Pattern Avoidance in Double Lists

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Abstract

In this paper, we consider pattern avoidance in a subset of words on $\{1, 1, 2, 2, \ldots, n, n\}$ called double lists. We enumerate double lists avoiding any permutation pattern of length at most 4 and completely determine the corresponding Wilf classes.

Keywords: permutation pattern, double list, Wilf class, Lucas number
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1 Introduction

Let $\mathcal{S}_n$ be the set of all permutations on $\{1, 2, \ldots, n\}$. Given $\pi \in \mathcal{S}_n$ and $\rho \in \mathcal{S}_m$ we say that π *contains* ρ as a pattern if there exists $1 \leq i_1 < i_2 <$

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$\cdots < i_m \leq n$ such that $\pi_{i_a} \leq \pi_{i_b}$ if and only if $\rho_a \leq \rho_b$. In this case we say that $\pi_{i_1} \cdots \pi_{i_m}$ is *order-isomorphic* to ρ, and that $\pi_{i_1} \cdots \pi_{i_m}$ is an *occurrence* of ρ in π. If π does not contain ρ, then we say that π *avoids* ρ. An *inversion* is an occurrence of the pattern 21, and a *coinversion* is an occurrence of the pattern 12. Pattern-avoiding permutations have been well-studied with applications to algebraic geometry, theoretical computer science, and more. Of particular interest are the sets $\mathcal{S}_n(\rho) = \{\pi \in \mathcal{S}_n \mid \pi \text{ avoids } \rho\}$. Let $\mathrm{s}_n(\rho) = |\mathcal{S}_n(\rho)|$. It is well known that $\mathrm{s}_n(\rho) = \frac{\binom{2n}{n}}{n+1}$ for $\rho \in \mathcal{S}_3$ [9]. For $\rho \in \mathcal{S}_4$, 3 different sequences are possible for $\{\mathrm{s}_n(\rho)\}_{n \geq 1}$. Two of these sequences are well-understood, but the computation of $\mathrm{s}_n(1324)$ remains open for $n \geq 37$ [5].

Pattern avoidance has been studied for a number of combinatorial objects other than permutations. The definition above extends naturally for patterns in words (i.e. permutations of multisets) and there have been several algorithmic approaches to determining the number of words avoiding various patterns [2, 3, 8, 10].

In another direction, a permutation may be viewed as a bijection on $[n] = \{1, \ldots, n\}$. When we graph the points (i, π_i) in the Cartesian plane, all points lie in the square $[0, n+1] \times [0, n+1]$, and thus we may apply various symmetries of the square to obtain involutions on the set $\mathcal{S}_n$. For $\pi \in \mathcal{S}_n$, let $\pi^r = \pi_n \cdots \pi_1$ and let $\pi^c = (n+1-\pi_1) \cdots (n+1-\pi_n)$, the reverse and complement of π respectively. For example, the graphs of $\pi = 1342$, $\pi^r = 2431$, and $\pi^c = 4213$ are shown in Figure 1. Pattern-avoidance in centrosymmetric permutations, i.e. permutations π such that $\pi^{rc} = \pi$ has been studied by Egge [6] and by Barnabei, Bonetti and Silimbani [1]. Ferrari [7] generalized this idea to pattern avoidance in centrosymmetric words. In all of these cases, knowing the first half of the word or permutation uniquely determines the second half.

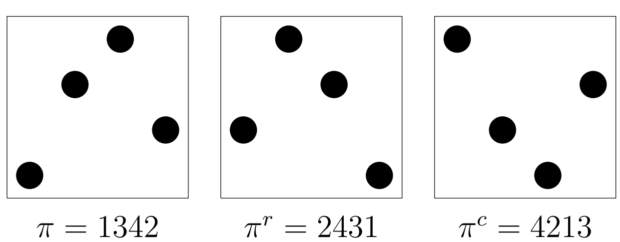

Figure 1: The graphs of $\pi = 1342$, $\pi^r = 2431$, and $\pi^c = 4213$

A final variation involves circular permutations. In a circular permutation $\pi_1 \cdots \pi_n$, we consider the last digit in the permutation to be adjacent to the first and two permutations are considered the same if they differ by only a rotation. For example, 1234, 2341, 3412, and 4123 are all the same circular permutation. A circular permutation π is said to contain ρ as a pattern if there exists a rotation of π that contains ρ. Circular permutations avoiding permutation patterns were studied by Callan [4] and Vella [13] who obtained a number of interesting enumeration sequences.

In this paper we consider a specific type of word that borrows ideas from centrosymmetric and circular permutations. In particular, we define the set of *double lists* on n letters to be

$$\mathcal{D}_n = \{\pi\pi \mid \pi \in \mathcal{S}_n\}.$$

In other words, a double list is a permutation of $\{1, \ldots, n\}$ concatenated with itself. We see immediately that $|\mathcal{D}_n| = n!$. As with centrosymmetric objects, knowing the first half of a double list determines the second half. As with circular permutations, we have taken a permutation and appended the end to the beginning. Yet, double lists are a new combinatorial object of interest in their own right. Consider

$$\mathcal{D}_n(\rho) = \{\sigma \in \mathcal{D}_n \mid \sigma \text{ avoids } \rho\},$$

and let $\mathrm{d}_n(\rho) = |\mathcal{D}_n(\rho)|$. We obtain a number of interesting enumeration sequences for $\{\mathrm{d}_n(\rho)\}_{n\geq 1}$ with connections to other combinatorial objects. The goal of this paper is to completely determine $\mathrm{d}_n(\rho)$ for $\rho \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$.

2 Avoiding patterns of length 1, 2, or 3

The main focus of this paper is avoidance of length 4 patterns, but for completeness we first consider shorter patterns. First, notice that the graph of a double list $\sigma \in \mathcal{D}_n$ is a set of points on the rectangle $[0, 2n+1] \times [0, n+1]$. Using the reverse and complement involutions described in Section 1, we see that

$$\sigma \in \mathcal{D}_n(\rho) \iff \sigma^r \in \mathcal{D}_n(\rho^r) \iff \sigma^c \in \mathcal{D}_n(\rho^c).$$

We will partition the set of permutation patterns of length m into equivalence classes where $\rho \sim \tau$ means that $\mathrm{d}_n(\rho) = \mathrm{d}_n(\tau)$ for $n \geq 1$. In this case ρ and τ are said to be *Wilf equivalent*. When this equivalence holds because of one of the symmetries of the rectangle, we say that ρ and τ are *trivially Wilf equivalent*. Using trivial Wilf equivalence we have that $12 \sim 21$, $123 \sim 321$ and $132 \sim 213 \sim 231 \sim 312$, so we need only consider 4 patterns in this section: 1, 12, 123, and 132.

Avoiding a pattern of length 1 or length 2 is trivial. It is straightforward to check that for $n \geq 1$, $\mathrm{d}_n(1) = 0$, and similarly

$$\mathrm{d}_n(12) = \mathrm{d}_n(21) = \begin{cases} 1 & n = 1 \\ 0 & n \geq 2. \end{cases}$$

With pattern-avoiding permutations, avoiding a pattern of length 3 is the first non-trivial enumeration, and for any pattern ρ of length 3, we have that $\mathrm{s}_n(\rho)$ is the nth Catalan number. Double lists are more restrictive, so we obtain simpler sequences for $\mathrm{d}_n(\rho)$. More strikingly, although $\mathrm{s}_n(123) = \mathrm{s}_n(132)$ for $n \geq 1$, we obtain two distinct sequences in this new context.

