children of $\rho = 1212$ are shown in figure 4.

We now have a well-defined way to label a row of $2n-1$ squares in Harris's nth construction with the children of any $\rho \in \{1212, 1221, 2121, 2112, 2211\}$.

This allows us to label $5 \cdot (2n-1)$ of the squares of Harris's nth construction. Analogous to the labeling of the unit cube given at the beginning of this section, let the row corresponding to 1212 be the labels for the bottom row of the front face of the nth solid, the row corresponding to 1221 be the labels for the right side, the row corresponding to 2121 the bottom row of the back face, the row corresponding to 2112 the left side, and the row corresponding to 2211 the bottom face. We have now labeled the squares that lie in the

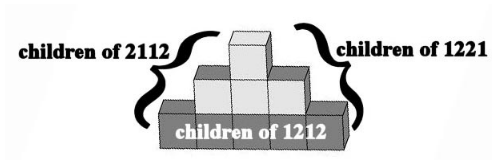

Figure 5: Labeled and unlabeled portions of Harris's third solid

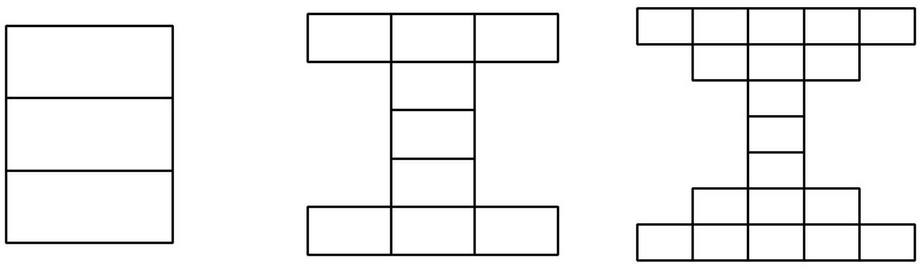

Figure 6: Region of Harris's first three solids to be filled with the children of 1122

dark gray area of Harris's nth solid in figure 5.

It now remains the label the light gray squares of figure 5 with the children of 1122. Notice that if we remove these lighter unlabeled squares from Harris's nth solid and flatten them, we need to fill a shape formed by $2n-1$ rows of cubes where the lengths of the rows are $2n-3, 2n-5, \ldots, 3, 1, 1, 1, 3, \ldots, 2n-5, 2n-3$ for $n \geq 2$. Indeed, the number of squares in this unlabeled portion of Harris's nth construction is $1 + 2 \cdot \sum_{i=2}^{n}(2i-3) = 2n^2 - 4n + 3$, and we know from Theorem 1 that there are $2n^2 - 4n + 3$ children of 1122 in $S_{n+1}^{(2)}(\{132, 231, 2134\})$. The shapes to be filled for $n = 2, 3$, and 4 are shown in figure 6.

First, notice that these shapes can be constructed recursively. Given one such shape, (i) replace the middle row of length 1 with three rows of length 1, and (ii), for all other rows, add one square to the beginning and one square to the end of the row. This process is illustrated in figure 7

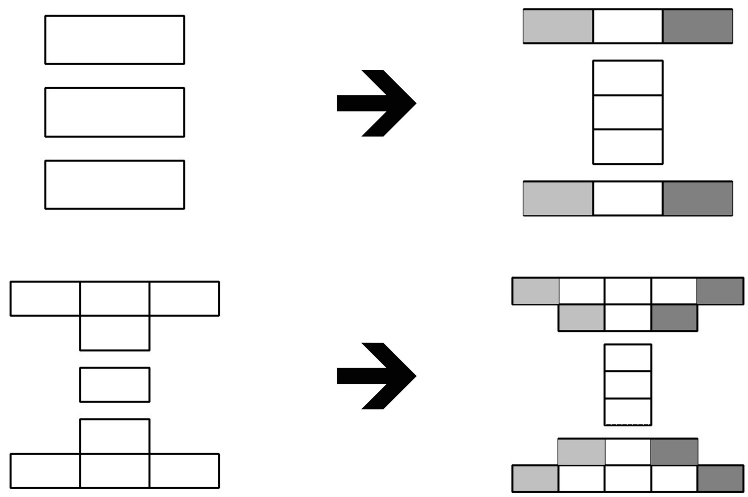

Figure 7: A recursive construction for the top, front, and back of Harris's solids

