ENUMERATION SCHEMES FOR PERMUTATIONS AVOIDING BARRED PATTERNS

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Abstract. We give the first comprehensive collection of enumeration results for permutations that avoid barred patterns of length ≤ 4. We then use the method of prefix enumeration schemes to find recurrences counting permutations that avoid a barred pattern of length > 4 or a set of barred patterns.

1. Introduction

Let $q = q_1 \cdots q_m$ be a finite string of numbers. The *reduction* of q, denoted $red(q)$, is the string obtained by replacing the ith smallest letter(s) of q with i. For example, the $red(2674425) = 1452213$. Given two permutations $p \in S_n$, $q \in S_m$, we say p *contains* q as a pattern if there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that $red(p_{i_1} \cdots p_{i_m}) = q$. Otherwise p *avoids* q. This definition of pattern avoidance appears in the characterization of 1-stack sortable permutations [10] and the characterization of smooth Schubert varieties [5]. Further, it introduces an interesting and well-studied enumeration problem; namely, count the elements of the set $S_n(Q) = \{p \in S_n \mid p \text{ avoids } q \text{ for all } q \in Q\}$.

The focus of this paper is not the study $S_n(Q)$, but a variation of pattern avoidance given by the following definitions. Given $q' \in S_m$, $b \in \{0,1\}^m$, the *barred permutation* q is the permutation obtained by copying the entries of q' and putting a bar over q'_i if and only if $b_i = 1$. Write $\overline{S_m}$ for the set of all barred permutations of length m. For example, the complete set of barred permutations of length 2 is $\{12, \bar{1}2, 1\bar{2}, \bar{1}\bar{2}, 21, \bar{2}1, 2\bar{1}, \bar{2}\bar{1}\}$.

A barred permutation compactly encodes two permutations, one of which contains the other. In particular, let $\underline{q}$ be the permutation formed by deleting all barred letters of q and then reducing the remaining (unbarred) letters, and let $\overline{q}$ be the permutation formed by all letters of q, with or without bars. For $p \in S_n$, $q \in \overline{S_m}$, we say p *contains* q as a barred pattern if every instance of $\underline{q}$ in p is part of an instance of $\overline{q}$ in p. In this case, we may say every instance of $\underline{q}$ *extends to* an instance of $\overline{q}$. For example if $q = \bar{1}32$, we have $\underline{q} = red(32) = 21$ and $\overline{q} = 132$. p avoids q if and only if every decreasing pair of numbers in p has a smaller number preceding it.

This variation of pattern avoidance also appears in several interesting applications.

- A permutation is two stack sortable if and only if it avoids 2341 and $3\bar{5}241$ [10].

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- A permutation is forest-like if and only if it avoids the patterns 1324 and $21\bar{3}54$ [2]. These permutations also characterize locally factorial Schubert varieties [11].

Beyond the special cases of barred pattern avoidance relevant to these applications, little is known beyond the work of Callan, where he completely enumerates permutations avoiding a single pattern of length 4 with one bar [3], and deals with the special case of $\{3\bar{5}241\}$-avoiding permutations [4]. The goal of this paper is to consider the problem of barred pattern avoidance in a more general and comprehensive context. We consider barred permutations of any length and with any number of bars. Several preliminary results are given, and we completely characterize permutations avoiding a barred pattern of length ≤ 5, before we modify the method of prefix enumeration schemes to the case of barred pattern avoidance, and discuss its success rate.

2. Enumeration

Before we consider results for specific sets of barred patterns, we derive a series of useful lemmas

Lemma 1. *Let $q \in \overline{S_m}$, such that* every *number of q is barred. Then $S_n(\{q\})$ is the set of permutations that contain $\overline{q}$.*

Proof. Notice that $\underline{q} = \emptyset$. That is for p to avoid q every instance of the empty permutation must be a part of a copy of $\overline{q}$, i.e. p contains $\overline{q}$. □

More specifically, this lemma illustrates that, in some sense, barred pattern avoidance bridges the gap from standard pattern avoidance (no bars) to standard pattern containment (all possible bars), with a range of intermediate cases. However, as the following propositions illustrate, a number of these intermediate cases may also be equivalent to standard pattern avoidance.

Lemma 2. *Let $q \in \overline{S_m}$ such that q_i is the only barred element and either (i) $q_{i+1} = q_i \pm 1$, or (ii) $q_{i-1} = q_i \pm 1$. Then,*

$$S_n(\{q_1 \cdots q_{i-1}\overline{q_i}q_{i+1} \cdots q_m\}) = S_n(\{red(q_1 \cdots q_{i-1}q_{i+1} \cdots q_m)\})$$

Proof. Clearly, if p avoids $q_1 \cdots q_{i-1}q_{i+1} \ldots q_m$, then it avoids $q_1 \cdots q_{i-1}\overline{q_i}q_{i+1} \cdots q_m$ since there are no instances of $q_1 \cdots q_{i-1}q_{i+1} \ldots q_m$ to expand to an instance of q. Thus,

$$S_n(\{q_1 \cdots q_{i-1}q_{i+1} \cdots q_m\}) \subseteq S_n(\{q_1 \cdots q_{i-1}\overline{q_i}q_{i+1} \cdots q_m\})$$

On the other hand, without loss of generality assume that $q_i = q_{i+1} \pm 1$, p avoids $q_1 \cdots q_{i-1}\overline{q_i}q_{i+1} \cdots q_m$ and there is an instance of $\underline{q} = q_1 \cdots q_{i-1}q_{i+1} \cdots q_m$ that extends to an instance of $\overline{q} = q_1 \cdots q_{i-1}q_iq_{i+1} \cdots q_m$. Choose the instance of $\overline{q}$ that uses the leftmost possible element of p for q_i. Then $q_1 \cdots q_{i-1}q_iq_{i+2} \cdots q_m$ is another instance of $\underline{q}$ that does not expand to $\overline{q}$. So every element of $S_n(\{q_1 \cdots q_{i-1}\overline{q_i}q_{i+1} \cdots q_m\})$ already avoids $red(q_1 \cdots q_{i-1}q_{i+1} \cdots q_m)$. □

Finally, we eliminate the case of having bars on all but one letter by the following observation.

Lemma 3. *Suppose that $q \in \overline{S_m}$ with only one unbarred letter. Then $|S_n(\{q\})| = 0$ for all $n \geq 1$.*

Proof. Notice that avoiding q means that every instance of a 1 pattern expands to an instance of $\overline{q}$. Without loss of generality, assume that q has barred entries after the lone unbarred letter. Then the final entry of any permutation is a copy of 1 that does not expand to $\overline{q}$. □

We now consider permutations avoiding patterns of length 1, 2, 3, 4, and 5 in turn, noting that many results follow almost directly from Lemmas 1 and 2. With the exception of the work of Callan [3] for patterns of length 4 with 1 bar, this is the first comprehensive list of such results.

2.1. **Avoiding barred patterns of length 1 or 2.** We begin with avoiding patterns of length 1.

It is well known that $|S_n(\{1\})| = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$.

We now see from Lemma 1 that $\left|S_n(\{\bar{1}\})\right| = \begin{cases} 0 & n = 0 \\ n! & n \geq 1 \end{cases}$.

For patterns of length 2, we observe that the Wilf equivalences $|S_n(\{q\})| = |S_n(\{q^r\})| = |S_n(\{q^c\})| = \left|S_n(\{q^{-1}\})\right|$ extend to barred patterns in the obvious way, where q^r denotes q reverse, q^c denotes q complement, and q^{-1} denotes q inverse [7].

Thus, we already have $|S_n(\{12\})| = |S_n(\{21\})| = 1, n \geq 0$.

Further, by Lemma 1, we have $\left|S_n(\{\bar{1}\bar{2}\})\right| = \left|S_n(\{\bar{2}\bar{1}\})\right| = n! - 1$.

Finally, $\left|S_n(\{\bar{1}2\})\right| = \left|S_n(\{2\bar{1}\})\right| = \left|S_n(\{1\bar{2}\})\right| = \left|S_n(\{\bar{2}1\})\right| = |S_n(\{1\})|$, where the first and third equalities are by reversal, the second equality is by complement, and the final equality is by Lemma 2.

2.2. **Avoiding barred patterns of length 3.** It is well known that $|S_n(\{q\})| = \frac{\binom{2n}{n}}{n+1}$ where q is any unbarred pattern of length 3 [7].

Thus, $|S_n(\{\overline{q}\})| = n! - \frac{\binom{2n}{n}}{n+1}$ where $\overline{q}$ is any pattern of length 3 with all bars.

By Lemma 3 it only remains to consider the case of patterns with 1 bar. The trivial Wilf equivalences and Lemma 2 give:

$$\left|S_n(\{\bar{1}23\})\right| = \left|S_n(\{32\bar{1}\})\right| = \left|S_n(\{12\bar{3}\})\right| = \left|S_n(\bar{3}21)\right| = |S_n(21)|\,, \text{ and}$$

$$\left|S_n(\{1\bar{2}3\})\right| = \left|S_n(3\bar{2}1)\right| = |S_n(21)|\,.$$

It is enough to consider the pattern 132 with bars on various elements to complete the characterization. If there is a bar on 3 or 2, we may make use of Lemma 2, so the remaining interesting case is that of $S_n(\{\bar{1}32\})$.