Proposition 1. $\mathrm{d}_n(123) = \mathrm{d}_n(321) = \begin{cases} n! & n \leq 2 \\ 1 & n \geq 3. \end{cases}$

Proof. For $n \leq 2$, all double lists avoid permutation patterns of length 3. However, for $n \geq 3$, the unique double list avoiding 123 is $n \cdots 1 n \cdots 1$. We verify this directly for the 6 members of $\mathcal{D}_3$, with a copy of 123 underlined in each of the other 5 double lists: $\underline{123}123$, $\underline{1}3\underline{2}1\underline{3}2$, $2\underline{1}3\underline{2}1\underline{3}$, $23\underline{123}1$, $3\underline{123}12$. Now, assume $\mathcal{D}_n(123) = \{n \cdots 1 n \cdots 1\}$ and consider $\mathcal{D}_{n+1}(123)$. Given $\sigma \in \mathcal{D}_{n+1}(123)$, let σ' be the double list obtained by deleting both copies of $n+1$ in σ. Since $\sigma \in \mathcal{D}_{n+1}(123)$, we know $\sigma' \in \mathcal{D}_n(123)$. By assumption, $\sigma' = n \cdots 1 n \cdots 1$. To construct σ, we must only reinsert the two copies of $n+1$ so that σ avoids 123. If $n+1$ is inserted after the initial n, then we have $1n(n+1)$ as a copy of 123 in σ where the 1 is in the first half of σ, and $n(n+1)$ are in the second half of σ. Therefore, $n+1$ must be inserted before the initial n, and $\mathcal{D}_{n+1}(123) = \{(n+1)n \cdots 1(n+1)n \cdots 1\}$. □

Finally, we consider double lists avoiding 132.

Proposition 2. $\mathrm{d}_n(132) = \mathrm{d}_n(213) = \mathrm{d}_n(231) = \mathrm{d}_n(312) = \begin{cases} n! & n \leq 2 \\ 1 & n = 3 \\ 0 & n = 4. \end{cases}$

Proof. For $n \leq 2$, all double lists avoid permutation patterns of length 3. However, for $n = 3$, the unique double list avoiding 132 is 231231. Indeed for the other 5 double lists in $\mathcal{D}_3$: $\underline{1}2\underline{3}1\underline{2}3$, $\underline{132}132$, $2\underline{132}13$, $3\underline{1}2\underline{3}1\underline{2}$, $32\underline{132}1$. Now, consider the 4 ways to insert 4 into 231231: $423\underline{142}31$, $243\underline{1}2\underline{43}1$, $\underline{2}3\underline{4}12\underline{3}41$, $23\underline{142}314$. We see (via the underlined occurrences) that each of these double lists contains a 132 pattern. If there are no 132-avoiding double lists of length n, then there are no 132-avoiding double lists of length $n + 1$, since deleting both occurrences of $n + 1$ in such a list should produce another 132-avoiding double list. □

At this point, we have completely characterized double lists avoiding a single pattern of length 1, 2, or 3. Although we obtained only trivial sequences, the fact that we obtained two distinct Wilf classes for avoiding patterns of length 3 is a noteworthy difference between avoidance in double lists and avoidance in permutations.

3 Avoiding patterns of length 4

The remainder of this paper is concerned with double lists avoiding a single pattern of length 4. Using the symmetries of the rectangle, we can partition the 24 patterns of length 4 into 8 trivial Wilf classes, as shown in Table 1. Notably, the trivial Wilf equivalences are the *only* Wilf equivalences for patterns of length 4. This is in contrast to the case for pattern-avoiding permutations. In that context, we have an additional trivial Wilf equivalence since $\mathrm{s}_n(\rho) = \mathrm{s}_n(\rho^{-1})$ for $n \geq 1$, so $\mathrm{s}_n(1342) = \mathrm{s}_n(1423)$. As it turns out, there are a number of non-trivial Wilf equivalences for pattern-avoiding permutations so that every length 4 pattern is equivalent to one of 1342, 1234, or 1324. For large n, we have that

$$\mathrm{s}_n(1342^{\bullet}) < \mathrm{s}_n(1234^{\dagger}) < \mathrm{s}_n(1324^{\circ}).$$

In Table 1 each pattern is marked according to its Wilf equivalence class for permutations; patterns equivalent to 1342 are marked with $\bullet$, those equivalent to 1234 are marked with $\dagger$, and those equivalent to 1324 are marked with $\circ$. A closer look at the table reveals a couple more subtleties of the pattern-avoiding double lists problem. For permutations, the monotone pattern 1234 is neither the hardest nor the easiest pattern to avoid; for double lists, it is the easiest pattern to avoid. Similarly, one might expect that all patterns equivalent to 1324 may produce smaller sequences than those avoiding 1234, which produce smaller sequences than those avoiding 1324, but this is also not the case. Other than the trivial equivalences of reverse and complement, Wilf equivalence in the context of double lists appears to be a very different phenomenon than equivalence in the context of permutations. We now consider each of these patterns in turn.

Pattern ρ	$\{\mathrm{d}_n(\rho)\}_{n=1}^{10}$
$1342^\bullet \sim 2431^\bullet \sim 3124^\bullet \sim 4213^\bullet$	1, 2, 6, 12, 15, 15, 15, 15, 15, 15
$2143^\dagger \sim 3412^\dagger$	1, 2, 6, 12, 13, 14, 16, 18, 20, 22
$1423^\bullet \sim 2314^\bullet \sim 3241^\bullet \sim 4132^\bullet$	1, 2, 6, 12, 17, 23, 27, 30, 33, 36
$1432^\dagger \sim 2341^\dagger \sim 3214^\dagger \sim 4123^\dagger$	1, 2, 6, 12, 17, 23, 31, 40, 50, 61
$1243^\dagger \sim 2134^\dagger \sim 3421^\dagger \sim 4312^\dagger$	1, 2, 6, 12, 19, 25, 34, 44, 55, 67
$2413^\bullet \sim 3142^\bullet$	1, 2, 6, 12, 18, 29, 47, 76, 123, 199
$1324^\circ \sim 4231^\circ$	1, 2, 6, 12, 21, 38, 69, 126, 232, 427
$1234^\dagger \sim 4321^\dagger$	1, 2, 6, 12, 27, 58, 121, 248, 503, 1014

Table 1: Enumeration of double lists avoiding a pattern of length 4

3.1 The pattern 1342

The pattern 1342 is the hardest permutation of length 4 to avoid, and, from initial data, is the easiest pattern for which to conjecture a general enumeration formula.

Theorem 1. $\mathrm{d}_n(1342) = \begin{cases} n! & n \leq 3 \\ 12 & n = 4 \\ 15 & n \geq 5. \end{cases}$

Proof. For $n \leq 3$, all double lists avoid 1342, and for $n = 4$, a check of the 24 members of $\mathcal{D}_n$ yields exactly 12 that avoid 1342. They are 12431243, 21342134, 23142314, 23412341, 24132413, 24312431, 31243124, 32143214, 32413241, 42314231, 43124312, 43214321.

We now consider $\mathcal{D}_n(1342)$ for $n \geq 5$ and make three key structural observations. Let $\sigma = \pi\pi \in \mathcal{D}_n(1342)$ and let $\sigma' = \pi'\pi' \in \mathcal{D}_{n-2}(1342)$ be the double list obtained by deleting both copies of n and both copies of $n-1$ from σ. Then

1. π' avoids 123.
2. π' contains at most one coinversion.
3. If π' contains a coinversion, then the coinversion is composed of the digits 1 and 2 or the digits 2 and 3.