By construction the shape to be filled always has 3 rows of length 1 at the center, corresponding to the back, top, and front of the top cube in Harris's construction. Conveniently, there are always 3 children of $\rho = 1122\cdots nn$. Place $(n+1)(n+1)\rho$ in the first, $\rho(n+1)(n+1)$ in the second, and $(n+1)\rho(n+1)$ in the third of these rows.

Now, consider the other members of $S_{n+1}^{(2)}(\{132, 231, 2134\})$. As we have seen, we can always obtain a member of $S_{n+1}^{(2)}(\{132, 231, 2134\})$ by prepending $(n+1)(n+1)$ to the front of some $\rho \in S_n^{(2)}(\{132, 231, 2134\})$. For each white square in the two triangles above, label the square with $(n+1)(n+1)\rho$ where ρ is the label of that square in the previous construction.

By construction the center permutation in each row is 213-avoiding, since it has the form $(n+1)(n+1)\cdots k112233\cdots l$ where $l = k$ or $l = k-1$. We may obtain two children from its immediate parent ρ on $\{1, 1, \ldots, n, n\}$ by taking $(n+1)\rho(n+1)$ and $\rho(n+1)(n+1)$. Let $\rho(n+1)(n+1)$ be the label of the light gray square at the left of the row with $(n+1)(n+1)\rho$ at the center, and let $(n+1)\rho(n+1)$ be the label of the dark gray square at the right of

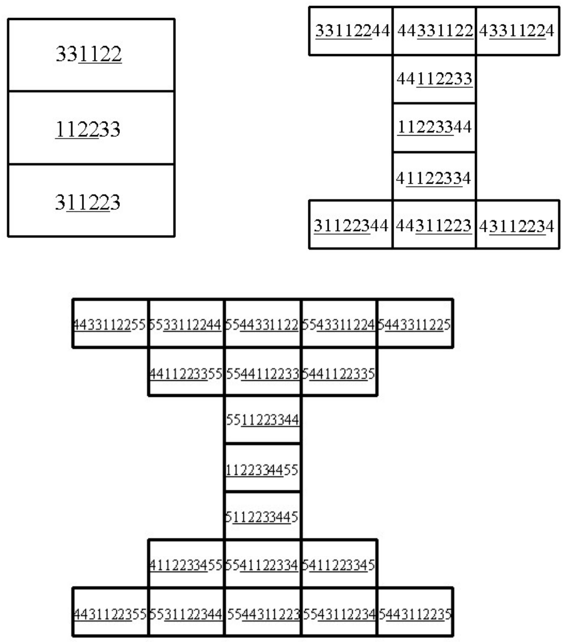

Figure 8: The children of $\rho = 1122$ on Harris's first three solids

the row with $(n+1)(n+1)\rho$ at the center.

In this way, we have labeled all of the squares in the figure described above with the children of 1122. The first few examples of such labelings are given in figure 8, with the immediate parent of each permutation underlined.

We now have a new way to label all squares on the surface of Harris's nth solid. The location of any permutation π indicates what its parent is in $S_2^{(2)}(\{132, 231, 2134\})$. This labeling also has the nice property that all 213-avoiding children of 1212 lie on the center column, and the number of columns a permutation is located away from the center column indicates how many immediate parents one would have to trace back to obtain a 213-avoiding permutation.

This establishes another clear bijection between the two problems in question, and further illustrates the nice and unexpected connection between a question of middle school geometry and enumerative combinatorics.

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