Proposition 1. $\left|S_n(\{\bar{1}32\})\right| = (n-1)!$ *for all* $n \geq 0$.

Proof. We claim that $S_n(\{\bar{1}32\})$ is precisely the set of permutations that begin with 1, thus giving the above enumeration.

First, notice that if p begins with 1, then p_1 cannot be involved in a 21 pattern since it is both the first and the smallest element of p. Thus, every 21 pattern is preceded by the smallest element, and so $p \in S_n(\{\bar{1}32\})$.

Now, if p does not begin with 1, then $p_1 1$ forms a 21 pattern that is not preceded by a smaller entry so $p \notin S_n(\{\bar{1}32\})$.

□

We have now finished the enumeration of permutations avoiding barred patterns of length ≤ 3.

2.3. **Avoiding barred patterns of length 4.** It is well known that for patterns with 0 bars, permutation patterns fall into the three classes of $|S_n(\{1234\})|$, $|S_n(\{1342\})|$, and $|S_n(\{1324\})|$, [1]. For the first of these, we have a closed form enumeration, for the second a generating function, and for the third a recurrence that allows enumeration up to $n = 20$ [1] [6].

As given by the Lemmas, we need only consider the case of 2 bars and 1 bar in turn.

For two bars, we have two cases: either the forbidden pattern contains a symmetry of a pair of consecutive numbers of the form $(c-1)\bar{c}$, or it does not.

If the forbidden pattern has a symmetry of $(c-1)\bar{c}$, it will be equivalent to avoiding a simpler pattern, similar to the argument of Lemma 2. So we need only consider the cases where this does NOT happen. They are the patterns $\overline{1}\overline{2}43$, $\overline{1}32\overline{4}$ and their symmetries.

Proposition 2. $\left|S_n(\{\overline{1}\overline{2}43\})\right| = \left|S_n(\{\overline{1}32\overline{4}\})\right| = (n-2)!$ *for all* $n \geq 2$

Proof. It is enough to show that $S_n(\{\overline{1}\overline{2}43\})$ is precisely the set of permutations that begin with the letters 12, and that $S_n(\{\overline{1}32\overline{4}\})$ is precisely the set of permutations that begin with the letter 1 and end with the letter n.

For the first, if p begins with 12, clearly neither of these letters is involved in a 21 pattern, and every 21 pattern is preceded by the smaller 12, thus $p \in S_n(\{\overline{1}\overline{2}43\})$. On the other hand, if p starts with 21, it clearly contains the forbidden pattern 21 not preceded by a smaller 12 pattern. Also, if one of the first two letters of p is ≥ 3 then that letter is part of a forbidden 21 pattern that is not preceded by a smaller 12 pattern, so we are done.

For the case of $S_n(\{\overline{1}32\overline{4}\})$, if p begins with 1 and ends with n, then neither of these can be involved in a 21 pattern, so every 21 pattern is preceded by a smaller number (the 1) and succeeded by a larger number (the n), so $p \in S_n(\{\overline{1}32\overline{4}\})$. On the other hand, if p begins with something other than 1, then $p_1 1$ forms a 21 pattern that is not preceded by a smaller number. Similarly if p ends with something other than n, then np_n forms a 21 pattern that is not succeeded by a larger number. $\square$

Finally, we consider the case of barred patterns of length 4 with precisely one bar. This was first comprehensively studied by Callan [3]. The following propositions are proved in a similar way to Callan's work, but with slightly modified notation, and are included for completeness.

Callan showed that permutations avoiding a barred pattern of length 4 with exactly one bar fall into 4 categories. By Lemma 2, 64 of these 96 ($= 4! \times 4$) patterns are equivalent to avoiding an unbarred pattern of length 3, thus yielding the Catalan numbers. The remaining 3 cases are those for which the sequence $\{|S_n(\{q\})|\}_{n\geq 0}$ gives the Bell numbers, OEIS Sequence A051295, and OEIS Sequence A137533 [8]. The data in Table 2.3, first computed by Callan [3], lists the 32 remaining patterns, grouped by Wilf equivalence class.

We consider one representative from each of these classes. Permutations that avoid other patterns but yield the same counting sequence can be enumerated by similar methods.

Table 1. Permutation Classes Avoiding a Pattern of Length 4 with 1 Bar

Representative	**Other Class Members**	**Sequence**
$1\bar{4}23$	$134\bar{2}, 23\bar{1}4, \bar{2}431, \bar{3}124, 32\bar{4}1, 4\bar{1}32, 421\bar{3}$	Bell
$\bar{2}413$	$2\bar{4}13, 24\bar{1}3, 241\bar{3}, \bar{3}142, 3\bar{1}42, 31\bar{4}2, 314\bar{2}$	Bell
$\bar{1}423$	$\bar{1}342, 241\bar{4}, 243\bar{1}, 312\bar{4}, 324\bar{1}\bar{4}132, \bar{4}213$	A051295
$\bar{1}324$	$132\bar{4}, 423\bar{1}, \bar{4}231$	A051295
$\bar{1}432$	$234\bar{1}, 321\bar{4}, \bar{4}123$	new

Proposition 3. $\left|S_n(\{1\bar{4}23\})\right|$ *satisfies the recurrence*

$$\left|S_n(\{1\bar{4}23\})\right| = \sum_{i=1}^{n} \binom{n-1}{i-1} \left|S_{n-i}(\{1\bar{4}23\})\right|$$

Proof. Let i be the position of the letter n in a $1\bar{4}23$-avoiding permutation. Then, the $i-1$ letters preceding n must be in decreasing order (otherwise $j < k < n$ forms a 123 pattern without a larger element between the j and k). The $n-k$ letters after n may be in any order, so long as they avoid $1\bar{4}23$. This gives a typical graph of a $1\bar{4}23$-avoiding permutation, considered as a function from $[n]$ to $[n]$ as in Figure 1.

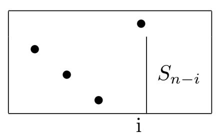


Figure 1. A generic $\{1\bar{4}23\}$-avoiding permutation

There are $\binom{n-1}{i-1}$ ways to choose the initial k elements of the permutation, and $\left|S_{n-i}(\{1\bar{4}23\})\right|$ ways to order the last $n-i$ elements, so summing over all possible positions i for the entry n, we obtain the above recurrence.

This is the same recurrence satisfied by the Bell numbers. Further, this proof gives a clear bijection with set partitions of $\{1, \ldots, n\}$. Given a $\{1\bar{4}23\}$-avoiding permutation p let the set containing n in the corresponding set partition be n and all elements that appear before n in p. Since the elements of p that after n have the same recursive $\{1\bar{4}23\}$-avoiding structure, the rest of the set partition can be computed similarly. □

Proposition 4. $\left|S_n(\{\bar{1}423\})\right|$ *satisfies the recurrence*

$$\left|S_n(\{\bar{1}423\})\right| = \sum_{i=1}^{n} (n-i)! \left|S_{i-1}(\{\bar{1}423\})\right|$$

Proof. Let i be the position of the letter 1. Then the $(n-i)$ entries following i may appear in any order. However, the letters before the 1 must all be smaller than the letters after the 1, otherwise $j1k$ with $j > k$ forms a 312 pattern without a smaller letter in front of it. The $i-1$ entries preceding i must merely avoid the forbidden pattern $\bar{1}423$, giving the graph of a typical $\bar{1}423$-avoiding permutation to be as in Figure 2.

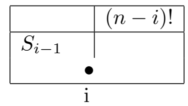


FIGURE 2. A generic $\{\bar{1}423\}$-avoiding permutation

There are $\left|S_{i-1}(\{\bar{1}423\})\right|$ ways to order the first $i-1$ elements, and $(n-i)!$ ways to order the last $n-i$ elements, so summing over all possible positions i for the letter n gives the above recurrence.

This recurrence gives sequence A051295 in the Online Encyclopedia of Integer Sequences. □

Proposition 5.

$$\left|S_n(\{\bar{1}432\})\right| = (n-1)! + \sum_{j=2}^{n} \frac{(n-2)!}{(j-2)!} + \sum_{i=3}^{n} \sum_{j=2}^{n-i+2} \sum_{l=j+i-2}^{n} \frac{(n-i)!(l-j-1)!}{(l-i)!(i-3)!}$$

Proof. We break the set $S_n(\{\bar{1}432\})$ into cases depending on the location of the letter 1.

If 1 is the first letter of a permutation p, then clearly $p \in S_n(\{\bar{1}432\})$ since 1 as the first letter cannot be involved in a forbidden 321 pattern, and every 321 pattern is preceded by the 1. Thus, there are $(n-1)!$ permutations avoiding $\bar{1}432$ and beginning with 1.