For the first observation, suppose to the contrary that π' contains 123 and the occurrence of 123 is formed by the digits $\pi'_a < \pi'_b < \pi'_c$. If n (resp. $n-1$) appears before π'_b or after π'_c in π, then $\pi'_a\pi'_c n\pi'_b$ (resp. $\pi'_a\pi'_c(n-1)\pi'_b$) is a copy of 1342 in $\sigma = \pi\pi$. Therefore, n and $n-1$ must both appear between π'_b and π'_c in π. If they are in increasing order, then $\pi'_a(n-1)n\pi'_c$ is a copy of 1342 in π, and thus in σ. If they are in decreasing order, then $\pi'_a(n-1)n\pi'_c$ is a copy of 1342 in σ. Since we have exhausted all possible options, it must be the case that π' avoids 123.

For the second observation, we know that π' avoids 123, so if π' contains two coinversions, either (a) π' contains the pattern 132, (b) π' contains the pattern 213, or (c) π' contains the pattern 3412. It can be shown that case (a) and case (b) are impossible by a similar analysis to the previous paragraph, conditioning on various possible positions of n and $n-1$. Case (c) is even more readily discounted, since $34\underline{1}2\underline{34}1\underline{2}$ already contains a copy of 1342.

Finally, if π' contains a coinversion, we show that it must use two consecutive digits and they must include the digit 2. Suppose on the contrary that we have the coinversion $\pi'_i < \pi'_j$ where $\left|\pi'_j - \pi'_i\right| > 1$. Then no matter the location of π'_i+1, it forms a coinversion with either π'_i or π'_j. This contradicts our previous observation that π' contains at most one coinversion. Therefore, the coinversion must use consecutive digits. Now suppose that the coinversion uses digits π'_i and π'_i+1 where $\pi'_i \geq 3$. To avoid other coinversions, it must be the case that $\pi' = (n-2)(n-3)(n-4)\cdots(\pi'_i+3)(\pi'_i+2)\pi'_i(\pi'_i+1)(\pi'_i-1)(\pi'_i-2)\cdots 21$. However, in this case, $1\pi'_i(\pi'_i+1)2$ is a copy of 1342 in σ. Therefore, any coinversion must either use the digits 1 and 2 or the digits 2 and 3.

Using these three observations, we see that there are only 3 possible forms for π'. They are: $(n-2)\cdots 1$ (the decreasing permutation), $(n-2)\cdots 4231$, or $(n-2)\cdots 312$. Now, we consider ways to reinsert n and $n-1$ into π' to form π so that $\sigma = \pi\pi$ is a member of $\mathcal{D}_n(1342)$. There are 6 ways to insert them into the decreasing permutation; namely,

$$\begin{array}{ccc} n\cdots 1, & (n-1)\cdots 1n, & (n-1)\cdots 2n1, \\ (n-2)\cdots 1n(n-1), & (n-2)\cdots 2n1(n-1), & (n-2)\cdots 2n(n-1)1. \end{array}$$

There are also 6 ways to insert them into $(n-2)\cdots 4231$; namely,

$$\begin{array}{rl} n\cdots 4231, & (n-1)\cdots 4231n, \\ (n-1)\cdots 423n1, & (n-2)\cdots 4231n(n-1), \\ (n-2)\cdots 423n1(n-1), & (n-2)\cdots 423n(n-1)1. \end{array}$$

Finally, there are only 3 ways to insert them into $(n-2)\cdots 312$; namely

$$n\cdots 312, \qquad (n-1)\cdots 312n, \qquad (n-2)\cdots 312n(n-1).$$

These 15 permutations π uniquely describe all possible members $\sigma = \pi\pi \in \mathcal{D}_n(1342)$ for $n \geq 5$. □

To illustrate, the 15 members of $\mathcal{D}_6(1342)$ are shown in Figure 2. While an eventually constant sequence is expected for smaller patterns, the constant sequence 15 is perhaps a bit more surprising in this context. Nonetheless the structural argument in this proof sets the stage for several of the proofs yet to come in the following subsections.

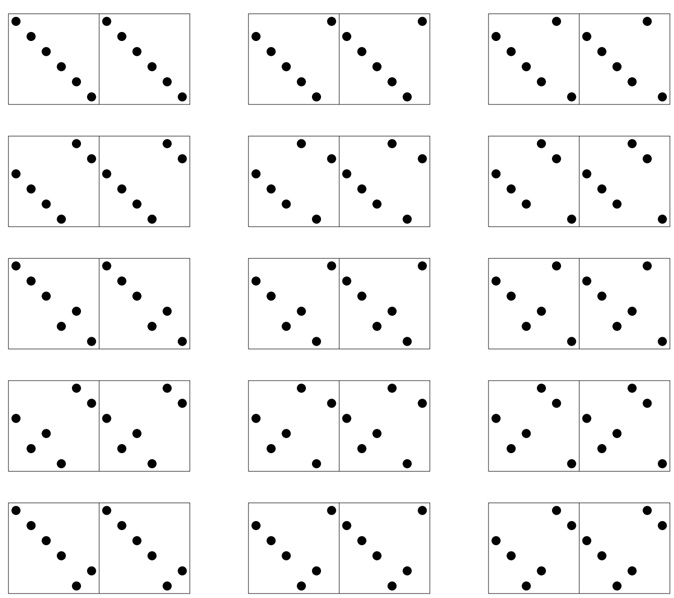

Figure 2: The members of $\mathcal{D}_6(1342)$

3.2 The patterns 2143 and 1423

Two of our patterns yield avoidance sequences that grow linearly.

Theorem 2. $$\mathrm{d}_n(2143) = \begin{cases} n! & n \leq 3 \\ 12 & n = 4 \\ 13 & n = 5 \\ 2(n+1) & n \geq 6. \end{cases}$$

Proof. The cases for $n \leq 5$ are easily verified by brute force methods, so we focus on the case where $n \geq 6$. Intuitively there are an even number of double lists avoiding 2143 for a geometric reason. We have that $2143^{rc} = 2143$, so ρ avoids 2143 if and only if ρ^{rc} avoids 2143. For $n \geq 6$, there are exactly two members $\sigma = \pi\pi$ of $\mathcal{D}_n(2143)$ that are reverse-complement invariant. If n is even, they are $\pi = 12\cdots n$ and $\pi = \frac{n+2}{2}\cdots n1\cdots\frac{n}{2}$; If n is odd, they are $\pi = 12\cdots n$ and $\pi = \frac{n+3}{2}\cdots n\frac{n+1}{2}1\cdots\frac{n-1}{2}$. All other 2143-avoiders come in pairs ρ and ρ^{rc}. However, it turns out that it is easier to characterize the members of $\mathcal{D}_n(2143)$ using other distinguishing features.

Notice that there are no inversions among elements after 1 and larger than 2 in π. Suppose to the contrary that $i < j < k$ where $\pi_i = 1$ and $\pi_j > \pi_k > 2$. Then $21\pi_j\pi_k$ forms an occurrence of 2143 in σ. Similarly, all elements before n and other than $n-1$ must appear in increasing order. Therefore, there are only 3 possible double lists $\sigma = \pi\pi$ where 1 precedes n: $\pi = 12\cdots n$, $\pi = 13\cdots n2$, and $\pi = (n-1)12\cdots(n-2)n$. So far, we have described 3 members of $\mathcal{D}_n(2143)$, as shown in Figure 3. It remains to consider when n precedes 1 in π.

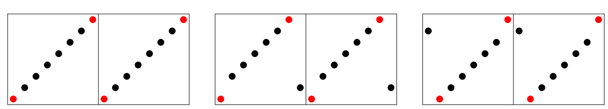

Figure 3: 2143-avoiding lists where 1 precedes n

If n precedes 1, then there is at most 1 element between n and 1. Suppose to the contrary that there are two elements $\pi_a > \pi_b$ that appear between n and 1 in π. Then $\pi_b 1 n \pi_a$ forms a 2143 pattern in σ, taking $\pi_b 1$ from the first copy of π and $n\pi_a$ from the second copy. We have two subcases: either $\pi_{j-1} = n$ and $\pi_{j+1} = 1$ or $\pi_{j-1} = n$ and $\pi_j = 1$.

