PROVING THAT THREE LINES ARE CONCURRENT

The role of elementary geometry in learning proofs is well established. Among the more challenging problems that a student may encounter, those asking to prove that three lines are concurrent occupy a special place. The common approach in solving these problems is finding a suitable triangle where the three lines are known cevians such as medians or perpendicular bisectors. Yet, many other problems, are even more general and involve arbitrary concurrent cevians. Ceva's Theorem is a standard approach in this case:

THEOREM 1. ***Ceva's Theorem***
Given a triangle ABC, and points A', B', and C' that lie on lines BC, CA, and AB respectively, the lines AA', BB' and CC' are concurrent if and only if

$$\frac{|C'A|}{|C'B|} \cdot \frac{|A'B|}{|A'C|} \cdot \frac{|B'C|}{|B'A|} = 1.$$

Finding a suitable triangle and expressing the ratios in the equality above is not always straightforward. We present an elementary solution to an interesting problem emphasizing two important steps in following with Ceva's Theorem:

Let ABC be a triangle. Construct on each side of the triangle three rectangles $ACDE$, $AFGB$ and $BHIC$. Prove that the perpendicular bisectors to the segments EF, GH and ID are concurrent.

This is a particular case of a more general set of problems related to rectangles attached to sides of a triangle [1]. The initial difficulty lies in the fact that the problem seems very general. Similar problems usually have additional restrictions, such as squares instead of rectangles or an initial triangle that is not arbitrary. Furthermore, the location of the intersection point of the three lines in the problem seems also arbitrary for different rectangle widths and it does not relate immediately to the triangle ABC.
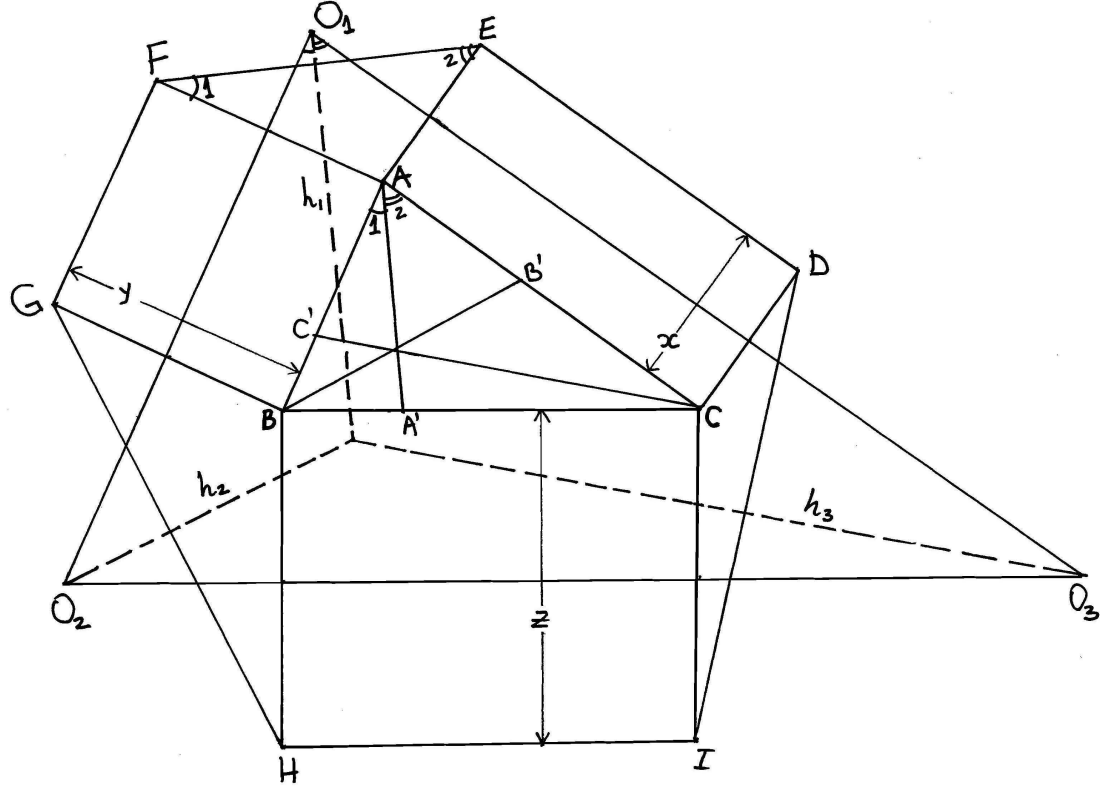
The overall proof strategy is to attempt to use strictly the given quantities only. These are the angles and the length of the sides of $\triangle ABC$ and the width of each rectangle, denoted by x, y and z.