If 1 is the second letter of a permutation p that begins with j, then we get the following graph:

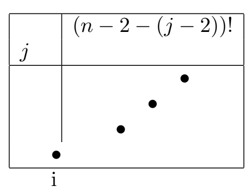


FIGURE 3. A $\{\bar{1}432\}$-avoiding permutation with 1 as the second letter

That is, all letters smaller than j must appear in increasing order (otherwise $j > a > b$ forms a 321 pattern without a smaller letter in front of it), so we may choose the positions of these letters but not their order. This can be done in $\binom{n-2}{j-2}$ ways. Further, the letters greater than j may appear in any order, but their positions are exactly the positions left over after choosing the places of the letters smaller than j. These larger letters can be ordered in $(n-2-(j-2))!$ ways. Summing over all possible values for j, we get the second term in the proposition.

Finally, we consider the case of 1 appearing in the third position or later. We obtain a permutation graph as in Figure 4.

That is, all letters before 1 must appear in increasing order, otherwise $a > b > 1$ is a 321 pattern not preceded by a smaller letter. If j is the smallest letter before 1 and l is the largest letter before 1, we may also conclude that

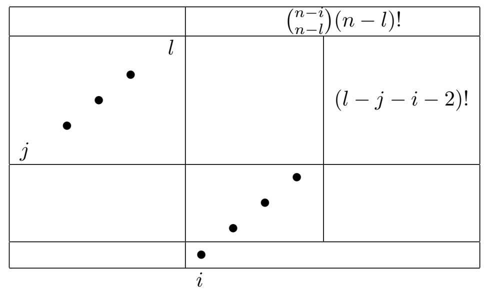


FIGURE 4. A $\{\bar{1}432\}$-avoiding permutation with 1 as the third letter or later

- The $n-l$ letters larger than l may appear in any order, so we may choose their positions in $\binom{n-i}{n-l}$ ways, and their order in $(n-l)!$ ways.
- The letters smaller than j must appear in increasing order, otherwise $j > a > b$ is a 321 pattern not preceded by a smaller letter.
- The letters smaller than j must appear strictly before the letters between j and l that are after the 1. Otherwise, let a be a letter $j < a < l$ that occurs before letter b, with $b < j$. Then lab is a 321 pattern not preceded by a smaller letter.
- Now, the positions of the remaining $(l-j-i-2)$ letters is determined, and they can be ordered in $(l-j-i-2)!$ ways.

Thus, summing over all possibilities for j, l, and i gives the third and final term in the proposition.

This formula produces sequence A137533 in the Online Encyclopedia of Integer Sequences. □

2.4. **Avoiding barred patterns of length 5.** A comprehensive study of permutations avoiding patterns of length 5 is not yet completed, however, computational data shows that a number of new non-degenerate cases remain to be studied. We give a survey of computational data for $n \leq 7$ and patterns with 1, 2, or 3 bars.

The symmetries of reverse, complement, and inverse give 89 distinct equivalence classes for the sequence $|S_n(\{q\})|$ when q is a pattern of length 5 with one bar. Of these classes, 52 are equivalent to avoiding a pattern of length 4 by Proposition 2.

For the 37 remaining classes, computation suggests that there are at least 17 different possible sequences for $|S_n(\{q\})|$. 15 of these are new to the literature. Table 2 below sorts these non-degenerate results by their first 7 terms.

Similarly, for patterns of length 5 with 2 bars, there are 172 equivalence classes and 150 of these reduce to avoiding an unbarred pattern of length 3. Of the 22 non-degenerate cases, we get at least 13 distinct sequences, 9 of these new to the literature. These sequences are given in Table 3.

Finally, for patterns of length 5 with 3 bars, all cases are degenerate to either $S_n(\{q\}) = 1$ or $S_n(\{q\}) = (n-3)!$.

Now that we have exhausted comprehensive case by case analysis of permutations avoiding a single barred permutation, we consider a method to compute recurrences for $S_n(Q)$ where Q is an arbitrary set of barred permutation patterns.

Pattern Class Representatives	**Sequence**	**OEIS Number**
$25\bar{3}14$, $35\bar{2}41$, $45\bar{3}12$, $51\bar{4}23$	1, 2, 6, 23, 104, 530, 2958	A117106
$352\bar{4}1$	1, 2, 6, 23, 104, 530, 2959	A137534 (new)
$1\bar{4}235$, $4\bar{2}513$	1, 2, 6, 23, 104, 531, 2977	A137535 (new)
$423\bar{1}5$, $425\bar{1}3$, $5314\bar{2}$(**)	1, 2, 6, 23, 104, 531, 2982	A110447
$\bar{4}2153$, $5\bar{1}423$	1, 2, 6, 23, 104, 532, 3002	A137536 (new)
$5134\bar{2}$	1, 2, 6, 23, 104, 532, 3003	A137537 (new)
$253\bar{1}4$, $\bar{3}1542$(*), $\bar{3}5214$(*), $\bar{3}5241$ $42\bar{5}13$, $43\bar{5}21$(*), $45\bar{1}32$(*)	1, 2, 6, 23, 104, 532, 3004	A137538 (new)
$1\bar{5}324$, $\bar{4}1523$	1, 2, 6, 23, 104, 532, 3005	A137539 (new)
$\bar{4}1253$	1, 2, 6, 23, 104, 533, 3026	A137540 (new)
$1\bar{5}234$, $412\bar{5}3$	1, 2, 6, 23, 104, 533, 3027	A137541 (new)
$\bar{1}3425$, $3524\bar{1}$	1, 2, 6, 23, 104, 533, 3038	A137542 (new)
$\bar{1}3245$, $3241\bar{5}$, $\bar{5}1432$, $\bar{5}3412$	1, 2, 6, 23, 104, 534, 3060	A137543 (new)
$\bar{5}1342$	1, 2, 6, 23, 104, 534, 3064	A137544 (new)
$\bar{5}2143$	1, 2, 6, 23, 104, 535, 3081	A137545 (new)
$5234\bar{1}$	1, 2, 6, 23, 104, 535, 3082	A137546 (new)
$\bar{5}1243$	1, 2, 6, 23, 104, 535, 3085	A137547 (new)
$\bar{5}1234$, $\bar{5}1324$	1, 2, 6, 23, 104, 535, 3088	A137548 (new)

(*) This sequence will be proven by the method of prefix enumeration schemes.
(**) This sequence has been proven by Callan in [4].

Table 2. Number of permutations avoiding a pattern of length 5 with one bar

3. Enumeration Schemes

Our goal in this section is to introduce a single method that works to enumerate many classes $S_n(Q)$ where Q is a set of barred permutation patterns. Following Zeilberger [12] [13] and Vatter [9] we derive an algorithm whose input is a set of permutation patterns Q, and whose output can be read as a recurrence counting $S_n(Q)$. Notation from the Zeilberger's and Vatter's original work with unbarred permutation patterns will be adapted as necessary.

In the following sections, we discuss in turn the notions of *refinement*, *reversibly deletable elements*, *gap vectors*, and *stop points*. These four concepts are combined to form an *enumeration scheme*, or recurrence counting $|S_n(Q)|$.

3.1. **Refinement.** Since the set $S_n(Q)$ may be complicated, we first partition $S_n(Q)$ into disjoint subsets and look for recurrences between these subsets.

One natural and useful way to partition the permutations of $S_n(Q)$ is by the patterns formed by the first i letters of its elements. The following notation will be useful to discuss this partitioning of $S_n(Q)$:

$$S_n(Q; p_1 \cdots p_i) = \{\pi \in S_n \mid \pi \text{ avoids } q \text{ for all } q \in Q, \pi_1 \cdots \pi_i \text{ reduces to } p_1 \dots p_i\}$$

Pattern Class Representatives	**Sequence**	**OEIS Number**
$2\bar{5}3\bar{1}4$, $\bar{3}5\bar{1}42$	1, 2, 5, 14, 43, 143, 509	A006789
$\bar{4}25\bar{1}3$, $5\bar{1}32\bar{4}$	1, 2, 5, 14, 43, 143, 510	A098569
$145\bar{3}\bar{2}$(*)	1, 2, 5, 14, 43, 143, 511	A137549 (new)
$251\bar{4}\bar{3}$(*), $415\bar{3}\bar{2}$	1, 2, 5, 14, 43, 144, 522	A137550 (new)
$\bar{3}\bar{1}542$	1, 2, 5, 14, 43, 144, 523	A047970
$\bar{2}413\bar{5}$, $425\bar{3}\bar{1}$, $42\bar{5}3\bar{1}$	1, 2, 5, 14, 43, 144, 525	A137551 (new)
$\bar{1}43\bar{5}2$	1, 2, 5, 14, 43, 145, 538	A122993
$\bar{1}5\bar{2}43$	1, 2, 5, 14, 43, 146, 550	A137552 (new)
$\bar{2}\bar{1}453$, $243\bar{1}\bar{5}$, $423\bar{1}\bar{5}$, $534\bar{2}\bar{1}$(*)	1, 2, 5, 14, 43, 146, 561	A137553 (new)
$354\bar{2}\bar{1}$(*), $\bar{5}324\bar{1}$	1, 2, 5, 14, 43, 147, 575	A137554 (new)
$\bar{4}\bar{5}123$	1, 2, 5, 14, 43, 147, 578	A137555 (new)
$\bar{1}432\bar{5}$	1, 2, 5, 14, 43, 148, 592	A137556 (new)
$345\bar{2}\bar{1}$	1, 2, 5, 14, 43, 150, 617	A137557 (new)

(*) This sequence will be proven by the method of prefix enumeration schemes later in this chapter.