In the case where $\pi_{j-1} = n$ and $\pi_{j+1} = 1$, let $i = \pi_j$. Consider elements π_a and π_b, such that $a < j-1$ and $b > j+1$. It must be the case that $\pi_a > \pi_j > \pi_b$; otherwise a case analysis shows that σ contains a 2143 pattern. Next, an inversion $\pi_a > \pi_b$ after π_{j+1} creates the 2143 occurrence $\pi_a\pi_b ni$ in σ, while an inversion $\pi_a > \pi_b$ before π_{j-1} creates the 2143 occurrence $i1\pi_a\pi_b$ in σ. Therefore, the only 2143-avoiders in this case are the $n-2$ lists where $\pi = (i+1)\cdots ni1\cdots(i-1)$ $(2 \le i \le n-1)$, as shown in Figure 4.

On the other hand, if $\pi_{j-1} = n$ and $\pi_j = 1$, if there is an inversion in $\pi_1 \cdots \pi_{j-2}$ or in $\pi_{j+1}\cdots\pi_n$, there is a 2143 pattern with two exceptions. The double lists where $\pi = 4\cdots n132$ or $\pi = (n-1)(n-2)n1\cdots(n-3)$ are 2143-avoiding. In addition, we obtain $n-1$ lists where $\pi = i\cdots n1\cdots i-1$ $(2 \le i \le n)$. There are $2 + (n-1) = n+1$ members of $\mathcal{D}_n(2143)$ where n immediately precedes 1, as shown in Figure 5.

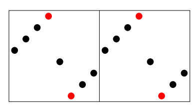

Figure 4: 2143-avoiders where n is two positions before 1

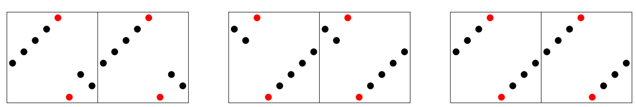

Figure 5: 2143-avoiders where n immediately precedes 1

We have now accounted for $(n-2)+(n+1) = 2n-1$ additional permutations π such that $\pi\pi \in \mathcal{D}_n(2143)$. Together with the original 3 lists we have $2n-1+3 = 2(n+1)$ double lists avoiding 2143. □

The number of 1423-avoiding double lists also grows linearly but for a different reason.

Theorem 3. $$\mathrm{d}_n(1423) = \begin{cases} n! & n \le 3 \\ 12 & n = 4 \\ 17 & n = 5 \\ 23 & n = 6 \\ 3(n+2) & n \ge 7. \end{cases}$$

Proof. Again, the cases for $n \leq 6$ are easily verified by brute force methods, so we focus on the case where $n \geq 7$. Now, we condition on which of the letters 1 and n comes first in $\sigma = \pi\pi \in \mathcal{D}_n(1423)$.

If 1 precedes n, then all other digits must appear in decreasing order in π; otherwise $1n$ in the first copy of π and any increasing pair in the second copy of π form a 1423 pattern in $\sigma = \pi\pi$. Further, n must be the last element of π. Since all other digits appear in decreasing order, if n is not the last digit of π, then $\pi_n = 2$, and $1n23$ is a 1423 pattern in σ. Since n is last, then either $\pi_{n-1} = 1$, $\pi_{n-2} = 1$, or $\pi_{n-3} = 1$. Otherwise $\pi_{n-3} > \pi_{n-2} > \pi_{n-1}$ and $1\pi_{n-3}\pi_{n-1}\pi_{n-2}$ is a copy of 1423 in σ, taking the first three digits from the first copy of π and the remaining digit from the second copy. There are exactly 3 double lists in $\mathcal{D}_n(1423)$ where 1 precedes n; namely, $\pi = (n-1)\cdots 4132n$, $\pi = (n-1)\cdots 312n$ or $\pi = (n-1)\cdots 1n$.

Now, suppose that n precedes 1. We quickly see that the digits after 1 in π must appear in decreasing order; otherwise, 1 from the first copy of π and n and the increasing pair from the second copy form a 1423 pattern. This implies there are at most 2 digits after 1 in π; otherwise we can form a 1423 pattern using $1\pi_{n-2}\pi_n$ from the first copy of π and π_{n-1} from the second copy of π. Similarly, all digits after n and larger than 1 in π must appear in decreasing order.

What can be said about the digits that appear before n? Two things: (a) either the only digit before n is $n-2$, or all digits before n are larger than all digits after n, and (b) if there are at least four digits before n, then they appear in decreasing order. For observation (a), if $\pi_1 = i$ and $\pi_2 = n$ where $i < n-2$ then $in(n-2)(n-1)$ forms a 1423 pattern in σ where the first three digits come from the first copy of π. Further, if there is more than one digit before n in π, let the first two digits of π be a and b where $a < b$. By assumption there exists a digit c that appears after n in π where $a < c$. We have that either $anbc$ or $ancb$ is a 1423 pattern in σ where in the first case, an comes from the first copy of π and in the second case, anc comes from the first copy of π. Therefore, observation (a) holds. A similar analysis supports observation (b). If there are two digits before n in π, they may appear in either order, and if there are 3 digits before n they may form either a 132 pattern or a 321 pattern as all other patterns lead to a 1423 pattern in σ.

Here, then, is the final enumeration. We have seen 3 double lists where $\pi_n = n$. We have also seen that if n precedes 1, we may choose the position of n, the arrangement of the digits before n, and the position of 1 (one of the last 3 digits), and then the rest of the double list is decreasing. Therefore, there are 3 double lists beginning with n, 3 beginning with $(n-1)n$, 3 beginning with $(n-2)n$, 3 beginning with $(n-2)(n-1)n$, 3 beginning with $(n-1)(n-2)n$, 3 beginning with $(n-3)(n-1)(n-2)n$, 3 beginning with $(n-1)(n-2)(n-3)n$, and 3 where $\pi_i = n$ for $5 \le i \le n-3$. Finally there are 2 lists where $\pi_{n-2} = n$ (since there are only two positions to place 1 following n), and 1 list where $\pi_{n-1} = n$. Adding these together, we have $3 \cdot 8 + 3 \cdot (n-7) + 3 = 3(n+2)$ double lists avoiding 1423. □

3.3 The patterns 1432 and 1243

The avoidance sequences for two patterns grow quadratically.

Theorem 4. $$\mathrm{d}_n(1432) = \begin{cases} n! & n \le 3 \\ 12 & n = 4 \\ 17 & n = 5 \\ \frac{n^2}{2} + \frac{3n}{2} - 4 & n \ge 6. \end{cases}$$

Proof. Again, the base cases are easily checked by brute force techniques, so we focus on the case where $n \ge 7$.

First, consider $\sigma' = \pi'\pi' \in \mathcal{D}_{n-1}(1432)$. Notice that all digits after $n-1$ in π' and larger than 1 must appear in increasing order, otherwise the 1 from the first copy of π' followed by $n-1$ and a decreasing pair from the second copy of π' form a 1432 pattern.