Step 1. *Identify a triangle where the given lines are cevians.*

- We construct the perpendicular bisectors of FA, EA and CI. These form a triangle $\triangle O_1 O_2 O_3$. O_1 is the circumcenter of $\triangle EAF$, O_2 is the circumcenter of $\triangle GHB$ and O_3 is the circumcenter of $\triangle CDI$. So the lines given in the problem, h_1, h_2 and h_3, are cevians in $\triangle O_1 O_2 O_3$ which is similar to $\triangle ABC$.
- We now reduce the original problem to one in $\triangle ABC$ by constructing the following parallels: $AA' \parallel h_1$, $BB' \parallel h_2$ and $CC' \parallel h_3$. All we have to do now is to prove that AA', BB' and CC' are concurrent.

Step 2. *Use Ceva's Theorem. To ensure that it will work we need to use only the elements in $\triangle ABC$ and x, y and z when we express the ratios in Ceva's Theorem.*

We will use repeatedly the Law of Sines as follows:
In $\triangle ABA'$ and $\triangle ACA'$:

$$\frac{|A'B|}{|AA'|} = \frac{\sin \angle A_1}{\sin \angle B}, \; \frac{|A'C|}{|AA'|} = \frac{\sin \angle A_2}{\sin \angle C}, \quad \text{which leads to} \quad \frac{|A'B|}{|A'C|} = \frac{\sin \angle A_1}{\sin \angle A_2} \cdot \frac{\sin \angle C}{\sin \angle B}.$$

Now notice that
$$\angle A_1 \equiv \angle(O_1O_2, h_1) \equiv \angle F_1,$$
$$\angle A_2 \equiv \angle(O_1O_3, h_1) \equiv \angle E_2.$$

Using the Law of Sines in $\triangle EFA$ we obtain
$$\frac{\sin \angle A_1}{\sin \angle A_2} = \frac{\sin \angle F_1}{\sin \angle E_2} = \frac{x}{y}.$$

Finally,

$$\frac{|A'B|}{|A'C|} = \frac{x}{y} \cdot \frac{\sin \angle C}{\sin \angle B}.$$

Now we are able to write directly the other two ratios in Ceva's Theorem using circular permutations:

$$\frac{|B'C|}{|B'A|} = \frac{y}{z} \cdot \frac{\sin \angle A}{\sin \angle C},$$

$$\frac{|C'A|}{|C'B|} = \frac{z}{x} \cdot \frac{\sin \angle B}{\sin \angle A}.$$

In conclusion,

$$\frac{|A'B|}{|A'C|} \cdot \frac{|B'C|}{|B'A|} \cdot \frac{|C'A|}{|C'B|} = \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x} \cdot \frac{\sin \angle C}{\sin \angle B} \cdot \frac{\sin \angle A}{\sin \angle C} \cdot \frac{\sin \angle B}{\sin \angle A} = 1.$$

which proves that AA', BB', CC' and, therefore, h_1, h_2, h_3 are concurrent.

As a final remark, once we identify the triangle where the given lines are cevians, the effort should be directed to express each ratio in Ceva's Theorem using the original quantities only. In other words, one should write the three ratios using strictly the given hypothesis and avoiding statements inferred from it. This almost always guarantees the circular permutation and the necessary cancellation in the final part of the proof.

References

[1] Nikolaos Dergiades and Floor van Lamoen. *Rectangles Attached to Sides of a Triangle*, Forum Geometricorum, Vol. 3 (2003), 145-159