Table 3. Number of permutations avoiding a pattern of length 5 with two bars

$$S_n\left(Q; \begin{matrix} p_1 \cdots p_l \\ l_1 \cdots l_i \end{matrix}\right) = \left\{ \pi \in S_n \;\middle|\; \begin{matrix} \pi \text{ avoids } Q, \\ \pi_1 \cdots \pi_i \text{ reduces to } p_1 \ldots p_i, \text{ and} \\ l_1, \ldots, l_i \text{ are the first } i \text{ letters of } \pi \end{matrix} \right\}.$$

For example, $S_3(\{132\}; 12) = \{123, 231\}$, i.e. of the 5 permutations of length 3 that avoid the pattern 132, only 123 and 231 begin with an increasing pair of letters. Similarly, $S_3\left(\{132\}; \begin{matrix} 12 \\ 23 \end{matrix}\right) = \{231\}$.

Given $p = p_1 \cdots p_i$, a *refinement* of p is a permutation $q = q_1 \cdots q_{i+1}$ such that $q_1 \cdots q_i$ reduces to p. For example, the refinements of $\emptyset$ are $\{1\}$. The refinements of 1 are $\{12, 21\}$. The refinements of 12 are $\{123, 132, 231\}$.

Finally, we have the following useful observation:

Proposition 6.

$$S_n(Q; p) = \bigcup_{q \in \textit{refinements of } p} S_n(Q; q), \textit{ and so}$$

$$|S_n(Q; p)| = \sum_{q \in \textit{refinements of } p} |S_n(Q; q)|.$$

Thus, for any set of patterns Q, we have $S_n(Q) = S_n(Q; 1) = S_n(Q; 12) \cup S_n(Q; 21) = \cdots$. This partitioning of $S_n(Q)$ into disjoint sets depending on the initial few letters is identical to the work of Zeilberger.

Graphically, we may represent refinement using a graph, where the vertices correspond to the sets $S_n(Q; p)$ and are labelled with the prefixes p. There is a directed edge from a prefix to each of its refinements. To count $S_n(Q)$ it is enough to count the subsets $S_n(Q; p)$ represented by the leaves of the graph. An example of such a graph of refinements is given in Figure 5.

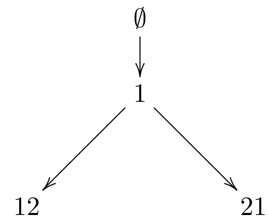


FIGURE 5. The graph of refinements for an arbitrary pattern set

Now that we have a way to partition $S_n(Q)$ into disjoint subsets, we consider ways to find recurrences between these subsets.

3.2. **Reversibly Deletable Elements.** The key tool for finding recurrences between $S_n(Q;p)$ for various prefixes p is the following:

Definition 1. *Given Q, a set of barred permutation patterns, and p, a prefix of length l, $l > 0$, we say that position r ($1 \leq r \leq l$) is* reversibly deletable *if and only if the action of removing p_r from a Q-avoiding permutation of length n and inserting p_r into a Q-avoiding permutation of length $n-1$ is a bijection between* $S_n\left(Q;\ \begin{matrix} p_1 \cdots p_l \\ i_1 \cdots i_l \end{matrix}\right)$ *and* $S_{n-1}\left(Q;\ \begin{matrix} p_1 \cdots p_{r-1} p_{r+1} \cdots p_l \\ i_1 \cdots i_{r-1} i_{r+1} \cdots i_l \end{matrix}\right)$.

In the case of unbarred pattern avoidance, it is enough to check that the insertion of p_r into a Q-avoiding permutation of length $n-1$ gives a Q-avoiding permutation of length n, since the deletion of a letter from a permutation cannot cause a bad pattern. More specifically, for unbarred pattern avoidance, p_r is reversibly deletable if and only if every forbidden pattern involving p_r implies the existence of a forbidden pattern without p_r.

For example, if $Q = \{123\}$, and $p = 21$, we have that $p_1 =$"2" is reversibly deletable, since the only way for p_1 to be involved in a 123 pattern is for there to be $21 \cdots p_s \cdots p_t$ with "2"$< p_s < p_t$. But this means that "1"$< p_s < p_t$, and $1 p_s p_t$ forms a forbidden 123 pattern without $p_1 =$"2". That is, every 123 pattern involving p_1 implies the existence of a 123 pattern without p_1. Thus, it is impossible to create a $\{123\}$-containing permutation by inserting p_1 into a $\{123\}$-avoiding permutation. So inserting and deleting p_1 is indeed a bijection between $\{123\}$-avoiding permutations of length $n-1$ and $\{123\}$-avoiding permutations of length n.

For the case of barred patterns, the definition of reversibly deletable elements is equivalent to the old definition with the added caveat that p_r cannot be the only letter to play the role of a barred letter in extending a forbidden $\underline{q}$ pattern to $\overline{q}$. That is, deleting p_r from a Q-avoiding permutation can only fail to produce another Q-avoiding permutation if p_r plays the role of a barred letter and its removal makes an instance of a forbidden pattern $\underline{q}$ no longer extendable to the larger barred pattern $\overline{q}$.

In summary, to check algorithmically that p_r is reversibly deletable, we must check two things.

1. Check that *inserting* p_r into a Q-avoiding permutation always produces a Q-avoiding permutation.

(2) Check that *deleting* p_r from a Q-avoiding permutation always produces a Q-avoiding permutation.

We discuss how to rigorously check each of these in turn.

(1) **Insertion**

For insertion, as in the unbarred case, we check that every possible occurrence of a forbidden pattern with p_r implies the existence of a forbidden pattern without p_r. This is easily seen to happen in a finite number of scenarios. First, choose the letters of the prefix p (including p_r) that will be involved in the forbidden pattern. Then, choose all the ways that the remaining letters of the forbidden pattern can be spaced between the letters of p.

For example, for $Q = \{\bar{1}423\}$, a forbidden pattern is a 312 pattern without a smaller letter before it. Consider $p = 123$, and reversibly deletable candidate $p_2 =$"2". Recall that p is a prefix, denoting that the first three letters of our permutation are increasing, not that they are specifically the letters 1, 2, and 3. So the only way for p_2 to be involved in a bad pattern is for p_2 to be followed by a smaller increasing pair. These letters may be (a) both less than p_1, (b) one less than p_1 and one greater than p_1, or (c) both greater than p_1 and less than p_2, as in the permutation graphs in Figure 6 below. We consider a permutation as a function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. We use ⋇ to mark p_2 to be deleted, ■ to denote the letters of $123ab$ that, together with p_2, form a forbidden pattern, and ⊡ to mark another letter that, together with the letters marked ■, forms a forbidden pattern without p_2.

We quickly eliminate case (c) since $123ab$ where $2ab$ is a 312 pattern, and "1"$< a < b <$"2", actually extends to the 1423 pattern $12ab$. Now, it is easy to check that deleting p_2 in each of cases (a) and (b) gives another $\{\bar{1}423\}$-containing permutation.

Additionally, to check that p_r is reversibly deletable for prefix p and forbidden pattern q where q has b bars, we must also check scenarios with b additional letters.

These additional scenarios are indeed necessary. For example, let $Q = \{134\bar{2}\}$, and $p = 21$, and consider p_1. p_1 being involved in a bad pattern with two letters after the prefix may happen in one way, namely:

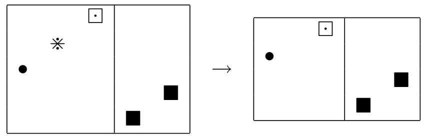
So, $21ab$ containing the forbidden pattern "2"$< a < b$ *does* imply that "1"$< a < b$ is a forbidden pattern without p_1. It seems from this that p_1 is reversibly deletable. However, $21abc$ containing the forbidden pattern $2ab$ does not imply that $1abc$ is bad (if "1"$< c <$"2"), since c may act as a barred letter extending the $1ab$ pattern, but not the $2ab$ pattern.

To show that this is always enough, note that if π contains a forbidden pattern $\underline{q}$ where q has b bars, then only b letters need be inserted to form a copy of the $\overline{q}$, so adding even more letters is redundant.

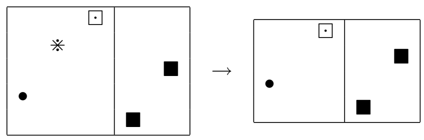
(2) **Deletion**

Now that we have rigorously shown that insertion of p_r is a map from Q-avoiding permutations to Q-avoiding permutations, we check that deletion also has this property.