Now, we claim that if $\sigma' = \pi'\pi' \in \mathcal{D}_{n-1}(1432)$, then inserting n immediately after $n-1$ produces a member $\sigma = \pi\pi$ of $\mathcal{D}_n(1432)$. Suppose to the contrary that inserting n immediately after $n-1$ creates a 1432 pattern. Then n must play the role of '4' in this new occurrence. If $n-1$ does not play the role of '3', then using $n-1$ instead of n would have been a 1432 pattern in σ'. Therefore, the new forbidden pattern must involve the n from the first copy of π and the $n-1$ from the second copy of π with two numbers a and b playing the roles of '1' and '2' respectively. Next, if $b < n-2$, we know that one copy of $n-2$ must occur somewhere between the two copies of $n-1$ in σ', so $a(n-1)(n-2)b$ would have been a forbidden pattern in σ'. Thus, $b = n-2$. If $a < n-3$, then one copy of $n-3$ must appear somewhere between the two copies of $n-2$ in σ', so $a(n-1)(n-2)(n-3)$ would have been a forbidden pattern in σ'. Thus, $a = n-3$. We now know that in π', the largest 4 digits appear in the order $(n-3)(n-1)n(n-2)$. We also assume that $n \ge 7$, so there are at least 3 smaller digits in π'. If any of these smaller digits d appears before $n-1$ in π', then $d(n-1)(n-2)(n-3)$ would have been a forbidden pattern in σ', so it must be the case that all digits smaller than $n-3$ appear after $n-1$ in π'. From the previous paragraph, we know that the digits $2, 3, \ldots n-4$ must appear in increasing order before $n-2$. Now, $1(n-2)$ from the first copy of π', followed by $(n-3)(n-4)$ from the second copy of π' form a forbidden pattern in σ'. In every case, we have shown that if σ' avoids 1432, then insertion of n immediately after $n-1$ results in σ avoiding 1432 as well.

Further, there is at most 1 digit after $n-1$ in π'. Suppose to the contrary that both digits b and c (with $b < c$) appear after $n-1$ in π'. Then $a(n-1)cb$ is a 1432 pattern in σ' where the first three digits come from the first copy of π'. Also, since we assume that $n \geq 7$, so there are at least 2 digits that appear before $n-1$ in π'. Pick one such digit d where $d \neq a$. If $d < a$, $d(n-1)ba$ is a forbidden pattern. If $d > a$, then $a(n-1)bd$ or $a(n-1)db$ is a forbidden pattern. In any case, we have shown that σ contains a forbidden pattern not including n, so $\sigma' \notin \mathcal{D}_{n-1}(1432)$, which is a contradiction.

Now, we must account for members $\sigma = \pi\pi$ of $\mathcal{D}_n(1432)$ where n does not immediately follow $n-1$ in π. We consider two cases: n follows $n-1$ and n precedes $n-1$.

If n follows $n-1$, but not immediately, there can be at most one digit between them. Otherwise, let $a < b$ be two digits between them in π. $an(n-1)b$ forms a 1432 pattern in σ. Further that one digit must smaller than all digits before $n-1$ and larger than all digits after n. Otherwise, suppose $a < b$ or $b < c$ where a is before $n-1$ b is between $n-1$ and n and c is after n. if $a < b$, then $an(n-1)b$ forms a forbidden pattern. If $b < c$, then $bn(n-1)c$ forms a forbidden pattern. Next, all digits before $n-1$ in π must appear in increasing order, otherwise bn from the first copy of π followed by the descent is a forbidden pattern. Finally, the only digit that can appear after n is 1. We already have seen that all digits after $n-1$ and smaller than $n-1$ and larger than 1 must appear in increasing order. A digit cannot be smaller than b and in increasing order with b at the same time. The only two lists of this form are when $\pi = 2 \cdots (n-1)1n$ or $\pi = 3 \cdots (n-1)2n1$.

If n precedes $n-1$, we have a different situation. We know everything after n and larger than 1 appears in increasing order, otherwise 1 from the first copy of π followed by n and the decreasing pair form a 1432 pattern. Finally we show that in this case, n must be the first digit of π. Suppose n is preceded by two digits $a < b$. Then $an(n-1)b$ is a forbidden pattern in σ where $an(n-1)$ come from the first copy of π and b comes from the second copy. Therefore, n must be the first or second digit in π. Suppose n is preceded by a digit a. If $a < n-2$ then $an(n-1)(n-2)$ is a forbidden pattern in σ. If $a = n-2$, recall all digits after n other than 1 must be in increasing order and $n \geq 6$ so $(n-4)(n-1)(n-2)(n-3)$ is a forbidden pattern. Thus if n precedes $n-1$, n is the first digit of π, and after choosing the position of 1 the rest of π is uniquely determined. There are $n-1$ choices for the position of 1, so we get $n-1$ double lists in this case.

In summary, we have shown that $\mathrm{d}_n(1432) = \mathrm{d}_{n-1}(1432) + 2 + (n-1) = \mathrm{d}_{n-1}(1432) + n + 1$, and after matching with the fact that $\mathrm{d}_6(1432) = 23$, we have the quadratic formula above. □

Theorem 5. $\mathrm{d}_n(1243) = \begin{cases} n! & n \leq 3 \\ 12 & n = 4 \\ 19 & n = 5 \\ \frac{n^2}{2} + \frac{5n}{2} - 8 & n \geq 6. \end{cases}$

Proof. Again, the base cases are easily checked by brute force techniques, so we focus on the case where $n \geq 7$.

We claim that if $\sigma' = \pi'\pi' \in \mathcal{D}_{n-1}(1243)$, then appending 1 to the end of π' and increasing all other digits by 1 produces a member $\sigma = \pi\pi$ of $\mathcal{D}_n(1243)$. Suppose to the contrary that σ contains a 1243 pattern but σ' does not. Then the 1 at the end of the first copy of π must play the role of '1' and π' contains a 132 pattern. Further, the digit 2 in the second copy of π must play the role of '1' in this 132 pattern otherwise taking 2 from the first copy of π followed by the 132 pattern in the second copy of π implies there is a 1243 pattern in σ'. Therefore the 1243 pattern in σ uses 1 from the first copy of π, 2 from the second copy of π and digits a and b playing the roles of '4' and '3' respectively.

Further, there are at most 2 digits between the 2 and the 1 in π. If the digits between 2 and 1 contain a 132 occurrence then 2 followed by this occurrence are a forbidden 1243 occurrence. We know that the only double list of length 3 or more that avoids 132 is 231231. If the digits between 2 and 1 contain the pattern 231231, then a sublist of σ is 2453124531 which contains the 1243 occurrence 1253. Now, since $n \geq 7$, there are at least three digits appearing before 2. If at least one of them, c, is less than b, then $2cab$ is a forbidden pattern in σ. If at least one of them d is greater then a, then $2bda$ is a forbidden pattern. If all three digits are greater than b and less than a and there is an decreasing pair $e > f$, then $2bef$ is a forbidden pattern, so we may assume that the three digits before 2 appear in increasing order with $e < f < g$ and are all between a and b in value. However, in this case $efag$ is a forbidden pattern. In all cases we have found a copy of 1243 in σ', so it must be the case that inserting a 1 at the end of π' and incrementing all other digits produces another 1243-avoiding double list.

Now, we consider members $\sigma = \pi\pi$ of $\mathcal{D}_n(1243)$ that do not end in 1. Notice that 1 must be one of the last three digits of π. If there were three digits after 1 with $a < b < c$, then in order for the digits $1, a, b, c$ to avoid 1243, we must have $1bca1bca$. Now consider d and e as digits before 1. If $d < a$ then $1dba$ is a forbidden pattern. If $d > b$ then $1adb$ is a forbidden pattern so we may assume that d and e are both between a and b in value. if $d > e$ appear in decreasing order, then $1ade$ is a forbidden pattern. If $d > e$ appear in increasing order, then $edcb$ is a forbidden pattern. Thus, it must be the case that there are at most two digits after 1.

Suppose then that 1 is followed by 2 digits in π. Let $a < b$ be those two digits. If $b < n$, then $1anb$ forms a forbidden pattern, so $b = n$. Further, we know that all digits larger than a must appear in increasing order in π lest we create a 1243 pattern using 1 and a as '1' and '2'. Thus, the last 3 digits of π are $1ab = 1an$. If there are at least four digits $c < d < e < f$ larger than a, then $cdfe$ is a 1243 pattern in σ. So, it must be the case that $a \geq n - 3$. If $a = n-3$ or $a = n-2$, then $1an(n-1)$ is a forbidden pattern, so the only option is to end in $1(n-1)n$. The digits before 1 must appear in decreasing order, otherwise, the increasing pair followed by $n(n-1)$ is a forbidden pattern. In this case, we get one double list where $\pi = (n-2)\cdots 1(n-1)n$.

Suppose that 1 is followed by exactly one digit in π. If π ends in $1i$ where $i \leq n-4$, then all numbers larger than i must be in increasing order in π and $(n-3)(n-2)n(n-1)$ is a forbidden pattern in σ. If 1 is followed by n, then we have $n-2$ choices for the location of $(n-1)$ and the rest of the digits must appear in decreasing order, lest we have a 1243 pattern. If 1 is followed by i where $(n-2) \leq i \leq (n-1)$, then n appears in position $n-i$ and the rest of the digits are decreasing. If 1 is followed by $(n-3)$, we have $\pi = (n-2)(n-1)n(n-4)\cdots 1(n-3)$. There are $1 + (n-2) + 3 = n+2$ possible double lists that do not end in 1.

In summary, $d_n(1243) = d_{n-1}(1243) + n + 2$, and putting this together with the base cases above, we achieve the desired enumeration. □

3.4 The patterns 1234, 2413, and 1324

The results of the previous sections make a stark contrast with pattern-avoiding permutations where most avoidance sequences grow exponentially. However, pattern avoidance in double lists is more restrictive, so it should not be surprising that we achieve such a variety of behaviors. We conclude by examining the three final patterns of length 4, each of whose avoidance sequences exhibits exponential growth.

We begin with the monotone pattern. In the context of permutations, 1234 is neither the hardest nor the easiest pattern to avoid, but for double lists it turns out that it is the easiest to avoid.

Theorem 6. $\mathrm{d}_n(1234) = \begin{cases} n! & n \leq 3 \\ 12 & n = 4 \\ 2^n - n & n \geq 5. \end{cases}$

Proof. If $\sigma = \pi\pi \in \mathcal{D}_n(1234)$ where $n \geq 5$, the digits of π may be partitioned into two subsequences: for some i where $0 \leq i \leq n$, the largest i digits appear in decreasing order in π, the smallest $n - i$ digits appear in decreasing order in π, and these two subsequences may be interleaved in any way. In either case, the permutation π may be encoded by a list of ℓs and ss for whether a digit belongs to the decreasing subsequence of larger digits or the decreasing subsequence of smaller digits. There are 2^n such encodings of a sequence of n ℓs and ss; however $n + 1$ of them (those of the form $\ell^i s^{n-i}$) encode the decreasing permutation, so we have overcounted by n. There are $2^n - n$ double lists avoiding the pattern 1234. □

The remaining two patterns also produce nice sequences that are characterized by linear recurrences with constant coefficients. Double lists avoiding 2413 are counted by the Lucas numbers L_n, where $L_0 = 2$, $L_1 = 1$, and for $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$.

Theorem 7. $\mathrm{d}_n(2413) = \begin{cases} n! & n \leq 3 \\ 12 & n = 4 \\ L_{n+1} & n \geq 5. \end{cases}$

Proof. As usual, it is straightforward to confirm the theorem via brute force techniques for specific small n. We show that $\mathrm{d}_n(2413) = \mathrm{d}_{n-1}(2413) + \mathrm{d}_{n-2}(2413)$ for $n \geq 7$.

We actually prove a more specific result. Let

$$\mathcal{D}_n^i = \{\sigma \in \mathcal{D}_n(2413) \mid \sigma_1 = i\}$$

and $\mathrm{d}_n^i(2413) = |\mathcal{D}_n^i(2413)|$. It turns out that $\mathrm{d}_n^i(2413) = 0$ if $i \notin \{1, n-2, n-1, n\}$, and for $n \geq 7$,

$$d_n^1(2413) = \mathrm{d}_{n-1}^1(2413) + \mathrm{d}_{n-2}^1(2413),$$

$$d_n^{n-2}(2413) = \mathrm{d}_{n-1}^{n-2}(2413) + \mathrm{d}_{n-2}^{n-2}(2413),$$
$$d_n^{n-1}(2413) = \mathrm{d}_{n-1}^{n-1}(2413) + \mathrm{d}_{n-2}^{n-1}(2413),$$
$$d_n^{n}(2413) = \mathrm{d}_{n-1}^{n}(2413) + \mathrm{d}_{n-2}^{n}(2413).$$

First, consider $\sigma = \pi\pi \in \mathcal{D}_n^i(2413)$ for $i \notin \{1, n-2, n-1, n\}$. If $n-2$ precedes n in π then $(n-2)ni(n-1)$ forms a forbidden pattern in σ where the first two digits come from the first copy of π and the last two digits come from the second copy. Therefore, $n-2$ comes after n. Now, $in1(n-2)$ forms a forbidden pattern where in come from the first copy of π, 1 comes from somewhere between the two copies of n and $n-2$ comes from the second copy of π. In every event, it is impossible to avoid 2413, so $\mathrm{d}_n^i(2413) = 0$ for $i \notin \{1, n-2, n-1, n\}$.

Next, consider $\sigma = \pi\pi \in \mathcal{D}_n^1(2413)$. Any coinversion in π that does not include the digit 1 must consist of a pair of consecutive digits and therefore must appear in consecutive positions. Suppose to the contrary there is a coinversion with $a < b$ such that $b \neq a+1$. Then $ab1(a+1)$ forms a forbidden pattern where the first two digits come from the first copy of π. If $a(a+1)$ is a coinversion in nonconsecutive positions we have the subsequence $ab(a+1)$ in π. If $b < a$ then $b(a+1)$ is another coinversion with nonconsecutive digits, which is not allowed. If $b > a+1$ then ab is another coinversion with nonconsecutive digits, which is still not allowed. We may only preserve these properties of coinversions by inserting $(n-1)n$ after 1 in any member of $\mathcal{D}_{n-2}^1(2413)$ or inserting n after 1 in any member of $\mathcal{D}_{n-1}^1(2413)$ to obtain σ.

Next, consider $\sigma = \pi\pi \in \mathcal{D}_n^{n-2}(2413)$. If $\pi_1 = n-2$, we claim that $\pi_2 = n-1$ and $\pi_n = n$. Suppose to the contrary that n precedes $n-1$. Then $(n-2)n1(n-1)$ is a forbidden pattern in σ. Now suppose that $\pi_2 = i < n-2$. Then $(n-2)ni(n-1)$ is a forbidden pattern so $\pi_2 = n-1$. Finally, suppose $\pi_n = i < n-2$. Then $(n-2)ni(n-1)$ is a forbidden pattern, so we know that $\pi_1 = n-2$, $\pi_2 = n-1$, and $\pi_n = n$. Now, the digits $n-2$, $n-1$, and

n can only play the role of '4' in a 2413 pattern so any coinversions amongst the digits $\{1, \ldots, n-3\}$ in π must appear between consecutive digits in consecutive positions as in the previous case. Given a member of $\mathcal{D}_{n-2}^{n-4}(2413)$, we may increment $n-4$, $n-3$, and $n-2$ by 2 and insert $(n-4)(n-3)$ in the third and fourth positions to obtain a member of $\mathcal{D}_n^{n-2}(2413)$. For example, $34215 \in \mathcal{D}_5^3(2413)$ produces $5634217 \in \mathcal{D}_7^5(2413)$. Given a member of $\mathcal{D}_{n-1}^{n-3}(2413)$, we may increment $n-3$, $n-2$, and $n-1$ by 1 and insert $n-3$ in the third position to obtain a member of $\mathcal{D}_n^{n-2}(2413)$. For example, $452316 \in \mathcal{D}_6^4(2413)$ produces $5642317 \in \mathcal{D}_7^5(2413)$.