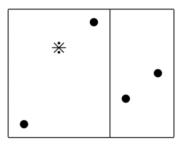
To do this, we check the scenarios for a forbidden pattern involving p_r as above. Namely, we want to show that if π^* begins with prefix $p^* =$



Case (a): both post-prefix letters less than p_1



Case (b): one post-prefix letter less than p_1 and one greater than p_1



Case (c): both post-prefix letters greater than p_1 and less than p_2

FIGURE 6. An example of checking that insertion is bijective

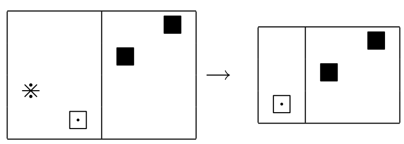


FIGURE 7. A $\{134\bar{2}\}$-avoiding permutation with prefix 21

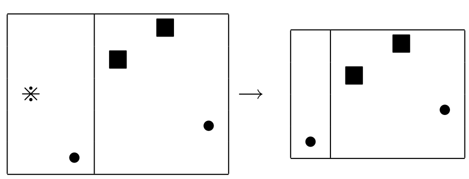


FIGURE 8. A $\{134\bar{2}\}$-containing permutation with prefix 21

$p_1 \cdots p_{r-1}p_{r+1} \cdots p_l$ and has a forbidden pattern, then π, beginning with $p = p_1 \cdots p_r \cdots p_l$ has a forbidden pattern. Thus, if we compute all the scenarios beginning with p^*, insert p_r, and check that each one contains a forbidden pattern, then we are done.

For example, if $Q = \{\bar{1}423\}$ and $p = 123$, we again consider p_2. There are a number of ways for p_1p_3 to be involved in a forbidden pattern that does *not* extend to 1423, namely:

Now, p_2 can be inserted into each of these scenarios in possibly multiple ways as in Figure 10.

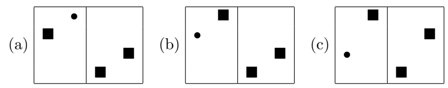
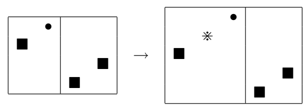
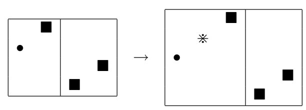


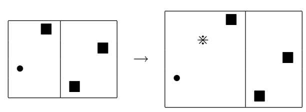
Figure 9. $\{\bar{1}423\}$-containing patterns with prefix 12.



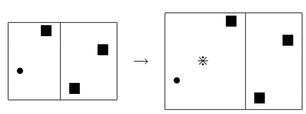
Case (a): inserting p_2 ($p_1 < p_2 < p_3$) into a $\{\bar{1}423\}$-avoiding permutation



Case (b): inserting p_2 ($p_1 < p_2 < p_3$) into a $\{\bar{1}423\}$-avoiding permutation



Case (c1): inserting p_2 ($p_1 < p_2 < p_3$) into a $\{\bar{1}423\}$-avoiding permutation



Case (c2): inserting p_2 ($p_1 < p_2 < p_3$) into a $\{\bar{1}423\}$-avoiding permutation

Figure 10. An example of checking that deletion is bijective

We may inspect that each of these resulting permutations contains a 312 pattern that does *not* extend to a 1423 pattern, and thus, p_2 is reversibly deletable.

To denote that p_r is reversibly deletable in the graphical notation, we draw a dotted arrow from p to p^* labelled with d_r, which denotes the deletion map of deleting the rth letter of π and reducing. For example, if $p = 21$ had p_1 reversibly deletable, we would encode this as in Figure 11

3.3. **Gap Vectors.** Unfortunately, the reversibly deletable elements are usually not sufficient to find recurrences counting the elements of $S_n(Q)$, so following Vatter [9], we introduce the notion of gap vectors. Given a set of forbidden patterns Q and prefix $p = p_1 \cdots p_i$, a *spacing vector* v is a vector in $\mathbb{N}^{i+1}$. We write $||v||$ to denote the sum of the entries of v. Spacing vectors help further narrow down the set $S_n(Q; p)$ into smaller subsets in the following way.

Definition 2. *Given Q, a set of forbidden patterns, p a prefix of length i, and v, a spacing vector of length $l + 1$, let $s_1 \cdots s_l$ be the permutation obtained by sorting*

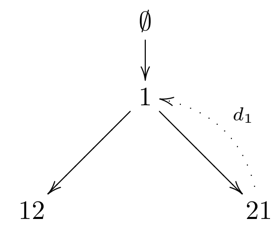


FIGURE 11. Representation of reversibly deletable elements

p. $S_n(Q;p;v)$ denotes the set of permutations of length n, avoiding Q, beginning with prefix p, and with exactly v_1 letters smaller than s_1, exactly v_j letters that are greater than s_{j-1} and smaller than s_j, and exactly v_{i+1} letters that are greater then s_l.

For example, $S_n(\{123\};12;\langle 0,1,2\rangle)$ denotes the set of permutations avoiding 123, beginning with an increasing pair of letters p_1p_2, with one letter a such that $p_1 < a < p_2$ and two letters bigger then p_2.

Definition 3. *A spacing vector v is a* gap vector *for $[Q,p]$ if there are no permutations avoiding Q with prefix p and spacing vector $\geq v$ (componentwise).*

For example, if $Q = \{123\}$ and $p = 12$, then $v = \langle 0,0,1\rangle$ is a gap vector since if we have at least one letter a larger than p_1 and p_2, then p_1p_2a forms a 123 pattern.

We check this similarly to the unbarred case, with yet another constraint.

To check that v is a gap vector in the unbarred case, we consider permutations starting with p, and name v_1 letters smaller than $s_1 = 1$ (say $\frac{1}{v_1+1},\ldots,\frac{v_1}{v_1+1}$), v_2 letters between $s_1 = 1$ and $s_2 = 2$ (say $1+\frac{1}{v_2+1},\ldots 1+\frac{v_2}{v_2+1}$), $\ldots$, and v_{l+1} letters bigger than $s_l = l$ (say $l+\frac{1}{v_{l+1}+1},\ldots,l+\frac{v_{l+1}}{v_{l+1}+1}$). Now, consider all permutations that begin with p and end with any of the $(v_1+\cdots+v_{l+1})!$ permutations of these fractional letters. If *each* of these permutations contains a forbidden pattern from Q, then v is a gap vector for $[Q,p]$.

In the barred case, while the algorithm in the previous paragraph is necessary to show that v is a gap vector, it is no longer sufficient. For example, when avoiding $Q = \{23\overline{1}\}$, with prefix $p = 1$, $v = \langle 0,1\rangle$ appears to be a gap vector when considering permutations of length 2. However, there are permutations of length 3 with spacing vector $v^* = \langle 1,1\rangle$ that avoid Q. That is, we may have a vector such that $|S_n(Q;p;v)| = 0$, but there is some $w > v$ (componentwise) such that $|S_n(Q;p;w)| > 0$. However, we want to find a basis for the set of vectors v such that $|S_n(Q;p;w)| = 0$ for all n and for all $w \geq v$.

In light of this complication, to show that v is a gap vector for $[Q,p]$, not only do we need to confirm that there are no Q-avoiding permutations with prefix p and spacing v, but also that there are no Q-avoiding permutations with spacing w, $w > v$ componentwise. More precisely, if we are concerned with finding the basis of gap vectors for $[Q,p]$, most of the time we proceed as in the unbarred case, with one important exception. More work needs to be done to check for gap vectors when avoiding a pattern of the form $q = q_1\cdots q_{m-i-1}\overline{q}_{m-i}\cdots\overline{q}_m$. We begin with the case when $i = 0$, i.e. q ends with exactly one barred letter.

Theorem. *Let* $q \in \overline{S}_m$ *such that* $\underline{q} = q_1 \cdots q_{m-1}$*. Then there are* no *basis gap vectors for* $[\{q\}, p]$ *for any prefix* p*.*

Proof. Assume that q is as in the proposition, and that v is a basis gap vector for $[\{q\}, p]$. Now, let $\pi = \pi_1 \cdots \pi_l \in S_{|p|+\|v\|}$ that has prefix p. Since v is a basis gap vector, π contains q, but if the last letter of π is deleted, then it avoids q. That is, by definition of basis gap vector, the last letter of π is involved in a forbidden $\underline{q}$ pattern that does not extend to a $\overline{q}$ pattern.

For each instance of $\underline{q}$ in π, choose a letter to append to π which will extend the instance to $\overline{q}$; write L for the set of such letters to be appended to π. Without loss of generality, assume that $q_{m-1} < q_m$. Then append the letters of L to π in increasing order, and call the resulting permutations π^*. We claim that either π^* is a $\{q\}$-avoiding permutation or can be further extended to be $\{q\}$-avoiding, with prefix p and spacing w, $w > v$, so v is not a gap vector, and by contradiction we are done.

To see that $\pi^* = \pi_1^* \cdots \pi_{l_2}^*$ is $\{q\}$-avoiding, we consider several cases.

First, by Lemma 2, we may assume that $\overline{q}$ is not a monotone pattern and that there exists q_c with $q_{m-1} < q_c < q_m$.