Next, consider $\sigma = \pi\pi \in \mathcal{D}_n^{n-1}(2413)$. Then either $\pi_2 = n$ or $\pi_n = n$. Suppose to the contrary that $\pi_i = n$ where $3 \leq i \leq n-1$. First, all digits between $n-1$ and n in π must be smaller than all digits after n in π; otherwise we have a 2413 pattern in σ. Since we assume $n \geq 7$, either there are at least 2 digits between $n-1$ and n in π or there are at least 2 digits after n in π. In the first case, suppose the digits between $n-1$ and n include $a < b$ and c is a digit after n in π. Then $bnac$ is a 2413 pattern in σ. If the digits after n in π include $a < b$ and c is a digit between $n-1$ and n then $a(n-1)cb$ is a forbidden pattern in σ. Therefore n is either the second or the last digit in π. In the first case, given $\sigma = \pi\pi \in \mathcal{D}_{n-2}^{n-3}(2413)$ where $\pi_2 = n-2$, we may prepend $(n-1)n$ to the front of π to obtain a 2413-avoiding member of $\mathcal{D}_n^{n-1}(2413)$. If $\sigma = \pi\pi \in \mathcal{D}_{n-1}^{n-2}(2413)$ where $\pi_2 = n-1$, then increment π_1 and π_2 and insert $(n-2)$ into the third position. For example, $563412 \in \mathcal{D}_6^5(2413)$ becomes $6753412 \in \mathcal{D}_7^6(2413)$. Now, if $\pi_n = n$ we approach the situation differently. If $\sigma' = \pi'\pi' \in \mathcal{D}_{n-2}^{n-3}(2413)$ with $\pi'_{n-2} = n-2$, then remove π'_1 and π'_{n-2} to obtain a permutation on $\{1, \ldots, n-4\}$ then create the new permutation $\pi = (n-1)(n-3)(n-2)\pi'_2 \cdots \pi'_{n-3}n$. By inspection, $\pi\pi \in \mathcal{D}_n^{n-1}(2413)$. If $\sigma' = \pi'\pi' \in \mathcal{D}_{n-1}^{n-2}(2413)$ with $\pi'_{n-1} = n-1$, then remove π'_1 and π'_{n-1} to obtain a permutation on $\{1, \ldots, n-3\}$ then create the new permutation $\pi = (n-1)(n-2)\pi'_2 \cdots \pi'_{n-2}n$, where again, by inspection, $\pi\pi \in \mathcal{D}_n^{n-1}(2413)$.

Finally, consider $\sigma = \pi\pi \in \mathcal{D}_n^n(2413)$. Given $\sigma' = \pi'\pi' \in \mathcal{D}_{n-2}^{n-2}(2413)$, delete π'_1 and create $n(n-2)(n-1)\pi'_2 \cdots \pi'_{n-2}n(n-2)(n-1)\pi'_2 \cdots \pi'_{n-2} \in \mathcal{D}_n^n(2413)$. If $\sigma' = \pi'\pi' \in \mathcal{D}_{n-1}^{n-1}(2413)$, prepend n to the front of π' to obtain a member σ of $\mathcal{D}_n^n(2413)$. □

The final sequence is perhaps the most surprising result. The task of enumerating 1324-avoiders in other contexts has proven especially challenging. For double lists, however, structure is evident beginning with the $n = 7$ term. It turns out these double lists satisfy a tribonacci recurrence.

Theorem 8. $d_n(1324) = \begin{cases} n! & n \leq 3 \\ 12 & n = 4 \\ 21 & n = 5 \\ 38 & n = 6 \\ 69 & n = 7 \\ 126 & n = 8 \\ 232 & n = 9 \\ \mathrm{d}_{n-1}(1324) + \mathrm{d}_{n-2}(1324) + \mathrm{d}_{n-3}(1324) & n \geq 10. \end{cases}$

Proof. As before, we focus on the $n \geq 10$ case, and leave the $n \leq 9$ cases to brute force verification.

First, given $\sigma = \pi\pi \in \mathcal{D}_n(1324)$ it is impossible for 1 to precede n if $n \geq 7$. Suppose to the contrary that 1 precedes n. All digits in $\{2, \ldots, n-1\}$ appear between the first 1 and the last n and must appear in increasing order to avoid 1324. Suppose two digits $a < b$ appear between 1 and n in π. Then $1ban$ is a 1324 pattern in σ. If there is just one digit i between 1 and n in π, then if $i > 2$, $1i2n$ is a forbidden pattern, and if $i = 2$, then $132n$ is a forbidden pattern. Therefore if 1 appears before n, 1 is immediately before n and the digits $\{2, \ldots, n-1\}$ appear in increasing order between the first occurrence of $1n$ and the second occurrence of $1n$ in σ. Since $n \geq 7$, there are either 3 digits $a < b < c$ before the first 1 (in which case $acbn$ is a forbidden pattern) or there are 3 digits $a < b < c$ in π after the first n (in which case $1bac$ is a forbidden pattern). In every event we have forced the occurrence of a 1324 pattern so it is impossible for 1 to precede n if $n \geq 7$.

Now, if n precedes 1, n must appear as one of the first 3 digits of π. Suppose that n appears in position $i \geq 4$. Then $\pi_1 \cdots \pi_{i-1}\pi_1 \cdots \pi_{i-1}$ must avoid 132. We have seen that this is impossible for $i - 1 \geq 4$, and the only way to do this if $i - 1 = 3$ is for $\pi_1\pi_2\pi_3$ to form a 231 pattern. However $\pi_3 < \pi_1 < \pi_2$ implies that $\pi_1\pi_2\pi_3 n 1 \pi_1\pi_2\pi_3 n 1$ contains the 1324 pattern $1\pi_2\pi_3 n$. Therefore n must appear in one of the first three positions.

Let $\mathcal{D}_n^i(1324) = \{\sigma \in \mathcal{D}_n(1324) \mid \sigma_i = n\}$ and let $\mathrm{d}_n^i(1324) = |\mathcal{D}_n^i(1324)|$. We claim that $\mathrm{d}_n^1(1324) = \mathrm{d}_n^2(1324)$ and $\mathrm{d}_n^3(1324) = \mathrm{d}_{n-2}^1(1324)$ for $n \geq 6$.

First we show that $\mathrm{d}_n^1(1324) = \mathrm{d}_n^2(1324)$ for $n \geq 6$. We claim that if $\pi\pi \in \mathcal{D}_n^2(1324)$, then π_1 and $\pi_2 = n$ can be transposed to produce a member of $\mathcal{D}_n^1(1324)$. Suppose to the contrary that $\pi\pi \in \mathcal{D}_n^2(1324)$ but $\pi_2\pi_1\pi_3 \cdots \pi_n\pi_2\pi_1\pi_3 \cdots \pi_n \notin \mathcal{D}_n^1(1324)$. In this case, we know that $\pi_1 < n-1$ since if $\pi_1\pi_2 = (n-1)n$, both $(n-1)$ and n can only play the role of '4' in a 1324 pattern and transposing them does not change their involvement. If $\pi_1 < n-1$ and it plays the role of a '1' in a pattern where n plays the role of '4', we must have used the first copy of π_1 and the second copy of n, so transposing them within each copy of π does not affect the existence of the 1324 patterns. The only other way for both to be involved in the same copy of 1324 that could possibly be destroyed by transposing π_1 and π_2 is for π_1 to play the role of '2' and n to play the role of '4' in a 1324 pattern in $\pi\pi$. In this case suppose the double list beginning with $\pi_1 n$ contains 1324 but the list beginning with $n\pi_1$ avoids 1324. Since $n\pi_1\pi_3\cdots\pi_n n\pi_1\pi_3\cdots\pi_n$ avoids 1324, all digits larger than π_1 must appear in increasing order immediately after π_1 and $\pi_1 \geq n-3$. Now, a case analysis shows that any σ beginning with $(n-3)n(n-2)(n-1)$ or $(n-2)n(n-1)$ cannot have σ_1 play the role of '2' in a 1324 pattern so it is the case that transposing π_1 and π_2 provides a bijection between $\mathcal{D}_n^1(1324)$ and $\mathcal{D}_n^2(1324)$.

To see that $\mathrm{d}_n^3(1324) = \mathrm{d}_{n-2}^1(1324)$ for $n \geq 6$ notice that if $\pi\pi \in \mathcal{D}_n^3(1324)$, then $\pi_1 = (n-2)$ and $\pi_2 = (n-1)$. We know these two numbers must appear in increasing order since 1 comes after n. If there exists i where $\pi_1 < i < \pi_2$, then $\pi_1\pi_2 i n$ is a forbidden pattern and if there exists i where $\pi_2 < i < n$, then $\pi_1 i \pi_2 n$ is a forbidden pattern. Since $\pi = (n-2)(n-1)n\pi_3\cdots\pi_n$, we may delete $(n-1)$ and n to obtain $\pi'\pi' \in \mathcal{D}_{n-2}^1(1324)$.

It remains to show that $\mathrm{d}_n^1(1324)$ satisfies the tribonacci recurrence (and thus so do $\mathrm{d}_n^2(1324)$, $\mathrm{d}_n^3(1324)$, and $\mathrm{d}_n(1324)$). For $\sigma' \in \mathcal{D}_{n-3}^1(1324)$, replace $n-3$ with $n(n-3)(n-2)(n-1)$ to obtain $\sigma \in \mathcal{D}_n^1(1324)$. For $\sigma' \in \mathcal{D}_{n-2}^1(1324)$, replace $n-2$ with $n(n-2)(n-1)$ to obtain $\sigma \in \mathcal{D}_n^1(1324)$. For $\sigma' \in \mathcal{D}_{n-1}^1(1324)$, prepend n to the front of each copy of π to obtain $\sigma \in \mathcal{D}_n^1(1324)$. This map sends members of $\mathcal{D}_{n-3}^1(1324) \cup \mathcal{D}_{n-2}^1(1324) \cup \mathcal{D}_{n-1}^1(1324)$ to $\mathcal{D}_n^1(1324)$.

Further, each of these operations is bijective. In other words, if $\sigma = \pi\pi \in \mathcal{D}_n^1(1324)$, then π either begins with $n(n-1)$, $n(n-2)(n-1)$, or $n(n-3)(n-2)(n-1)$. Indeed, if $\pi_2 \leq n-4$, then $n-1$, $n-2$, and $n-3$ appear in increasing order in π, and $(n-4)(n-2)(n-3)(n-1)$ is a 1324 pattern in σ, so $\pi_2 \geq n-3$. If $\pi_2 = n-2$ and $\pi_3 \neq n-1$, then $\pi_n = n-1$. If not, then we see all digits between π_2 and $n-1$ must be larger than all digits after $n-1$ in π, to avoid a 1324 pattern where $n-1$ plays the role of '3' and n plays the role of '4'. However, if $a < n-2$ is before $n-1$ in π and $b < a$ is after $n-1$ in π, then $b(n-2)a(n-1)$ is a copy of 1324 in σ. Therefore, if $\pi_2 = n-2$ and $\pi_3 \neq n-1$, then $\pi_n = n-1$. Now, since we assume $n \geq 6$, let $a < b < c$ be three digits less than $n-2$ in π.

If $\pi_n = n-1$, then $a(n-2)c(n-1)$ is a 1324 pattern in σ, so it must be the case that $\pi_3 = n-1$ if $\pi_2 = n-2$. Finally, if $\pi_2 = n-3$, then $n-2$ appears before $n-1$ in π (or $(n-3)(n-1)(n-2)n$ is a 1324 pattern in σ). If $\pi_3 < (n-3)$ then $\pi_3(n-2)(n-3)(n-1)$ is a 1324 pattern in σ. Now that we know $\pi_2 = n-3$ implies that $\pi_3 = n-2$, a similar analysis to the case where $\pi_2 = n-2$ shows that $\pi_4 = n-1$ as well.

We now have a bijection between $\mathcal{D}^1_n(1324)$ and $\mathcal{D}^1_{n-3}(1324) \cup \mathcal{D}^1_{n-2}(1324) \cup \mathcal{D}^1_{n-1}(1324)$ by editing appropriate prefixes so $\mathrm{d}^1_n(1324)$ satisfies the tribonacci recurrence. Because $\mathcal{D}^1_n(1324)$ is in bijection with $\mathcal{D}^2_n(1324)$ and $\mathcal{D}^3_n(1324)$ is in bijection with $\mathcal{D}^1_{n-2}(1324)$, $\mathrm{d}^2_n(1324)$ and $\mathrm{d}^3_n(1324)$ also satisfy the tribonacci recurrence. Finally, since $\mathrm{d}_n(1324) = \mathrm{d}^1_n(1324)+\mathrm{d}^2_n(1324)+\mathrm{d}^3_n(1324)$, $\mathrm{d}_n(1324)$ satisfies the tribonacci recurrence as well, which is what we wanted to show.

□

4 Summary

We have now completely characterized $\mathrm{d}_n(\rho)$ where ρ is a permutation pattern of length at most 4. The corresponding results are given in Table 2. These results provide an interesting contrast to pattern avoiding permutations. First, the only Wilf equivalences are the trivial ones. Second, the monotone pattern is the easiest pattern to avoid in the context of double lists. Finally, we obtained a variety of behaviors (constant, linear, quadratic, and exponential), as opposed to permutation pattern sequences which only grow exponentially.

The variety of sequence behaviors and the complete classification for length 4 patterns are both exciting developments, but this work raises additional possibilities for future work. In particular,

1. Is $1 \cdots n$ the easiest pattern of length n to avoid for all n? Can we characterize the hardest pattern of length n to avoid in general?

2. All of the sequences in Table 2 have rational generating functions. Do there exist patterns ρ where the sequence $\{\mathrm{d}_n(\rho)\}$ does not have a rational generating function?

3. With the exception of the proof of Theorem 6, the proofs in this paper were the result of detailed case analysis. While this is a thorough

Pattern ρ	$\mathrm{d}_n(\rho)$		OEIS
1342, 2431, 3124, 4213	15	$(n \geq 5)$	A010854
2143, 3412	$2n+2$	$(n \geq 6)$	A005843
1423, 2314, 3241, 4132	$3n+6$	$(n \geq 7)$	A008585
1432, 2341, 3214, 4123	$\frac{1}{2}n^2 + \frac{3}{2}n - 4$	$(n \geq 6)$	A052905
1243, 2134, 3421, 4312	$\frac{1}{2}n^2 + \frac{5}{2}n - 8$	$(n \geq 6)$	A183897
2413, 3142	L_{n+1}	$(n \geq 5)$	A000032
1324, 4231	$\lvert\mathcal{D}_{n-1}(\rho)\rvert + \lvert\mathcal{D}_{n-2}(\rho)\rvert + \lvert\mathcal{D}_{n-3}(\rho)\rvert$	$(n \geq 10)$	
1234, 4321	$2^n - n$	$(n \geq 4)$	A000325

Table 2: Formulas for $\mathrm{d}_n(\rho)$ where $\rho \in \mathcal{S}_4$

treatment that reveals much about the structure of pattern-avoiding double lists, it is not the most elegant approach. What are alternate proofs of these results?

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