Construct π^* by appending each letter of L individually. Suppose that $\pi_1 \cdots \pi_{l+i}$ contains a forbidden $\underline{q}$ pattern that uses π_{l+i}. Then either:

- The rest of the forbidden pattern consists of letters π_j with $j \leq l$. In this case, the letter of L that was meant to extend the bad pattern formed by replacing π_{l+i} with π_l has yet to be appended to π, and will extend this instance of $\underline{q}$ to an instance of $\overline{q}$ as well.
- If the rest of the forbidden pattern consists of both letters from $\pi_1 \cdots \pi_l$ and letters from L, then we note that by Lemma 2, there exists π_c in the instance of the forbidden pattern with $\pi_{l+j} < \pi_c < \pi_{l+i}$, $j < i$, $c < l$, so we may append another letter π_{l+i+1} to π^* extending this to a copy of $\overline{q}$. Again, there must be π_{c_2} with $c_2 < l$ so that $\pi_{l+i} < \pi_{c_2} < \pi_{l+i+1}$. If this letter π_{l+i+1} is involved in another instance of $\underline{q}$, repeat. We know this process terminates because there are a finite number of letters in π, so there is a maximum letter of $\{\pi_1, \ldots, \pi_l\}$ to play the role of π_{c_i} in this construction.

In both cases, we have shown that it is possible to append enough letters to π to make it $\{q\}$-avoiding, and thus there are no gap vectors for $[\{q\}, p]$. □

As an example of this construction, consider permutations that avoid the pattern $q = 241\overline{3}$. For the prefix 123, there are no permutations avoiding q with spacing $\langle 1, 0, 0, 0 \rangle$, and there are no permutations avoiding q with a spacing vector of weight 2; however, we can construct a permutation with prefix $p = 123$ and a spacing vector of weight 3 that avoids q. Notice that in the language of the proposition, $\pi = 2341$, which contains $\overline{q} = 231$ in several places, namely, 231, 241, and 341. Thus, we require the addition of a letter greater than "2" and less than "3", a letter greater than "2" and less than "4", and a letter greater than "3" and less than "4", to extend each of these copies of $\underline{q}$ to a copy of $\overline{q}$. Choosing two letters, a and b with "2" $< a <$ "3" $< b <$ "4" suffices. We append a and b to the end of π to obtain $246135 \in S_n(\{241\overline{3}\}; 123; \langle 1, 1, 1, 0 \rangle)$. Note that 246135 is $\{241\overline{3}\}$-avoiding, with prefix 123 and a spacing vector which is greater than $\langle 1, 0, 0, 0 \rangle$.

We note that Theorem 3.3 is necessary. In general, we can find p and q so that the smallest weight of a spacing vector v where $S_n(\{q\};p;v) \neq \emptyset$ is arbitrarily large, and checking all the appropriate scenarios would be time-consuming. With this Theorem, we need not consider all of these scenarios, but rather return that the set of gap vectors for $[\{q\},p]$ is empty. Since gap vectors were included in the scheme algorithm to help find recurrences, eliminating gap vectors in this case may seem to limit the success of our algorithm. However, we note that via the symmetries of the square, if we cannot find an enumeration scheme for $S_n(\{q\})$ where q ends in a barred letter, we may still find an enumeration scheme for $S_n(\{q^r\})$ where q^r does not end in a barred letter. Further, if q is part of a set of forbidden patterns Q, the other patterns not ending in a barred letter may still help find gap vectors for the enumeration scheme for $S_n(Q)$.

We also note that this construction does not necessarily generalize to patterns of the form $q = q_1 \cdots q_{m-i-1}\overline{q}_{m-i} \cdots \overline{q}_m$ where $i > 0$. For example, if $q = 351\overline{42}$ and $p = 231$, we note that $v = \langle 0,0,0,0 \rangle$ is a gap vector because two letters a and b, with $p_3 < b < p_1 < a < p_2$ must be appended to p to extend p to an instance of 35142. But now, p_1ab is a new forbidden 351 pattern that requires two letters c and d to be appended with $b < d < p_1 < c < a$ to extend p_1ab to an instance of 35142. This process continues indefinitely.

For the case of patterns which consist of a block of unbarred letters followed by a block of more than one barred letter, we must check extra scenarios to determine if v is a gap vector. Namely, if $S_n(Q;p;v) = \emptyset$, we must also check that $S_n(Q;p;w) = \emptyset$ for all $w > v$ with $\|w\| = \|v\| +$ (total number of bars in all patterns in Q)$\cdot$ (total number of occurrences of $q_1 \cdots q_{m-i-1}$ in p}) before concluding that v is in fact a gap vector.

With this more specific definition of gap vector, we have an ideal in $\mathbb{N}^{l+1}$ which necessarily has a finite basis. We have also exhibited a method to find basis vectors for a scheme. These serve to narrow down the cases we must consider to decide if an element is reversibly deletable.

Graphically, we write a basis for the gap vectors corresponding to p below p. For example, if $\langle 0,0,1 \rangle$ is a gap vector for the prefix 12, and this causes p_2 to be reversibly deletable, we would represent this situation as in Figure 12

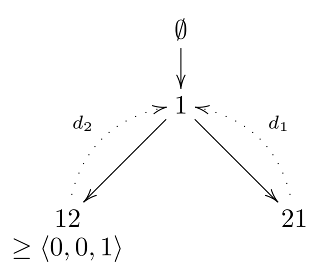


Figure 12. Representation of gap vectors

A final remark for gap vectors concerns the vector $v_0 = \langle 0,\ldots,0 \rangle$. Note that if v_0 is a gap vector for $[Q,p]$, then there are no permutations of any length avoiding Q and beginning with prefix p. Since the gap vector v_0 already indicates that $|S_n(Q;p)| = 0$, it is unnecessary to write $S_n(Q;p)$ in terms of smaller sets. However, for completeness of the definition of enumeration scheme (below), if v_0 is a gap vector, we allow any one of the p_i to be "exceptionally" reversibly deletable. We will return to this remark later.

3.4. **Stop Points.** As we have observed with reversibly deletable elements and with gap vectors, barred patterns require added considerations to find a rigorous enumeration schemes. While we have introduced enough notation to find recurrences between the subsets $S_n(Q;p)$, we require one extra tool to find the base cases for these recurrences, i.e. *stop points.*

The key observation is that there may be *no* permutations of length n that avoid Q and begin with prefix p, but there may be such permutations of length $n+k$ for some $k>0$. For example, if $Q=\{23\bar{1}\}$, there are no permutations of length 2 avoiding Q, but 231 is a permutation of length 3, beginning with a 12 pattern. Thus, in the notation of the enumeration scheme, we require a mechanism to indicate at what length we may begin to consider permutations beginning with that prefix.

Definition 4. *Given a set of forbidden patterns Q, and a prefix p* without *reversibly deletable elements, we say $s \geq |p|$ is a stop point for $[Q,p]$ if there are* no *permutations of length $\leq s$ that avoid Q and begin with prefix p*

For example, the set of stop points for $(\{23\bar{1}\}, 12)$ is $\{2\}$.

Proposition 7. *Given Q and p, the set S of stop points is finite.*

Proof. Notice that since S is a set of positive integers, it is enough to show that S has a well defined maximum element.

It is enough to note that stop points are only defined for prefixes with no reversibly deletable elements. If there were *no* permutations beginning with prefix p, we would obtain the gap vector $\langle 0,\ldots,0\rangle$, and by convention, position 1 is reversibly deletable, so p by definition has an empty set of stop points.

Since p has no reversibly deletable elements, then, we know that there is a permutation π of minimal length that begins with p and avoids Q. The set of stop points has maximum $|\pi|-1$. □

The simplest example of a scheme that requires stop points is the scheme for permutations avoiding $\{123,321,23\bar{1}\}$. Graphically, we represent stop points as a set after an asterisk, listed next to the permutation prefix p, as by $p=12$ in the scheme for $S_n(\{123,321,23\bar{1}\})$ in Figure 13

From this scheme, we have

- $|S_0(Q)|=1$
- $|S_1(Q)|=1$
- $|S_2(Q)|=|S_2(Q;12)|+|S_2(Q;21)|=0+|S_1(Q;1)|=0+1=1$
- $|S_3(Q)| = |S_3(Q;123)|+|S_3(Q;132)|+|S_3(Q;231)|+|S_3(Q;21)|$
  $= 0+0+|S_2(Q;21)|+0$
  $= 0+0+|S_1(Q;1)|+0$
  $= 0+0+1+0=1$
- $|S_n(Q)|=0$ for all $n\geq 4$

Without stop points, we would have computed $|S_2(Q)|=2$.

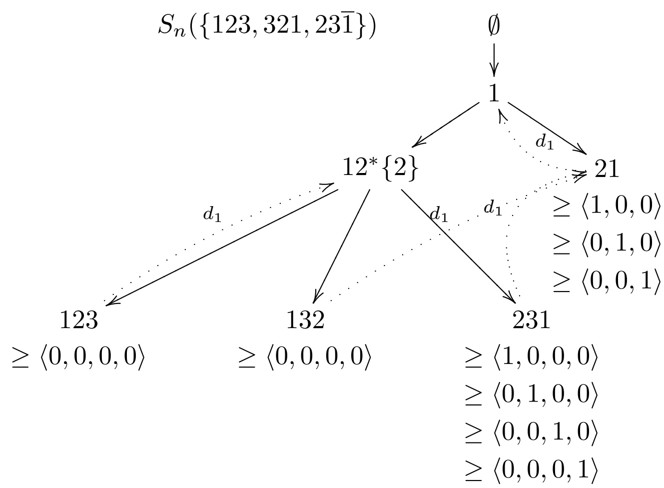


FIGURE 13. A scheme involving stop points

3.5. **Enumeration Schemes.** Finally, we have all the necessary tools to algorithmically find recurrences counting the elements of $S_n(Q)$ where Q contains barred permutation patterns. More specifically:

Definition 5. *An* enumeration scheme $\mathbb{S}$ *is a set of* 4*-tuples* $t = [p_j, R_j, G_j, S_j]$ *such that for each* t*:*

- p_j *is a reduced prefix of length* i*.*
- R_j *a (possibly empty) subset of* $\{1, \ldots, i\}$*.*
- G_j *is a (possibly empty) set of vectors of length* $i+1$*.*
- S_j *is a (possibly empty) finite set of positive integers whose minimum element is* $\geq |p_j|$
  and
- *either* R_j *is non-empty, or all refinements of* p_j *are also in the scheme.*

We have detailed how to find each of the elements of such a 4-tuple, namely if p_j is a prefix, denoting the set $S_n(Q;p_j)$, then R_j, G_j, and S_j are the corresponding reversibly deletable elements, set of gap vectors, and set of stop points.

The last condition ensures that the enumeration scheme can be read as a recurrence counting the elements of $S_n(Q)$. Recall that if R_j is non-empty, then we have a bijection between $S_n(Q;p)$ and $S_{n-1}(Q;p^*)$ for some p^*. If R_j is empty, then we require all refinements of p_j to be in the scheme for completeness.

Given an enumeration scheme $\mathbb{S}$ corresponding to pattern set Q, we can compute $|S_n(Q)|$ in the following way:

(1) Let P be the set of p_j such that either (i) p_j is a prefix of length $\leq n$ with reversibly deletable elements or (ii) p_j is a prefix of length n without reversibly deletable elements. We have $|S_n(Q)| = \sum_{p_j \in P} |S_n(Q;p_j)|$.
(2) For each $p_j \in P$, if $n \in S_j$, then we have $|S_n(Q;p_j)| = 0$.
(3) For each remaining $p_j \in P$, associate the set of spacing vectors v_j^* of all vectors of length $|p_j|+1$ and weight $n - |p_j|$ minus the set of gap vectors G_j. We have $|S_n(Q;p_j)| = \sum_{v \in v_j^*} |S_n(Q;p_j;v)|$.

(4) For each $p_j \in P$, and $v \in v_j^*$, if R_j is non-empty, we have $|S_n(Q; p_j; v)| = \left|S_{n-1}(Q; p_j^*, v^*)\right|$ for prefix p_j^* (p_j with letter r deleted) and vector $v^* = \langle v_1, \ldots, v_{r-1} + v_{r+1}, \ldots, v_{n+1}\rangle$. If R_j is empty, then $\left|S_n(Q; p_j^*)\right| = 1$.

4. The Maple Package bVATTER

The algorithms both (i) to find a scheme and (ii) to read a scheme into a sequence have been programmed in the `Maple` package `bVATTER`, available from the author's website: `http://www.math.rutgers.edu/~lpudwell/maple.html`. The main functions are `SchemeImage`, `SeqS`, `Sipur`.

`SchemeImage` inputs a set of patterns Q, a maximum depth scheme to search for, and a maximum weight of gap vectors to search for, and outputs a concrete enumeration scheme for words avoiding Q of the specified maximum depth. If it cannot find a scheme for Q, it searches for a scheme for a symmetry-equivalent pattern set and returns that scheme instead.

`SeqS` inputs a scheme and an integer K, and uses the scheme to compute $|S_i(Q)|$ for $1 \leq i \leq K$.

`Sipur` inputs a list $[L]$ of pairs of integers, a maximum scheme depth, a maximum weight of gap vectors, and an integer K. It outputs all information about schemes for permutations avoiding one pattern of each length in L where each pair is of the form [length,number of bars]. For example, `Sipur([[4,1]],4,2,30)` outputs all information about permutations avoiding one pattern of length 4 with 1 bar. It will search for schemes of depth 4 with maximum gap vector weight 2 and will output the first 30 terms of the sequence $|S_i(Q)|$ given by each scheme it finds.

`Sipur` has been run on $[L]$ for various lists of the form $[[3, x_i]^a, [4, y_i]^b, [5, z_i]^c]$, and the output is available from the author's website.

5. Success Rate

In this section, we consider the success rate of prefix enumeration schemes for $S_n(Q)$ where Q is a set of barred permutation patterns.

Recall that sets of permutation patterns can be put into equivalence classes based on the permutation involutions of reverse, complement and inverse. We measure success in terms of the number of such equivalence classes for which there is an enumeration scheme. In the Table 4, pattern lengths denotes the lengths of patterns, as well as the number of bars. For example pattern lengths [4,0],[4,1] denotes two patterns of length 4, one without bars, and one with precisely one bar. Specific schemes for the data in the table can be found at the author's website. As for words, it should be noted that pattern sets that are counted as unsuccessful do not necessarily lack an enumeration scheme; they may have enumerations schemes of greater depth than the computer has searched.

5.1. Examples.

We now examine the enumeration schemes for permutations avoiding some specific barred patterns of length 5, thus exhibiting recurrence relations for five of the new sequences given in the tables of Section 2.4.

First, we consider the four classes of patterns of length 5 with one bar that give the sequence 1, 2, 6, 23, 104, 532, 3004. These are the classes with representatives $25\bar{1}43$, $25\bar{1}34$, $43\bar{5}21$, and $43\bar{5}12$.

For the class represented by $25\bar{1}43$ we have the scheme in Figure 14:

Pattern Lengths	Success Rate	Pattern Lengths	Success Rate
[2,0]	1/1 (100%)	[3,0],[3,0],[3,0]	5/5 (100%)
[2,1]	1/1 (100%)	[3,0],[3,0],[3,1]	43/45 (95.6%)
[2,0],[2,0]	1/1 (100%)	[3,0],[3,0],[3,2]	45/45 (100%)
[2,1],[2,0]	2/2 (100%)	[3,0],[3,1],[3,1]	135/138 (97.8%)
[2,1],[2,1]	2/2 (100%)	[3,0],[3,1],[3,2]	280/280 (100%)
[3,0]	2/2 (100%)	[3,0],[3,2],[3,2]	138/138 (100%)
[3,1]	4/4 (100%)	[3,1],[3,1],[3,1]	115/118 (97.5%)
[3,2]	4/4 (100%)	[3,1],[3,1],[3,2]	378/378 (100%)
[3,0],[3,0]	5/5 (100%)	[3,1],[3,2],[3,2]	378/378 (100%)
[3,0],[3,1]	18/20 (90%)	[3,2],[3,2],[3,2]	118/118 (100%)
[3,0],[3,2]	20/20 (100%)	[4,0]	2/7 (28.6%)
[3,1],[3,1]	27/28 (96.4%)	[4,1]	12/16 (75%)
[3,1],[3,2]	50/50 (100%)	[4,2]	25/26 (96.2%)
[3,2],[3,2]	28/28 (100%)	[4,3]	16/16 (100%)
[3,1],[4,0]	59/71 (83.1%)	[5,1]	15/89 (16.9%)
[3,1],[4,1]	229/240 (95.4%)	[5,2]	(in progress)
[3,1],[4,2]	355/364 (97.5%)		
[3,0],[4,1]	84/88 (95.5%)		
[3,0],[4,2]	133/136 (97.8%)		
[4,0],[5,1]	(in progress)		

TABLE 4. Success rate of schemes for various sets of barred patterns

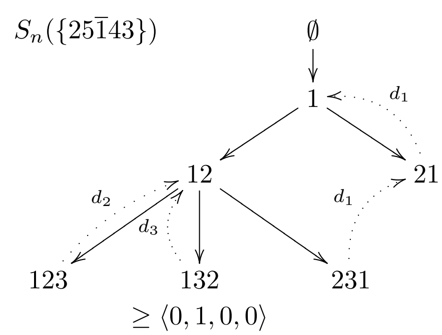


FIGURE 14. The scheme for $S_n(\{25\bar{1}43\})$

This scheme gives the sequence 1, 2, 6, 23, 104, 532, 3004, 18426, 121393, 851810, 6325151, 49448313, 405298482, 3470885747, 30965656442 for $S_n(\{25\bar{1}43\})$ with $1 \leq n \leq 15$.

Next, the equivalence class represented by the pattern $25\bar{1}34$ has the scheme in Figure 15.

This differs from the scheme for $S_n(\{25\bar{1}43\})$ only by the gap vector associated to 132, and yields the same sequence.

The equivalence class with representative $43\bar{5}21$ has the scheme in Figure 16.

This is also symmetric to the previous schemes and yields the same sequence.

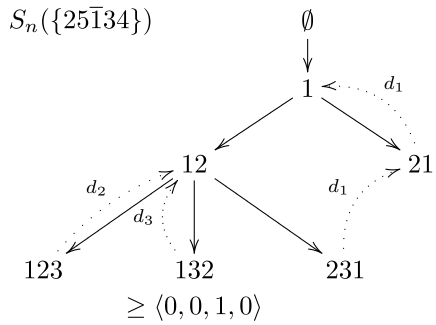


FIGURE 15. The scheme for $S_n(\{25\bar{1}34\})$

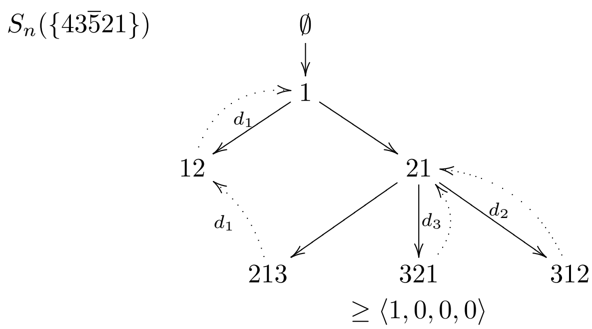


FIGURE 16. The scheme for $S_n(\{43\bar{5}21\})$

Finally, the equivalence class with representative $43\bar{5}12$ has the scheme in Figure 17.

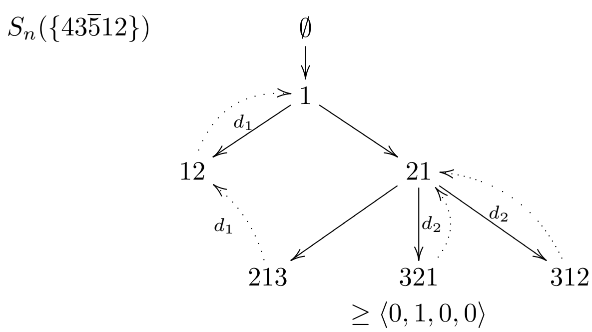


FIGURE 17. The scheme for $S_n(\{43\bar{5}12\})$

Again, since the only difference from the scheme for $S_n(\{43\bar{5}21\})$ is the gap vector associated with prefix 321, so we get the same sequence yet again.

Now, we consider schemes for the 4 patterns of length 5 with two bars that yield new sequences.

The pattern $5\overline{1}\overline{2}43$ has the scheme in Figure 18. This yields the new sequence 1, 2, 5, 14, 43, 143, 511, 1950, 7903, 33848, 152529, 720466, 3555715, 18285538, 97752779 for $S_n(\{5\overline{1}\overline{2}43\})$, $1 \leq n \leq 15$.

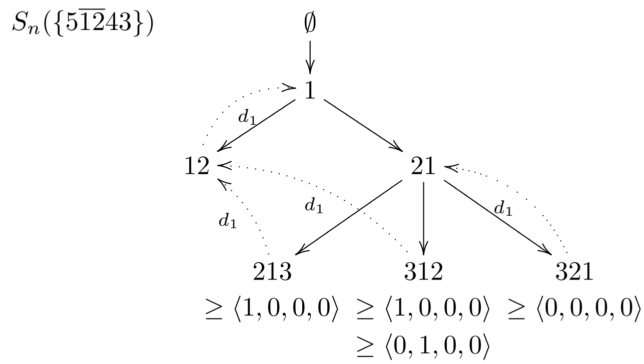


FIGURE 18. The scheme for $S_n(\{5\overline{1}\overline{2}43\})$

The equivalence class with representative $31\overline{5}\overline{4}2$ has the scheme in Figure 19, which yields the new sequence 1, 1, 2, 5, 14, 43, 144, 522, 2030, 8398, 36714, 168793, 813112, 4091735, 21451972, 116891160 for $S_n(\{31\overline{5}\overline{4}2\})$, $1 \leq n \leq 15$.

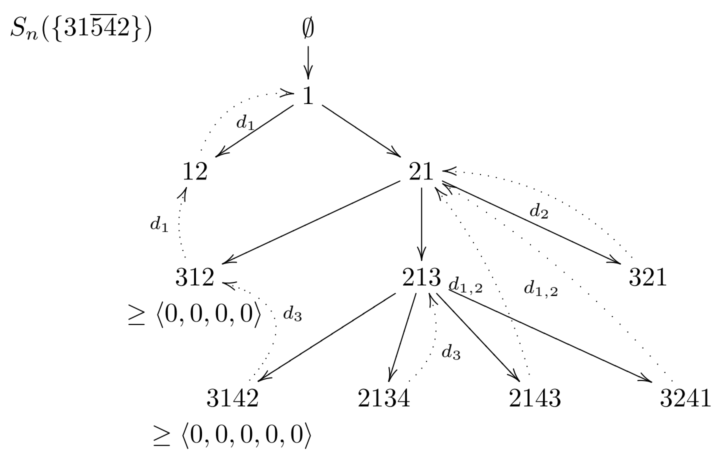


FIGURE 19. The scheme for $S_n(\{31\overline{5}\overline{4}2\})$

The equivalence class with representative $\overline{5}\overline{4}231$ has the scheme in Figure 20, which gives the new sequence 1, 2, 5, 14, 43, 146, 561, 2518, 13563, 88354, 686137, 6191526, 63330147, 720314930, 8985750097 for $S_n(\{\overline{5}\overline{4}231\})$, $1 \leq n \leq 15$.

Finally, the pattern $\overline{5}\overline{4}132$ has the scheme in Figure 21, which gives the new sequence 1, 1, 2, 5, 14, 43, 147, 575, 2648, 14617, 96696, 754585, 6794015, 69116493, 781266266, 9688636317 for $S_n(\{\overline{5}\overline{4}132\})$, $1 \leq n \leq 15$.

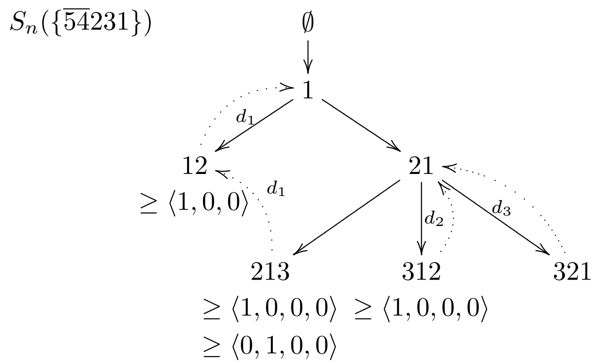


Figure 20. The scheme for $S_n(\{\overline{54}231\})$

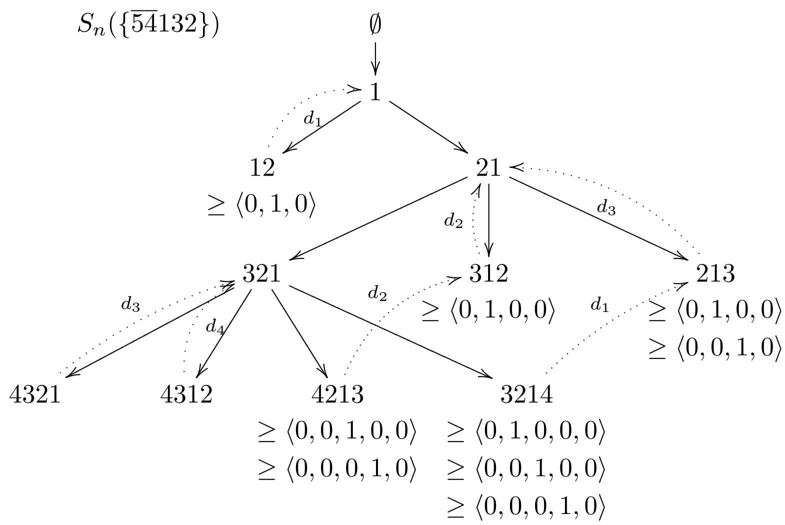


Figure 21. The scheme for $S_n(\{\overline{54}132\})$

In light of the preceding discussion, each of these schemes can be considered as a rigorously proven recurrence counting pattern-avoiding permutations, each sequence completely new to the literature.

6. Summary and Future Work

Now that we have completed our discussion of permutations avoiding barred patterns, and explored the success of prefix schemes for completing this enumeration, we have discovered recurrences for several new sequences.

It still remains to determine if there are "nice" closed forms or generating functions for these sequences, and to find ways to count permutations which avoid barred patterns where enumeration schemes have not yet succeeded.

However, it is important to note that this method of enumeration schemes, already very successful for counting pattern-avoiding permutations and pattern-avoiding words in the standard sense extends nicely to enumerate permutations avoiding barred patterns as well. Moreover, this is the first such method for counting many classes of permutations avoiding barred patterns.

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